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湖 南 科 学 技 术 出 版 社

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湖南科学技术出版社出版

（长沙市展览馆路14号）

湖南省新华书店发行　湖南省新华印刷二厂印刷

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1981年6月第1版　1986年11月第2版第2次印刷

开本：850×1168毫米 1/32　印张：12.5　插页：5　字数：323,000

印数：3,001——4,300

统一书号：13204·35　定价：5.00元

征订期号：湖南新书目 86——7(25)

内 容 简 介

构造论是可列马尔科夫过程理论中的一个核心课题。本书是作者在构造论方面研究成果的总结。

全书共六篇：第一、二、三篇阐述构造论的分析方法和结果；第四、五篇阐述构造论的概率方法和结果，以及两种方法的联系；第六篇阐述与构造论相联系的马尔科夫过程的某些性质。书中对双边生灭过程和单边生灭过程给予了较高的重视。

本书可供科技工作者、高等院校理工科师生，特别是概率论专门化的师生和数学工作者阅读和参考。

第二版序

第二版在以下几方面对第一版作了较大的补充和修改：第一，给出了唯一性准则的简洁证明；第二，充实了马亭边界理论，对该理论作了详细的叙述和证明；第三，改写了双有限（有限非保守和有限流出）Q 过程的构造；第四，增加了逼近马氏链和逼近最小过程，以及 DV 型和 $(DV)^*$ 型延拓过程，广义DV型和广义$(DV)^*$型延拓过程．此外，还有许多小的改动．

本书第一版于1981年出版后，获得许多同志的热情关心、欢迎和鼓励．特别令人激动的是，不少同志提出了宝贵的修改意见．对此，我深表感谢．我还衷心感谢湖南科学技术出版社，给本书提供第二版的机会．

杨向群

1984年4月5日于湘潭大学

序

马尔科夫过程在随机过程中占有十分重要的地位。可列马尔科夫过程是马尔科夫过程的一个非常活跃而且理论比较完整的分支，它在科学技术的许多领域中都具有广泛的应用。不少著名的概率论学者，例如柯尔莫果洛夫（A.H.Колмогоров），杜勃(J. L. Doob)，费勒（W. Feller），钟开莱等，都长期在这一分支上开展工作，并作出了重要的贡献。二十多年来，我国的概率论工作者对这一领域进行了广泛深入的研究，目前已有《生灭过程和马尔科夫链》、《齐次可列马尔可夫过程》、《可逆马尔可夫过程》等三部专著出版。

1958年，南开大学王梓坤教授开始发表他在这一领域的研究成果，随后他和他的学生以及同事们深入并开拓了这方面的工作。由于长期不懈的努力，他们在这一学科的几个主要课题上都取得了很好的成绩。本书是杨向群教授的一部专著，总结了他二十年来对可列马尔科夫过程构造论的研究成果，其中有些成果还是首次发表的。

可列马尔科夫过程的一个核心课题是构造问题。杨向群的研究主要集中在构造论中生灭过程的构造，有限边界的 Q过程的构造，以及构造论中的概率方法及其与分析方法的联系等三个方面，并在这些方面取得了达到国际先进水平的成果。

生灭过程在国际上有不少人研究。卡林（S. Karlin）等给出了最小解的积分表现。费勒（W. Feller）用分析方法构造了同时满足向后和向前方程组的全部生灭过程。王梓坤用概率方法构造了全部不中断的生灭过程。杨向群用两种方法构造了全部生灭过

程并细致地考察了这些生灭过程的性质。因此，生灭过程的构造问题得以圆满解决。

可列马尔科夫过程的一般构造是十分困难的问题。1957年费勒在Q保守、有限流出、有限流入条件下，构造了满足向前方程组的全部Q过程，后来杨向群在同样条件下构造了全部Q过程。威利姆斯（D. Williams），钟开莱分别在Q保守、有限流出情形下求出了全部Q过程。最近，杨向群对这一问题又有了新的发展。他在更广泛的条件——Q的非保守状态和流出边界都有限——下求出了全部Q过程。

构造论中的两种方法，即概率方法和分析方法，各有其优点和不足，均取得一定的成果。两种方法的结果在形式上相差很远。杨向群就生灭过程找到了这两种方法的联系，把这两种不同形式的结果统一起来，既使分析结果赋以明确的概率意义，又使概率结果表达到同样简洁的形式。由杨向群开始的这个方面的工作还有待深入，它是一个发展前途广阔的领域。

本书不是作者成果的简单汇编，而是经过精心整理和编排的一本著作。我想，读者在阅读了王梓坤的专著《生灭过程和马尔科夫链》第一、二、三章后，就可以比较顺利地阅读完本书而达到这一领域的研究前沿。因此，本书的出版一定会对我国概率论的发展起促进作用。

在本书出版之际，我略抒所见，为学识所限，不免有不妥之处，请同志们批评指正。

侯振挺

1980年3月于长沙铁道学院

前　言

本书是作者在可列马尔科夫过程构造论方面研究成果的一个总结．第一章介绍构造论的分析基础，然后逐步进入专题．

构造论是马尔科夫过程理论中一个核心课题．它着眼于根据某些已知条件，构造出马尔科夫过程；或者说，将马尔科夫过程一个一个地刻划出来．这样可以根据每个马尔科夫过程的共性和个性来研究其性质．例如，基于构造论，可以比较顺利地从构造出来的马尔科夫过程的类型中，挑选出具有可逆性的过程，如同专著《可逆马尔科夫过程》中做过的那样．

目前解决构造论的方法有两种，一种是分析方法，一种是概率方法（各有其优点和不足），均取得了一定成果．本书第一章第三节对此作了概括的叙述．

全书共六篇，基本上分三部分．第一、二、三篇阐述构造论的分析方法和结果．第四、五篇阐述构造论的概率方法和结果，以及两种方法的联系．第六篇阐述与构造论相联系的马尔科夫过程的某些性质．对生灭过程给以较高的重视，这不仅因为生灭过程有它本身的理论和应用价值，而且它是产生解决一般问题的思想和方法的源泉．

第一篇论述构造论的分析基础．首先研究作为转移概率的 Q 过程的分析性质，最小解的构造和性质，Q 过程的一般形式；然后，对简单情形的Q过程进行了直接构造；最后讨论了Q过程的唯一性问题．

第二篇专论生灭过程构造论．我们构造了全部双边生灭过程和全部单边生灭过程，结果完整而富有启发性．掌握生灭过程构

造论对于理解Q过程的一般构造，是十分有益的。

第三篇研究Q过程的马亭（Martin）边界及其在构造论中的应用。首先，我们将离散参数马尔科夫链的马亭边界理论应用于Q过程，展开了广泛和深入的讨论。其次，我们引进了最小Q过程的马亭流出边界，并借助流出边界对Q过程的一般形式作了进一步的刻划。最后，对有限非保守、有限流出的情况构造了全部Q过程。

第四篇着重分析概率的Q过程的轨道结构。首先，我们引进W变换和强极限概念。W变换可以把一般的Q过程变换成各种轨道结构较简单的过程，这样便于从各个侧面来研究过程的轨道。强极限定理表明，较复杂的Q过程的轨道可以用较简单的Q过程的轨道来逼近。其次，我们引进飞跃区间的概念来研究过程的流出和流入问题。讨论了飞跃区间、飞跃点和柯氏方程组之间的关系，导出了流入分解定理。最后，研究了过程的延拓，直接地构造出简单过程的样本轨道。我们主要考虑D型延拓。为了保证延拓过程保持Q矩阵不变，还考虑了D^*型延拓。

第五篇专论生灭过程的概率方法构造。由于过程特殊，因而结果也较深刻。每个生灭过程不仅是一列杜勃（Doob）过程的强极限，而且它还与一列特征数列相对应。我们建立了两种生灭过程构造论的联系，使分析方法构造出来的过程有了明确的概率结构，使概率方法构造出来的过程具有简洁的分析表达形式。这样，便可以发挥每种方法的长处。两种方法配合使用，效果更为显著。

第六篇考察了马尔科夫过程的某些性质，主要是常返性和遍历性。这些性质紧密地依赖于构造论。

作者衷心地感谢王梓坤老师的辛勤指导，没有他的指导，这些研究成果是不可能取得的。侯振挺教授常与作者讨论，使作者获益非浅。他还仔细地阅读了本书的底稿并提出许多宝贵的改进意见；郭青峰副教授、吴荣、墨文川、陈木法等同志常与作者讨论并给予很大的支持和鼓励。作者谨向以上诸位表示感谢。

杨向群

1980年3月于湘潭大学

目　　录

第一篇　构造论的一般理论

第一章　构造论引论

§1. 引　言

本章中我们介绍构造论的基本结果，主要有：过程的分析性质，如连续性，Q矩阵的存在性，可微分性；过程满足柯氏方程组的条件；最小解的构造和性质；Q过程的一般形式等等。大部分内容取自 *Reuter*[1]。§4和§5以及定理6.2取自 *Chung*[1]。§8的结论来自 *Reuter*[2, 3]和 *Feller*[3]。§§10—12来自杨向群[12]。

§2. 记号和定义

设E为一可列指标集，称为状态空间。定义在E上的有界列矢量（或称有界函数）组成的巴拿赫空间记为$\boldsymbol{m}$，$f\in\boldsymbol{m}$的范数记为$\|f\|=\sup\limits_{i\in E}|f_i|$。定义在$E$上的可和行矢量组成的巴拿赫空间记为$\boldsymbol{l}$，$g\in\boldsymbol{l}$的范数记为$\|g\|=\sum\limits_{i\in E}|g_i|$。如$f\in\boldsymbol{m}$，$g\in\boldsymbol{l}$，其内

注　“(1)”和“定理1”分别表示同一节中的第一式和定理1；
“(2.1)”和“定理2.1”分别表示同一章第二节中的第一式和定理1；
“(3.2.1)”和“定理3.2.1”分别表示第三章第二节中的第一式和定理1，其余类推。(引理编号也是如此)。

积定义为

$$[g,f]=\sum_{j\in E}g_jf_j. \tag{1}$$

我们将采用矩阵记号，矩阵的极限将在逐元的意义下理解。从分析的观点看，一个马尔科夫过程或简称过程，是指一族满足下列条件的实值矩阵$P(t)=\{p_{ij}(t)\}$ $(i,j\in E,\ t\geqslant 0)$：

$$P(t)\geqslant \mathbf{0},\quad P(t)\mathbf{1}\leqslant \mathbf{1}; \tag{A}$$

$$P(t+s)=P(t)P(s); \tag{B}$$

$$\lim_{t\downarrow 0}P(t)=P(0)=I; \tag{C}$$

$\mathbf{0}$也表示零矩阵，有时也表示零列（或行）矢量，$\mathbf{1}$表示分量为1的单位列矢量，这并不会引起混淆。I表示么矩阵。

用矩阵$P(t)$的元素表示，条件(A)(B)(C)成为：对任意$i,j\in E$，$s,t\geqslant 0$有

$$p_{ij}(t)\geqslant 0,\quad \sum_j p_{ij}(t)\leqslant 1; \tag{A}$$

$$p_{ij}(t+s)=\sum_k p_{ik}(t)p_{kj}(s); \tag{B}$$

$$\lim_{t\downarrow 0}p_{ij}(t)=p_{ij}(0)=\delta_{ij}=\begin{cases}1, & \text{如} i=j,\\ 0, & \text{如} i\neq j.\end{cases} \tag{C}$$

其中求和号展布在E上，通常称(B)为柯尔莫果洛夫——切普曼(Колмогоров-Chapman)方程。

称

$$d_i(t)=1-\sum_j p_{ij}(t) \tag{2}$$

为过程$P(t)$的中断函数。

引理1 $d_i(t)$是t的非降函数。如果对某个$t>0$，对一切$i\in E$有$d_i(t)=0$，则必定对一切$t\geqslant 0$成立。

证 对s，$t>0$，由(B)(C)，

$$d_i(s+t)=1-\sum_j p_{ij}(s+t)$$

$$=1-\sum_{j}\sum_{k}p_{ik}(s)p_{kj}(t)$$

$$=1-\sum_{k}p_{ik}(s)\sum_{j}p_{kj}(t)\geqslant 1-\sum_{k}p_{ik}(s)$$

$$=d_i(s).$$

由此得$d_i(t)$的非降性．由上式看出，如果对某个 $t>0$对一切k有 $\sum_j p_{kj}(t)=1$，则$d_i(s+t)=0$ $(s>0)$，证毕。

当对某个（从而一切）$t>0$有$d_i(t)=0$ $(i\in E)$，即

$$P(t)\mathbf{1}=\mathbf{1},\ t>0. \tag{D}$$

或等价地

$$\sum_j P_{ij}(t)=1,\ i\in E,\ t>0. \tag{D}$$

时，称过程$P(t)$为不中断的，否则称为中断的。

对于中断过程$P(t)$，可以按下面方式扩大状态空间E而得到不中断的过程$\widetilde{P}(t)$：任取指标$\Delta\overline{\in}E$，记$\widetilde{E}=E\cup\{\Delta\}$，令

$$\left.\begin{aligned}&\widetilde{p}_{ij}(t)=p_{ij}(t),\ \widetilde{p}_{i\Delta}(t)=d_i(t),\ i,\ j\in E,\\&\widetilde{p}_{\Delta j}(t)=\begin{cases}1, & \text{当}j=\Delta\text{时},\\0, & \text{当}j\in E\text{时}.\end{cases}\end{aligned}\right\} \tag{3}$$

直接验证便得

引理2 设$P(t)=\{p_{ij}(t)\}$ $(i,j\in E,\ t\geqslant 0)$ 是中断过程，则 $\widetilde{P}(t)=\{\widetilde{p}_{ij}(t)\}$ $(i,\ j\in\widetilde{E},\ t\geqslant 0)$是不中断过程。

一切过程$P(t)$组成的类记为$\mathscr{P}$。

对每个$P(t)\in\mathscr{P}$，在$t=0$的右导数$P'(0)$存在，即极限

$$q_{ij}=p'_{ij}(0)=\lim_{t\downarrow 0}\frac{p_{ij}(t)-p_{ij}(0)}{t} \tag{4}$$

存在，而且

$$0\leqslant q_{ij}<\infty\,(i\neq j),\quad \sum_{j\neq i}q_{ij}\leqslant -q_{ii}\equiv q_i\leqslant\infty. \tag{5}$$

我们称矩阵$Q=(q_{ij})$ $(i, j\in E)$ 为过程$P(t)$的Q矩阵。

当$q_i<\infty$时，称状态i稳定。我们将研究一切状态都稳定即Q矩阵有限的过程。这种过程组成的类记为$\mathscr{P}_s$，即其Q矩阵满足下面条件：

$$0\leqslant q_{ij}<\infty \quad (i\neq j), q_i\equiv -q_{ii}<\infty, \quad d_i\equiv q_i-\sum_{j\neq i} q_{ij}\geqslant 0. \tag{6}$$

为了强调$P(t)\in\mathscr{P}_s$与有限矩阵Q的关系(4)，即

$$P'(0)=Q, \tag{7}$$

我们称$P(t)$ 为Q过程。我们特别强调指出：凡Q过程$P(t)$ 的一切状态都是稳定的。有相同Q矩阵的Q过程类记为$\mathscr{P}_s(Q)$。

§3. 构造问题

构造问题是相反的问题。除了 Williams[2]外，迄今对构造问题的研究几乎都是对满足 (2.6) 的Q进行的。

设给定矩阵Q满足(2.6)，称$\boldsymbol{d}=(d_i)_{i\in E}$为$Q$的非保守列矢量。如果$d_i=0$，称$i$为保守状态。称

$$H=\{i\mid d_i>0\} \tag{1}$$

为非保守状态集。如果H为空集，称Q为保守的。

构造问题的提法是：给定矩阵Q满足(2.6)。问题一，是否存在过程$P(t)$ 满足 (2.7)？换言之，Q过程是否存在？问题二，如果Q过程存在，是否唯一？问题三，如果Q过程不唯一，如何构造全部Q过程？

构造问题最早由Kolmogorov[1]于1931年提出，他首先导出向后微分方程组

$$p'_{ij}(t)=\sum_k q_{ik}p_{kj}(t), \quad i,j\in E, \quad t\geqslant 0. \tag{KB}$$

和向前微分方程组

$$p'_{ij}(t)=\sum_k p_{ik}(t)q_{kj}, \quad i,j\in E, \quad t\geqslant 0. \tag{KF}$$

1940年，Feller[1]证明，Q过程总是存在的。他并且构造了一个最小Q过程，从而完满地解决了问题一。

1945年Doob[1]证明：对保守的Q，或者只有一个Q过程，这就是最小Q过程，或者有无穷多个Q过程。Reuter[1]对保守Q找出了Q过程唯一的充要条件。从而对保守的矩阵Q，完满地解决了问题二。对于一般的Q，1974年，侯振挺教授[1]得到了Q过程的唯一性准则，从而，使问题二得到彻底解决。

对于问题三，距彻底解决尚远。目前试图解决此问题的方法大致有两种。一种是分析方法，例如Reuter[1—4]，Feller[1—5]，Williams[1—2]，孙振祖[1]，杨向群[1,2,6]，胡迪鹤[1—3]，主要地是使用分析工具和方法求解满足柯氏向后或向前方程组的Q过程，或求出Q过程$P(t)$所产生的压缩半群的无穷小算子，或求出Q过程的预解算子。另一种方法是概率方法，即极限过渡法。这种方法由王梓坤教授[7]于1958年提出，并成功地解决了生灭过程的构造问题（王梓坤[3]，王梓坤与杨向群[1，2]），其基本思想是：用结构比较简单的杜勃过程的样本轨道逼近Q过程的样本轨道，即Q过程的样本轨道是一列杜勃过程的样本轨道的强极限。这一方法后来为侯振挺教授[2]和作者[7—10]所深入。

问题三的分析方法解决情况如下。假定Q保守。Reuter[2]和孙振祖[1]对Q单流出时构造了全部Q过程。Feller[3]在有限流出和有限流入的情况下，求出了同时满足柯氏向后和向前方程组的全部Q过程。作者[1]在与费勒同样的假设下求出了全部Q过程。Williams[1]和钟开莱[2]在有限流出假设下，求出了全部Q过程。

因为当Q保守时，任何Q过程都满足柯氏向后方程组，而当Q非保守时，Q过程可以不必满足向后方程组，因而对于一般情形的Q的构造问题，较少文献涉及。

对于仅在状态0可以不必保守的生灭矩阵Q，Feller[5]求出了同时满足柯氏向后、向前方程组的生灭过程，作者[2]求出了全部生灭过程，即既可以满足两个方程组之一，也可以两个方程组都不满足的生灭过程。在本书第三章中，对于一般的Q，当Q

单流出时我们构造了满足向后方程组的全部Q过程，当Q单流入时我们构造了满足向前方程组的全部Q过程。特别地，对于零流出且仅有一个非保守状态时，求出了全部Q过程。在本书第七章中，我们对有限流出有限非保守的Q，求出了全部Q过程。

国内还有一些同志在构造论方面作了许多工作，可见书后的参考文献。

§4. 连续性

定理1 设$P(t)\in\mathscr{P}$，则对任意$i,j\in E$及$h>0$有

$$|p_{ij}(t\pm h)-p_{ij}(t)|\leqslant 1-p_{ii}(h), \tag{1}$$

$$|d_i(t\pm h)-d_i(t)|\leqslant 1-p_{ii}(h). \tag{2}$$

而且中断函数$d_i(t)$ 和$p_{ij}(t)$ 在$[0,\infty)$上一致连续。

证 只对加号证明即可。由(2.B)，

$$p_{ij}(t+h)-p_{ij}(t)=\sum_k p_{ik}(h)p_{kj}(t)-p_{ij}(t)$$

$$=[p_{ii}(h)-1]p_{ij}(t)+\sum_{k\neq i}p_{ik}(h)p_{kj}(t). \tag{3}$$

前一项非正，后一项非负且$\leqslant\sum\limits_{k\neq i}p_{ik}(h)\leqslant 1-p_{ii}(h)$，从而得(1)。将$P(t)$按(2.3)扩充为$\widetilde{P}(t)$ 后并应用$\widetilde{p}_{i\Delta}(t)$于(1)得(2)。证毕。

定理2 在有限区间$[0,b]$上，级数$\sum\limits_j p_{ij}(t)$ 一致收敛。

证 由定理1，$p_{ij}(t)$ 和$1-d_i(t)$ 都是连续函数，从而只需引用地尼 (Dini)定理(梯其玛希[1]第14页)，即可。

定理3 设$h>0$。和式

$$\sum_j|p_{ij}(t+h)-p_{ij}(t)| \tag{4}$$

当t增加时非增。当$h\to 0$时，上式在$t\geqslant\delta>0$上一致地趋于0。特别地，对每个$\delta>0$，$p_{ij}(t)$ 在$[\delta,\infty)$ 上一致连续。

证 设$0\leqslant s<t$。由（2.A）（2.B），

$$\sum_j|p_{ij}(t+h)-p_{ij}(t)|$$

$$=\sum_j\left|\sum_k[p_{ik}(s+h)-p_{ik}(s)]p_{kj}(t-s)\right|$$

$$\leqslant\sum_k|p_{ik}(s+h)-p_{ik}(s)|\sum_j p_{kj}(t-s)$$

$$\leqslant\sum_k|p_{ik}(s+h)-p_{ik}(s)|。\tag{5}$$

得证第一结论。其次，由于定理1，$p_{ij}(t)$ 连续，因而上式可以对$s\in[0,\ \delta]$积分。故$t\geqslant\delta>0$时，

$$\sum_j|p_{ij}(t+h)-p_{ij}(t)|\leqslant\sum_k\frac{1}{\delta}\int_0^\delta|p_{ik}(s+h)-p_{ik}(s)|ds,$$

但当$0\leqslant h\leqslant\sigma$时，后一级数被级数

$$\sum_k\frac{2}{\delta}\int_0^{2\delta}p_{ik}(s)ds$$

所控制，因而对$h\in[0,\ \delta]$一致连续。但依（梯其玛希［1］，第404页）一个定理，对每个k有

$$\lim_{h\to0}\int_0^\delta|p_{ik}(s+h)-p_{ik}(s)|ds=0。\tag{6}$$

从而对$t\geqslant\delta$一致地有

$$\overline{\lim_{h\to0}}\sum_j|p_{ij}(t+h)-p_{ij}(t)|$$

$$\leqslant\sum_k\frac{1}{\delta}\lim_{h\to0}\int_0^\delta|p_{ik}(s+h)-p_{ik}(s)|ds$$

$$=\sum_k\frac{1}{\delta}0\cdot=0。$$ 证毕。

定理4 在（0，∞）上，恒有$p_{ii}(t)>0$；对$i\neq j$，则恒有$p_{ij}(t)=0$或$p_{ij}(t)>0$。

证 （一）由(2.B)，对任意$t>0$，有

$$p_{ii}(t)\geqslant\left[p_{ii}\left(\frac{t}{n}\right)\right]^n，对一切n。\tag{7}$$

由(2.C)得$p_{ii}(t)>0$。

(二) 设$i\neq j$。由 (2.B)，

$$p_{ij}(t+s)\geqslant p_{ii}(s)p_{ij}(t)。\tag{8}$$

故如果对某$t_1>0$有$p_{ij}(t_1)>0$，则对一切$t\geqslant t_1$有$p_{ij}(t)>0$。

(三)谬设存在$t_0>0$使

$$p_{il}(t)=0\ \ (0<t\leqslant t_0)，\ p_{il}(2t_0)=c>0。\tag{9}$$

为了叙述方便可设E为非负整数集。依定理2，存在正整数N使

$$\sum_{j>N}p_{ij}(t)<\frac{c}{4}，\ 0<t\leqslant 2t_0。\tag{10}$$

令 $s=\dfrac{t_0}{2N}$，并定义

$$A_m=\{k|p_{ik}(ms)>0\}，\ m\geqslant 1。\tag{11}$$

由(二)，$A_m\subset A_{m+1}$。记$B_1=A_1$，$B_m=A_m-A_{m-1}$ $(m\geqslant 2)$。如果$k\overline{\in}A_m$，则

$$0=p_{ik}(ms)=\sum_j p_{ij}\{(m-1)s\}p_{jk}(s)$$

$$=\sum_{j\in A_{m-1}}p_{ij}\{(m-1)s\}p_{jk}(s)。\tag{12}$$

故 $\quad p_{jk}(s)=0，\ j\in A_{m-1}，\ k\overline{\in}A_m。\tag{13}$

倘若对某m：$1<m\leqslant 2N$有$A_m=A_{m-1}$。则由上式得

$$p_{ik}\{(m+1)s\}=\sum_{j\in A_m}p_{ij}(ms)p_{jk}(s)=0，\ k\overline{\in}A_m，$$

故$A_{m+1}=A_m$。重复论证得$A_n=A_m(n>m)$，特别有$A_{2N}=A_{4N}$。但由 (9)，$l\overline{\in}A_{2N}$，$l\in A_{4N}$。这一矛盾说明B_m $(1\leqslant m\leqslant 2N)$ 均非空集且互不相交。

令$1\leqslant m\leqslant 2N$，则$A_m\subset A_{2N}$。如果$k\overline{\in}A_m$，则由 (13)，对每个$n\geqslant 1$，

$$p_{ik}\{(n+1)s\}=\Big(\sum_{j\in A_m}+\sum_{j\in B_m}+\sum_{j\in A_{m-1}}\Big)p_{ij}(ns)p_{jk}(s).$$

由于(13)，第三个和为0，从而

$$\sum_{k\in A_m}p_{ik}\{(n+1)s\}\leqslant\sum_{j\in A_m}p_{ij}(ns)+\sum_{j\in B_m}p_{ij}(ns).$$

对n从1到$4N-1$求和得

$$\sum_{k\in A_m}p_{ik}(4Ns)\leqslant\sum_{n=1}^{4N}\sum_{j\in B_m}p_{ij}(ns).$$

因$1\in A_{2N}$，左方至少等于$p_{i1}(4Ns)=c$，故

$$c\leqslant\sum_{n=1}^{4N}\sum_{j\in B_m}p_{ij}(ns),\quad 1\leqslant m\leqslant 2N. \tag{14}$$

由于B_1，$B_2\cdots$，B_{2N}非空且互不相交，故至少存在其中N个$B_n(1\leqslant n\leqslant 2N)$，其并集$B$与集（1，2，…，$N$）不相交。因此

$$Nc\leqslant\sum_{n=1}^{4N}\sum_{j\in B}p_{ij}(ns). \tag{15}$$

另一方面，从(10)，

$$\sum_{j\in B}p_{ij}(ns)\leqslant\sum_{j>N}p_{ij}(ns)<\frac{c}{4}.$$

故(15)右方严格小于$4N\cdot\frac{c}{4}=Nc$。这一矛盾证明(9)不可能成立，证毕。

系 任意固定i，$d_i(t)$在$(0,\infty)$上恒等于零，或恒大于零。

证 结合引理2.2和定理4.4即可。

§5. Q矩阵的存在性

定理1 设$P(t)\in\mathscr{P}$，则对每个$i\in E$，

$$-p'_{ii}(0)=\lim_{t\downarrow 0}\frac{1-p_{ii}(t)}{t} \tag{1}$$

存在，但可能无穷。

证 由定理4.4，可以令

$$\Phi(t)=-\ln p_{ii}(t), \tag{2}$$

且$\Phi(t)$非负有限。由(2.A)(2.B)有

$$p_{ii}(s+t)\geqslant p_{ii}(s)p_{ii}(t). \tag{3}$$

由此推出

$$\Phi(s+t)\leqslant\Phi(s)+\Phi(t). \tag{4}$$

令
$$q_i\equiv\sup_{t>0}\frac{\Phi(t)}{t}. \tag{5}$$

如果$q_i<\infty$，则对任意$\varepsilon>0$，存在$t_0>0$使$\frac{\Phi(t_0)}{t_0}>q_i-\varepsilon$。而对每个$t>0$，可写$t_0=nt+\delta(0\leqslant\delta<t)$。故

$$q_i-\varepsilon\leqslant\frac{\Phi(t_0)}{t_0}\leqslant\frac{n\Phi(t)+\Phi(\delta)}{t_0}=\frac{nt}{t_0}\cdot\frac{\Phi(t)}{t}+\frac{\Phi(\delta)}{t_0}.$$

当$t\to0$时，$\frac{nt}{t_0}\to1$。由(2.C)有$\Phi(\delta)\to0$，从而

$$q_i-\varepsilon\leqslant\varliminf_{t\downarrow0}\frac{\Phi(t)}{t}\leqslant\varlimsup_{t\downarrow0}\frac{\Phi(t)}{t}\leqslant q_i.$$

由于ε任意，故$\lim_{t\downarrow0}\frac{\Phi(t)}{t}=q_i$。如果$q_i=\infty$，代替$q_i-\varepsilon$为任意大的正数$M$进行讨论，仍然有$\lim_{t\downarrow0}\frac{\Phi(t)}{t}=\infty$。

而
$$\lim_{t\downarrow0}\frac{\Phi(t)}{t}=\lim_{t\downarrow0}\frac{-\ln\{1-(1-p_{ii}(t))\}}{t}$$
$$=\lim_{t\downarrow0}\frac{1-p_{ii}(t)}{t}.$$

由此得证定理。

定理2 对任意$i\neq j$，

$$p'_{ij}(0)=\lim_{t\downarrow0}\frac{p_{ij}(t)}{t} \tag{6}$$

存在而且非负有限。

证 指定$i\neq j$及$h>0$。定义${}_jp_{ii}^{0}(h)=1$，

$${}_jp_{ii}^{n}(h)=\sum p_{ik_1}(h)p_{k_1k_2}(h)\cdots p_{k_{n-1}i}(h), \tag{7}$$

$$f_{ij}^{n}(h)=\sum p_{ik_1}(h)p_{k_1k_2}(h)\cdots p_{k_{n-1}j}(h)。\tag{8}$$

求和号都展布在$k_1 \neq j$，$k_2 \neq j$，…，$k_{n-1}\neq j$上。虽然我们用分析表达式定义 ${}_jp_{ii}^{n}(h)$ 及 $f_{ij}^{n}(h)$，但实质上它们分别是以 $\{p_{ij}(h)\}$ 为一步转移概率矩阵的马氏链，从i出发不经过j而于第n步到达i的概率，以及从i出发于第n步首达j的概率。

由(2.A)(2.B)，可验证

$$p_{ij}(nh)\geqslant\sum_{m=0}^{n-1}{}_jp_{ii}^{m}(h)p_{ij}(h)p_{jj}\{(n-m-1)h\},\tag{9}$$

$${}_jp_{ii}(mh)={}_jp_{ii}^{m}(h)+\sum_{a=1}^{m-1}f_{ij}^{a}(h)p_{ji}\{(m-a)h\},\tag{10}$$

$$p_{ij}(mh)=\sum_{a=1}^{m}f_{ij}^{a}(h)p_{jj}\{(m-a)h\}。\tag{11}$$

如果从概率意义考虑，上面式子更为显然。由（2.C），对

$\varepsilon<\frac{1}{2}$存在$t_0>0$使

$$\left.\begin{array}{l}\max\limits_{0\leqslant t\leqslant t_0}p_{ij}(t)<\varepsilon,\ \max\limits_{0\leqslant t\leqslant t_0}p_{ji}(t)<\varepsilon,\\ \min\limits_{0\leqslant t\leqslant t_0}p_{ii}(t)>1-\varepsilon,\ \min\limits_{0\leqslant t\leqslant t_0}p_{jj}(t)>1-\varepsilon。\end{array}\right\}\tag{12}$$

由此及(11)得

$$\sum_{a=1}^{m}f_{ij}^{a}(h)(1-\varepsilon)<\varepsilon,$$

从而

$$\sum_{a=1}^{m}f_{ij}^{a}(h)\leqslant 1。\tag{13}$$

因此从(10)得

$${}_jp_{ii}^{m}(h)\geqslant p_{ii}(mh)-\max_{1\leqslant a\leqslant m}p_{ji}\{(m-a)h\}。$$

由此，如果$mh<t_0$，则${}_jp_{ii}^{m}(h)>1-2\varepsilon$。于是从（9）得

$$p_{ij}(nh)>(1-2\varepsilon)\sum_{m=0}^{n-1}p_{ij}(h)(1-\varepsilon)\geqslant(1-3\varepsilon)np_{ij}(h),$$

$$\frac{p_{ij}(nh)}{nh}>(1-3\varepsilon)\frac{p_{ij}(h)}{h},\ 如果nh<t_0. \tag{14}$$

令

$$q_{ij}=\lim_{t\downarrow 0}\frac{p_{ij}(t)}{t}.$$

则由（14）有$q_{ij}<\infty$，而且存在 t_1，$0<t_1<\frac{t_0}{2}$使

$$\frac{p_{ij}(t_1)}{t_1}<q_{ij}+\varepsilon.$$

左方关于t_1是连续的，故存在$h_0>0$使

$$\frac{p_{ij}(t)}{t}<q_{ij}+2\varepsilon,\ |t-t_1|<h_0. \tag{15}$$

对任意的$h\in(0,\ \min(h_0,\ t_1))$，存在$n$使$t_1\leqslant nh<t_1+h<t_0$。由（14）、（15）得

$$(1-3\varepsilon)\frac{p_{ij}(h)}{h}<\frac{p_{ij}(nh)}{nh}<q_{ij}+2\varepsilon.$$

由于ε任意，我们得$\overline{\lim_{t\downarrow 0}}\frac{p_{ij}(t)}{t}\leqslant q_{ij}$。证毕。

系 对一切$i\in E$，右导数

$$D_i\equiv d_i'(0)=\lim_{t\downarrow 0}\frac{d_i(t)}{t}\geqslant 0 \tag{16}$$

存在而且有限。

证 由于引理2.2，只需将定理2应用于$\overleftarrow{p}_{i\Delta}(t)=d_i(t)$，即得(16)。

定理3 对任意$i\in E$，有

$$\sum_{j\neq i}p_{ij}'(0)+d_i'(0)\leqslant -p_{ii}'(0). \tag{17}$$

证 因

$$\sum_{j\neq i}\frac{p_{ij}(t)}{t}+\frac{d_i(t)}{t}=\frac{1-p_{ii}(t)}{t},$$

用Fatou引理便得(17)。

§6. 可微分性

设$P(t)\in\mathscr{P}$，$Q=(q_{ij})=\{p'_{ij}(0)\}$为 Q矩阵，且$q_i=-q_{ii}$，$D_i=d'_i(0)$。

定理1 设对指定i有$q_i<\infty$[1)]。则在$[0,\infty)$中$p_{ij}(t)$(对一切$j\in E$)和$d_i(t)$有有穷的连续导数，且

$$|p_{ij}(t+h)-p_{ij}(t)|\leqslant q_i h,\quad t\geqslant 0,\ h\geqslant 0. \tag{1}$$

$$|d_i(t+h)-d_i(t)|\leqslant q_i h,\quad t\geqslant 0,\ h\geqslant 0. \tag{2}$$

$$\sum_j |p'_{ij}(t)|+d'_i(t)\leqslant 2q_i,\quad t>0. \tag{3}$$

$$\sum_j p'_{ij}(t)+d'_i(t)=0,\quad t>0. \tag{4}$$

$$p'_{ij}(t_1+t_2)=\sum_k p'_{ik}(t_1)p_{kj}(t_2),\quad t_1>0,\ t_2\geqslant 0. \tag{5}$$

证 (一) 从 (4.7) 及q_i的定义得

$$p_{ii}(t)\geqslant e^{-q_it}\geqslant 1-q_it. \tag{6}$$

由此及 (4.1)(4.2) 得 (1)、(2)。

(1)、(2) 说明$p_{ij}(t)$ 和$d_i(t)$满足李普希兹条件，因而是绝对连续的，从而对几乎一切$t\geqslant 0$，$p'_{ij}(t)$，$d'_i(t)$存在，并且

$$p_{ij}(t)=\delta_{ij}+\int_0^t p'_{ij}(u)du, \tag{7}$$

$$d_i(t)=\int_0^t d'_i(u)du. \tag{8}$$

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1) 当$q_i=\infty$时，也可以得出$p_{ij}(t)(j\in E)$及$d_i(t)$在$(0,\infty)$上有有穷的连续导数，且(5)对$t_1>0$，$t_2>0$成立。见王梓坤[2, §2.2, 定理2]。

（二） 令$\Delta_{ij}(t,\ t+s)=\dfrac{p_{ij}(t+s)-p_{ij}(t)}{s}$， $t\geqslant 0$， $s>0$。

(9)

$$\Delta_i(t,\ t+s)=\frac{d_i(t+s)-d_i(t)}{s},\quad t\geqslant 0,\quad s>0. \tag{10}$$

由于$\sum\limits_j p_{ij}(t)+d_i(t)=1$及$d_i(t)$的非降性，我们有

$$\sum_j \Delta_{ij}(t,\ t+s)+\Delta_i(t,\ t+s)=0, \tag{11}$$

$$\Delta_i(t,\ t+s)\geqslant 0. \tag{12}$$

由（4.8）及（6）得

$$\Delta_{ij}(t,\ t+s)\geqslant -\frac{1-p_{ii}(s)}{s}p_{ij}(t)\geqslant -q_i p_{ij}(t). \tag{13}$$

故对任意集合$A\subset E$有

$$\sum_{j\in A}\Delta_{ij}(t,\ t+s)\geqslant -q_i\sum_{j\in A}p_{ij}(t)\geqslant -q_i. \tag{14}$$

由此并注意（12），从（11）得

$$\sum_{j\in A}\Delta_{ij}(t,\ t+s)$$

$$=-\sum_{j\in E-A}\Delta_{ij}(t,\ t+s)-\Delta_i(t,t+s)\leqslant q_i. \tag{15}$$

于是对任意$A\subset E$，

$$\left|\sum_{j\in A}\Delta_{ij}(t,\ t+s)\right|\leqslant q_i, \tag{16}$$

还有

$$0\leqslant\Delta_i(t,\ t+s)=-\sum_j\Delta_{ij}(t,\ t+s)\leqslant q_i. \tag{17}$$

在（14）（15）中取$A=\{j|\triangle_{ij}\geqslant 0\}$，则得

$$\sum_j|\Delta_{ij}|=\sum_{j\in A}\triangle_{ij}+\sum_{j\in E-A}(-\Delta_{ij})\leqslant 2q_i, \tag{18}$$

由此得

$$\sum_j |p'_{ij}(t)| \leqslant 2q_i, \tag{19}$$

只要牵涉的导数存在。特别地，上式对几乎一切$t \geqslant 0$成立。

在（7）中对j求和后再与（8）相加，由于（19），可以交换求和与积分的次序，于是

$$\sum_j p_{ij}(t) + d_i(t) = 1 + \sum_j \int_0^t p'_{ij}(u)du + \int_0^t d'_i(u)du$$

$$= 1 + \int_0^t \Big[\sum_j p'_{ij}(u) + d'_i(u)\Big]du.$$

故

$$\int_0^t \Big[\sum_j p'_{ij}(u) + d'_i(u)\Big]du = 0, \quad t \geqslant 0,$$

从而对几乎一切$t \geqslant 0$有

$$\sum_j p'_{ij}(t) + d'_i(t) = 0. \tag{20}$$

（三）设t使$p'_{ij}(t)$，$d'_i(t)$存在，且使（20）成立。往证

$$\sum(s) \equiv \sum_j |\Delta_{ij}(t, \ t \pm s) - p'_{ij}(t)| \to 0, \ s \downarrow 0. \tag{21}$$

先对加号证明。由于（19），任给 $\varepsilon > 0$，可以选取有限集$A \subset E$使

$$q_i \sum_{j \in A} p_{ij}(t) + \sum_{j \bar{\in} A} |p'_{ij}(t)| < \varepsilon. \tag{22}$$

则$\sum(s) \leqslant \sum_{j \in A} |\Delta_{ij}(t, t+s) - p'_{ij}(t)| + \sum_{j \bar{\in} A} |\Delta_{ij}(t, t+s)| + \varepsilon.$

用$\sum{}'$表示对$\Delta_{ij} < 0$的指标j求和，则由（14）、（22），

$$\sum_{j \bar{\in} A} |\Delta_{ij}| = \sum_{j \bar{\in} A} \Delta_{ij} - 2 \sum_{j \bar{\in} A}{}' \Delta_{ij}$$

$$\leqslant \sum_{j \bar{\in} A} \Delta_{ij} + 2q_i \sum_{j \bar{\in} A} p_{ij}(t)$$

$$\leqslant \sum_{j \overline{\in} A} \Delta_{ij} + 2\varepsilon, \tag{23}$$

从而由（11），

$$\sum(s) \leqslant \sum_{j \overline{\in} A} |\Delta_{ij}(t, t+s) - p'_{ij}(t)| - \sum_{j \in A} \Delta_{ij} - \Delta_i + 3\varepsilon,$$

由（20)（22）得

$$\overline{\lim_{s \downarrow 0}} \sum(s) \leqslant 0 - \sum_{j \in A} p'_{ij}(t) - d'_i(t) + 3\varepsilon$$

$$= \sum_{j \overline{\in} A} p'_{ij}(t) + 3\varepsilon < 4\varepsilon.$$

得证（21）中加号情形。

对于减号的情形，上面的论证稍作修改后仍然有效．代替（22）式的将是

$$q_i \sum_{j \overline{\in} A} p_{ij}(t-s) + \sum_{j \overline{\in} A} |p'_{ij}(t)| < \varepsilon, \ (s \leqslant t). \tag{24}$$

由于定理4.2，上式是成立的．（23）中的$p_{ij}(t)$应改为$p_{ij}(t-s)$而仍然成立．

（四）设$t>0$使$p'_{ij}(t)$，$d'_i(t)$存在，且（20）成立．则对任意$u>t$，由（21），

$$\sum_j |\Delta_{ij}(u, u+s) - \sum_k p'_{ik}(t) p_{kj}(u-t)|$$

$$= \sum_j \left| \sum_k [\Delta_{ik}(t, t+s) - p'_{ik}(t)] p_{kj}(u-t) \right|$$

$$\leqslant \sum_k |\Delta_{ik}(t, t+s) - p'_{ik}(t)| \to 0, \ s \downarrow 0. \tag{25}$$

由此得

$$\left| \Delta_i(u, u+s) - \sum_j \left[- \sum_k p'_{ik}(t) p_{kj}(u-t) \right] \right|$$

$$\leqslant \sum_{k} |\Delta_{ik}(t, t+s) - p'_{ik}(t)| \to 0, \quad s \downarrow 0.$$

故$p_{ij}(u)$，$d_i(u)$的右导数存在，类似可证左导数也存在。因此$p'_{ij}(u)$，$d'_i(u)$在(t, ∞)上存在，而且

$$p'_{ij}(u) = \sum_{k} p'_{ik}(t) p_{kj}(u-t), \tag{26}$$

$$d'_i(u) = -\sum_{k} p'_{ik}(u). \tag{27}$$

由(19)及上面二式可见$p'_{ij}(u)$，$d'_i(u)$是$u \in (t, \infty)$的连续函数。既然由(一)，上述的$t>0$可以任意地小，因此$p'_{ij}(t)$，$d'_i(t)$在$(0, \infty)$上存在，而且是连续函数，并且(4)(5)成立。因为(19)对一切$t>0$都成立，根据引理2.2，应用结论(19)于$\widetilde{P}(t)$便得(3)。

(五)往证

$$\lim_{t \downarrow 0} p'_{ij}(t) = p'_{ij}(0), \quad \lim_{t \downarrow 0} d'_i(t) = d'_i(0). \tag{28}$$

只证前一式即可，因为按照引理2.2，应用前一式的结论于$\widetilde{P}(t)$即得后一式。

由(6)，

$$\begin{aligned} p_{ij}(t+h) - p_{ij}(t) &\geqslant [p_{ii}(h) - 1] p_{ij}(t) \\ &\geqslant -q_i h p_{ij}(t). \end{aligned} \tag{29}$$

故

$$R_{ij}(t) \equiv p'_{ij}(t) + q_i p_{ij}(t) \geqslant 0, \tag{30}$$

且由(5)及(2.B)有

$$R_{ij}(t+s) = \sum_{k} R_{ik}(t) p_{kj}(s), \quad s>0, \quad t>0. \tag{31}$$

于是

$$R_{ij}(t+s) \geqslant R_{ij}(t) p_{jj}(s),$$

由$p'_{ij}(t)$在$(0, \infty)$上的连续性，得

$$R_{ij}(s) \geqslant \overline{\lim_{t \downarrow 0}} R_{ij}(t) p_{jj}(s), \quad \underline{\lim_{s \downarrow 0}} R_{ij}(s) \geqslant \overline{\lim_{t \downarrow 0}} R_{ij}(t).$$

故$\lim\limits_{t\downarrow 0}R_{ij}(t)$存在，即$\lim\limits_{t\downarrow 0}p'_{ij}(t)$存在。再由数学分析中的中值定理得

$\lim\limits_{t\downarrow 0}p'_{ij}(t)=p'_{ij}(0)$，证毕。

系 如$q_i<\infty$，则对任意$\delta>0$，对$u\geqslant\delta$一致地有

$$\lim_{s\to 0}\sum_j\left|\frac{p_{ij}(u+s)-p_{ij}(u)}{s}-p'_{ij}(u)\right|=0. \tag{32}$$

证 由（25）及（5）得出。

定理2 设对指定j有 $q_j<\infty$。则对一切 $i\in E$，$p_{ij}(t)$在$[0,\infty)$上有有穷的连续导数，而且

$$p'_{ij}(s+t)=\sum_k p_{ik}(t)p'_{kj}(s),\quad t\geqslant 0,\quad s>0. \tag{33}$$

证 由（4.8）及（6），

$$p_{ij}(t+h)-p_{ij}(t)\geqslant p_{ij}(t)[p_{jj}(h)-1]\geqslant -p_{ij}(t)q_jh,$$

故

$$D[p_{ij}(t)e^{q_jt}]=[Dp_{ij}(t)+p_{ij}(t)q_j]e^{q_jt}\geqslant 0.$$

这里D表示右下导数，因而当t增加时$p_{ij}(t)e^{q_jt}$非降，而且$Dp_{ij}(t)$几乎处处有限。

令 $$v_{ij}(t)=Dp_{ij}(t)+p_{ij}(t)q_j\geqslant 0. \tag{34}$$

改写（2.B）为

$$p_{ij}(s+t)e^{q_j(s+t)}=e^{q_jt}\sum_k p_{ik}(t)p_{kj}(s)e^{q_js},$$

按s微分并用富比尼(Fubini)定理得：对每个$t\geqslant 0$及几乎一切s有

$$v_{ij}(s+t)=\sum_k p_{ik}(t)v_{kj}(s). \tag{35}$$

如果用法都（Fatou）引理，则对一切s，$t\geqslant 0$有

$$v_{ij}(s+t)\geqslant\sum_k p_{ik}(t)v_{kj}(s). \tag{36}$$

特别地

$$\infty > v_{ij}(t_0) \geqslant p_{ii}(t_0 - s) v_{ij}(s)$$

对几乎一切t_0及一切$s \leqslant t_0$，故$v_{ij}(s)$在任何有限区间上有界。由富比尼定理得（35）对$s \overline{\in} Z$及$t \overline{\in} Z_s$成立，其中Z和Z_s的测度为零。对某个$s_0 \overline{\in} Z$，假定对某个t有

$$v_{ij}(t + s_0) > \sum_k p_{ik}(t) v_{kj}(s_0), \tag{37}$$

则对$t' > t$有

$$v_{ij}(t' + s_0) \geqslant \sum_l p_{il}(t' - t) v_{lj}(t + s_0)$$

$$> \sum_l p_{il}(t' - t) \sum_k p_{lk}(t) v_{kj}(s_0)$$

$$= \sum_k p_{ik}(t') v_{kj}(s_0).$$

由于$s_0 \overline{\in} Z$时（35）对几乎一切t成立，上式是不可能的。从而当$s \overline{\in} Z$时Z_s是空集。其次，设$s > 0$任意，$0 < s' < s$且$s' \overline{\in} Z$，则

$$v_{ij}(t + s) = v_{ij}(t + s - s' + s')$$

$$= \sum_k p_{ik}(t + s - s') v_{kj}(s')$$

$$= \sum_k \sum_l p_{il}(t) p_{lk}(s - s') v_{kj}(s') = \sum_l p_{il}(t) v_{lj}(s),$$

从而Z也是空集。因此（35）对一切$t \geqslant 0$，$s > 0$成立。

设$t \geqslant 0$，且$t_n \downarrow 0$，$t'_n \downarrow 0$使$v_{ij}(t + t_n) \to a_{ij}$，$v_{ij}(t + t'_n) \to a'_{ij}$。不妨设$t'_n < t_n$。由（35），

$$v_{ij}(t + t_n) \geqslant p_{ii}(t_n - t'_n) v_{ij}(t + t'_n).$$

故$a_{ij} \geqslant a'_{ij}$。由对称性，$a_{ij} = a'_{ij}$。即$v_{ij}(t + 0)(t \geqslant 0)$存在，类似地可证$v'_{ij}(t - 0)(t > 0)$存在，而且由$v_{ij}(t + t_n) \geqslant p_{ii}(t_n) v_{ij}(t)$得$v_{ij}(t + 0) \geqslant v_{ij}(t)(t \geqslant 0)$。由$v_{ij}(t) \geqslant p_{ii}(t_n) v_{ij}(t - t_n)$得$v_{ij}(t) \geqslant v_{ij}(t - 0)(t > 0)$。于是$v_{ij}(t + 0) \geqslant v_{ij}(t) \geqslant v_{ij}(t - 0)$。对$t > 0$，可取$0 < s < t$，则由于

$$v_{ij}(t-t_n)=\sum_k p_{ik}(t-s-t_n)v_{kj}(s),$$

$$v_{ij}(t-0)\geqslant\sum_k p_{ik}(t-s)v_{kj}(s)=v_{ij}(t).$$

这样，$v_{ij}(t)$在（0，∞）上连续。因而$p_{ij}(t)$在（0,∞）上有连续的地尼导数$Dp_{ij}(t)$，事实上在(0，∞)上有连续的导数$p'_{ij}(t)$(Saks[1]，第204页)。刚才已指出$v_{ij}(0+)$存在，即$p'_{ij}(0+)=\lim\limits_{t\downarrow 0}p'_{ij}(t)$存在。由中值定理得$p'_{ij}(0+)=p'_{ij}(0)$。于是$p'_{ij}(t)$在[0，∞）上连续，并且对一切$t\geqslant 0$，$s>0$成立的（35）成为(33)，证毕。

§7. 柯氏方程组

设$P(t)\in\mathscr{P}$。(6.5）对$t_1=0$，(6.33）对$s=0$未必成立，即当$q_i<\infty$时，

$$p'_{ij}(t)=\sum_k q_{ik}p_{kj}(t),\quad t\geqslant 0,\ j\in E.\quad (KB_i)$$

未必成立。当$q_j<\infty$时，

$$p'_{ij}(t)=\sum_k p_{ik}(t)q_{kj},\quad t\geqslant 0,\ i\in E.\quad (KF_j)$$

未必成立。但由

$$\Delta_{ij}(t,\ t+s)=\sum_k\Delta_{ik}(0,\ s)p_{kj}(t)$$

$$=\sum_k p_{ik}(t)\Delta_{kj}(0,\ s)$$

及法都引理有：当$q_i<\infty$时

$$p'_{ij}(t)\geqslant\sum_k q_{ik}p_{kj}(t),\quad t\geqslant 0,\ j\in E.\qquad (1)$$

当$q_j<\infty$时，

$$p'_{ij}(t) \geqslant \sum_k p_{ik}(t) q_{kj}, \quad t \geqslant 0, \quad i \in E. \tag{2}$$

分别称（1）和（2）为向后不等式组和向前不等式组。而称方程组$(KB_i)(i \in E)$即(3.KB)为向后方程组，称方程组$(KF_j)(j \in E)$即(3.KF)为向前方程组。

应用引理2.2中的$\widetilde{P}(t)$到（1），我们得

$$d'_i(t) \geqslant \sum_k q_{ik} d_k(t) + D_i. \tag{3}$$

其中$D_i = d'_i(0) \geqslant 0$。

如果$P(t) \in \mathscr{P}$，并且(KB_i)或(KF_j)成立，则q_i或q_j有限，并且$p'_{ij}(0) = q_{ij}$($j \in E$或$i \in E$)。特别如果$p(t)$满足向后或向前方程组，则$p(t)$是Q过程，即$P(t) \in \mathscr{P}_s(Q)$。实际上，由$(KB_i)$，

$$p'_{ii}(t) = q_{ii} p_{ii}(t) + \sum_{k \neq i} q_{ik} p_{ki}(t).$$

故q_{ii}有限，同样由(KF_j)得q_{jj}有限。在(KB_i)或(KF_j)中取$t=0$得$p'_{ij}(0) = q_{ij}$。

自然要问：Q过程$P(t) \in \mathscr{P}_s(Q)$满足向后或向前方程组的条件是什么？我们将讨论这一问题。

引理1 设$P(t) \in \mathscr{P}$，(KB_i)对几乎一切t成立。则(KB_i)对一切$t \geqslant 0$成立。

证 由假设得q_{ii}有限，而且

$$p_{ij}(t) = \delta_{ij} + \int_0^t \left[\sum_k q_{ik} p_{kj}(u) \right] du, \quad t \geqslant 0. \tag{4}$$

由于（2.6），被积表达式连续，对t微分得(KB_i)对一切$t \geqslant 0$成立。

定理2 设$P(t) \in \mathscr{P}$。则对指定i，(KB_i)成立的充要条件是q_i有限，且

$$\sum_j q_{ij} + D_i = 0. \tag{5}$$

其中$D_i = d'_i(0)$。

证 设(KB_i)成立，则q_i有限且

$$\sum_j p'_{ij}(t)=\sum_k q_{ik}\sum_j p_{kj}(t),$$

由（6.4），上式即

$$-d'_i(t)=\sum_k q_{ik}[1-d_k(t)].$$

由定理6.1，令$t\to 0$得

$$-D_i=\sum_k q_{ik},$$

即（5）。

反之，设q_i有限且（5）成立，则由（1）、（3）及（6.4）得

$$0=\sum_j p'_{ij}(t)+d'_i(t)\geqslant\sum_k q_{ik}[1-d_k(t)]$$

$$+\sum_k q_{ik}d_k(t)+D_i=\sum_k q_{ik}+D_i=0.$$

因此（1）中必成立等式，即(KB_i)成立，证完。

当状态i保守，即

$$q_i=\sum_{j\neq i}q_{ij}<\infty \tag{6}$$

时，由定理5.3必定$D_i=0$。故（5）成立。因此作为定理2的推论，我们有

定理3 当（6）成立时，(KB_i)成立。特别地，当Q保守时，每个Q过程满足向后方程组。

因此，当Q保守时，构造问题较为简单，求出全部Q过程的问题成为求出满足向后方程组的全部Q过程的问题。

定理4 设$P(t)\in\mathscr{P}_i(Q)$。向后方程组成立的充要条件是对任意$t\geqslant 0$，

$$\lim_{h\to 0}\sum_j\left|\frac{p_{ij}(t+h)-q_{ij}(t)}{h}-\sum_k q_{ik}p_{kj}(t)\right|=0. \tag{7}$$

向前方程组成立的充要条件是对任意$t\geqslant 0$，

$$\lim_{h\to 0}\sum_j\left|\frac{p_{ij}(t+h)-p_{ij}(t)}{h}-\sum_k p_{ik}(t)q_{kj}\right|=0. \tag{8}$$

证 必要性由定理6.1的系推出，充分性是明显的。

定理5 设$P(t)\in\mathscr{P}$。如果向前方程组（3.KF）对几乎一切$t\geqslant 0$成立，则必定对一切$t\geqslant 0$成立。

证 设Z为零测集，且$t\overline{\in}$Z时，(3．KF)成立。由此得

$$p_{ij}(t)=\delta_{ij}+\int_0^t\left[\sum_k p_{ik}(u)q_{kj}\right]du$$

$$\geqslant\delta_{ij}+q_{ij}\int_0^t p_{ii}(u)du+(1-\delta_{ij})q_{jj}\int_0^t p_{ij}(u)du,$$

$$\frac{p_{ij}(t)-\delta_{ij}}{t}\geqslant q_{ij}\frac{1}{t}\int_0^t p_{ii}(u)du$$

$$+(1-\delta_{ij})q_{jj}\frac{1}{t}\int_0^t p_{ij}(u)du.$$

由(2.C)及定理5.1和5.2，故

$$\bar{q}_{ij}\equiv p'_{ij}(0)\geqslant q_{ij}+(1-\delta_{ij})q_{jj}\delta_{ij}=q_{ij}. \tag{9}$$

因此$\bar{q}_{ii}\geqslant q_{ii}>-\infty$，从而一切$\bar{q}_{ij}$有穷，并且由向前不等式（2）有

$$p'_{ij}(t)\geqslant\sum_k p_{ik}(t)\bar{q}_{kj},\quad t\geqslant 0.$$

结合$t\overline{\in}$Z的(3.KF)得

$$\sum_k p_{ik}(t)(\bar{q}_{kj}-q_{kj})\leqslant 0,\quad t\overline{\in}Z,\ i,\ j\in E.$$

由于每一项非负，特别有$p_{ii}(t)(\bar{q}_{ij}-q_{ij})=0$（$t\overline{\in}$Z），从而$\bar{q}_{ij}=q_{ij}$，即$p'_{ij}(0)=q_{ij}$。于是当$t=0$时，(3.$KF$)是成立的。

设$u>0$且$u\in Z$。可找$t>0$使 $t\overline{\in}Z$，$0<u-t\overline{\in}Z$。于是由定理6.2，

$$p'_{ij}(u)=\sum_l p_{il}(t)p'_{lj}(u-t)$$

$$= \sum_l p_{il}(t) \sum_k p_{lk}(u-t) q_{kj}$$

$$= \sum_k \left[\sum_l p_{il}(t)\ p_{lk}(u-t) \right] q_{kj} = \sum_k p_{ik}(u) q_{kj}.$$

定理6 设 $P(t) \in \mathscr{P}$。如果向前方程组成立，则 $\sum_k p_{ik}(t) q_{kj}$ 在任何有限区间$[0, T]$上一致收敛。

证 由定理4.1和6.2，$p'_{ij}(t)$和$p_{ik}(t)q_{kj}$都连续，然后只需引用地尼定理即可。

我们指出：如果只构造满足向后方程组的Q过程，可以将非保守的情形化为保守的情形。

定理7 设$P(t) \in \mathscr{P}$，(Q)满足向后方程组，$\widetilde{P}(t) = \{\widetilde{p}_{ij}(t)\}$ $(i, j \in \widetilde{E})$ 按 (2.3) 确定。则矩阵 $\widetilde{Q} = \widetilde{p}'(0)$保守，$\widetilde{P}(t)$ 满足向后方程组。

证 显然 $\widetilde{Q} = \begin{pmatrix} Q & D \\ 0 & 0 \end{pmatrix}$，其中 $D = \{D_i\}_{i \in E}$ 为列矢量。根据定理2，$\widetilde{Q}$ 保守。根据(3.KB)，可见

$$\widetilde{p}_{ij}(t) = \sum_{k \in \widetilde{E}} \widetilde{q}_{ik}\ \widetilde{p}_{kj}(t), \quad i,\ j \in \widetilde{E}, \tag{10}$$

对i，$j \in E$是成立的。对 $i = \Delta$，$j \in \widetilde{E}$，上式显然成立。对 $i \in E$，$j = \Delta$，(10)成为

$$d'_i(t) = \sum_k q_{ik} d_k(t) + D_i. \tag{11}$$

当$t = 0$时，上式显然成立。当 $t > 0$ 时，上式可以从 (3.KB) 及 (6.4) 得出，证毕。

§8. 预解算子

设$P(t) \in \mathscr{P}$。考虑$P(t)$的拉普拉斯变换$\psi(\lambda) = \{\psi_{ij}(\lambda)\}$ $(i,$

$j\in E$, $\lambda>0$)：

$$\psi_{ij}(\lambda)=\int_0^\infty e^{-\lambda t}p_{ij}(t)dt,\ \lambda>0. \tag{1}$$

称$\psi(\lambda)$ ($\lambda>0$) 为过程$P(t)$的预解算子。

由(2.A—C)，我们得：对任意i，$j\in E$，λ，$\mu>0$，

$$\psi_{ij}(\lambda)\geqslant 0,\ \lambda\sum_j\psi_{ij}(\lambda)\leqslant 1; \tag{2}$$

$$\psi_{ij}(\lambda)-\psi_{ij}(\mu)+(\lambda-\mu)\sum_k\psi_{ik}(\lambda)\psi_{kj}(\mu)=0; \tag{3}$$

$$\lim_{\lambda\to\infty}\lambda\psi_{ij}(\lambda)=\delta_{ij}. \tag{4}$$

称(2)为范条件，称(3)为预解方程，称(4)为连续性条件。(2)(3)(4)的矩阵形式为

$$\psi(\lambda)\geqslant \mathbf{0},\ \lambda\psi(\lambda)\mathbf{1}\leqslant\mathbf{1}; \tag{5}$$

$$\psi(\lambda)-\psi(\mu)+(\lambda-\mu)\psi(\lambda)\psi(\mu)=\mathbf{0}; \tag{6}$$

$$\lim_{\lambda\to\infty}\lambda\psi(\lambda)=I. \tag{7}$$

由(2.D)得

$$\lambda\sum_j\psi_{ij}(\lambda)=1, \tag{8}$$

或

$$\lambda\psi(\lambda)\mathbf{1}=\mathbf{1}. \tag{9}$$

向后和向前不等式组(7.1)和(7.2)成为

$$\lambda\psi_{ij}(\lambda)-\delta_{ij}\geqslant\sum_k q_{ik}\psi_{kj}(\lambda), \tag{10}$$

$$\lambda\psi_{ij}(\lambda)-\delta_{ij}\geqslant\sum_k\psi_{ik}(\lambda)q_{kj}. \tag{11}$$

由定理7.3，

$$\lambda\psi_{ij}(\lambda)-\sum_k q_{ik}\psi_{kj}(\lambda)=\delta_{ij},\ i\in E-H. \tag{12}$$

这里H是非保守状态集。

定理1. $\psi(\lambda)$ ($\lambda>0$)是过程$P(t)\in\mathscr{P}$的预解算子的充分必要条件是范条件、预解方程和连续性条件成立。过程$P(t)$不中断的

充要条件是(9)成立。

证．必要性已证明，下面证充分性。

视$\psi(\lambda)$ 为作用在§2开头叙述的巴拿赫空间$\boldsymbol{l}$上的线性算子：$g\in\boldsymbol{l}$，$g\psi(\lambda)\in\boldsymbol{l}$。

$$[g\psi(\lambda)]_j=\sum_i g_i\psi_{ij}(\lambda)。\tag{13}$$

由于范条件，$\psi(\lambda)$是$\boldsymbol{l}$到$\boldsymbol{l}$中的非负线性算子，其范数界于λ^{-1}。由于预解方程，在一致算子拓扑中，

$$\left(-\frac{d}{d\lambda}\right)^n\psi(\lambda)=n![\psi(\lambda)]^{n+1}。\tag{14}$$

于是有

$$0\leqslant\left(-\frac{d}{d\lambda}\right)^n\psi_{ij}(\lambda)\leqslant\frac{n!}{\lambda^{n+1}},\tag{15}$$

$$0\leqslant\left(-\frac{d}{d\lambda}\right)^n\sum_j\psi_{ij}(\lambda)\leqslant\frac{n!}{\lambda^{n+1}}。\tag{16}$$

利用完全单调函数理论（例如，见 Feller[6]，第二卷，415－418），从(15)得出：$\psi_{ij}(\lambda)$ 是某个可测函数$f_{ij}(t)$的拉普拉斯变换。由（15）（16）得下面二式

$$0\leqslant f_{ij}(t)\leqslant 1,\tag{17}$$

$$0\leqslant\sum_j f_{ij}(t)\leqslant 1\tag{18}$$

对几乎一切 $t\geqslant 0$ 成立。在一个勒贝格测度为零的 t 集上，修正 $f_{ij}(t)$ 的值，可以使得（17）（18）对一切 $t>0$ 成立。我们假定已作了此修正。

其次，我们证明，按平面上的勒贝格测度，对几乎一切非负的s和t，有

$$f_{ij}(s+t)=\sum_k f_{ik}(s)f_{kj}(t)。\tag{19}$$

由双重拉普拉斯变换的唯一性定理，只要证（12）两边的双重拉普拉斯变换

$$\int_0^\infty\int_0^\infty \cdots e^{-\lambda s-\nu t}dsdt$$

相等。当$\lambda\neq\nu$时，(19) 两边的双重拉普拉斯变换是

$$(\nu-\lambda)^{-1}(\psi_{ij}(\lambda)-\psi_{ij}(\nu))\text{和}\ \sum_k \psi_{ik}(\lambda)\psi_{kj}(\nu)。$$

依预解方程，它们是相等的。令$\lambda\to\nu$，由 (14) 知 (19) 两边的双重拉普拉斯变换当$\lambda=\nu$时也是相等的。

现在，我们将在一零测集上改变$f_{ij}(t)$的值而得$p_{ij}(t)$，使$(p_{ij}(t))$是马尔科夫过程。对 $t=0$，我们定义 $p_{ij}(0)=\delta_{ij}$，对$t>0$，我们定义

$$p_{ij}(t)=t^{-1}\sum_k\int_0^t f_{ik}(u)f_{kj}(t-u)du$$

$$=t^{-1}\int_0^t\Big[\sum_k f_{ik}(u)f_{kj}(t-u)\Big]du。\tag{20}$$

因为 (19) 对几乎一切 (s, t) 成立，故对几乎一切$t>0$，(14) 中被积表达式对几乎一切$u\in(0, t)$等于$f_{ij}(t)$，因而

$$p_{ij}(t)=f_{ij}(t)\quad\text{对几乎一切}t>0。\tag{21}$$

其次，函数

$$g_k(t)=\int_0^t f_{ik}(u)f_{kj}(t-u)du$$

作为两个有界可测函数f_{ik}和f_{kj}的卷积，因而是连续的。而级数$\sum\limits_k g_k(t)$被级数

$$\sum_k\int_0^t f_{ik}(u)du\tag{22}$$

逐项控制，又级数 (22) 的每一项都是连续的。因此，依地尼定理，级数 (22) 在任何有限区间 $[0, T]$ 中一致收敛，从而级数$\sum\limits_k g_k(t)$在$[0, T]$中一致收敛，于是$p_{ij}(t)=t^{-1}\sum\limits_k g_k(t)$在$(0, T)$ 中连续，故

$$p_{ij}(t)\quad 在(0,\ \infty)中连续。\tag{23}$$

由（21）（23），从（17）得

$$0\leqslant p_{ij}(t)\leqslant 1\qquad 对一切 t>0。\tag{24}$$

从（18）（21）得

$$\sum_j p_{ij}(t)\leqslant 1\tag{25}$$

对几乎一切$t>0$成立。由(23)，利用 Fatou引理，可得上式对一切$t>0$成立。

我们已经证明$p(t)=(p_{ij}(t))$满足（2.A）和（2.C）。往证$p(t)$满足（2.B）。考虑到（21）以及（19）对几乎一切$(s,\ t)$成立，我们有

$$p_{ij}(s+t)=\sum_k p_{ik}(s)p_{kj}(t),对几乎一切(s,\ t)。\tag{26}$$

依（23），上式左方对$s>0$，$t>0$是连续的。因此，为证（26)对一切$s>0$，$t>0$成立，只需证明：指定$s>0$,（26）右方对$t>0$连续；指定$t>0$，（26）右方对$s>0$连续。由(23)（25），前一事实是正确的。为证后一事实，只需说明指定$t>0$时，级数

$$\sum_k p_{ik}(s)p_{kj}(t)$$

在任何区间$[a,\ b]$（$0<a<b$）中一致收敛。由于（24），只需说明级数$\sum\limits_k p_{ik}(s)$在$[a,\ b]$中一致收敛。实际上，由(20)，

$$\sum_j p_{ij}(t)=t^{-1}\int_0^t\Big[\sum_k\ \ \sum_j f_{ik}(u)f_{kj}(t-u)\Big]du$$

$$=t^{-1}\sum_k\int_0^t f_{ik}(u)\Big[\sum_j f_{kj}(t-u)\Big]du.$$

由(18)，上面的级数$\sum\limits_k$被级数(22)所控制，而已指出级数(22)在任何有限区间$[0,\ T]$中一致收敛，因而级数$\sum\limits_j p_{ij}(t)$在区间

$[a, b]$中一致收敛。

我们已证明$P(t)=(p_{ij}(t))$的元素是可测函数且（2.A）和（2.B）成立，即$P(t)=(p_{ij}(t))$是可测的广转移矩阵，因而极限$\lim\limits_{t\to 0+} p_{ij}(t)=u_{ij}$存在（见王梓坤[2]，§2.1定理1）。根据熟知的关于拉普拉斯变换的阿贝尔性质有$\lim\limits_{\lambda\to\infty}\lambda\psi_{ij}(\lambda)=\lim\limits_{t\to 0+} p_{ij}(t)$，再注意$\psi(\lambda)$的连续性条件（7），因而（2.C）成立。这样，$p(t)=(p_{ij}(t))$是马尔科夫过程，它的预解矩阵就是已给的$\psi(\lambda)$。充分性证完。

设$P(t)$不中断，显然（9）成立。反之，设（9）成立。如果对某个$t>0$有$\sum\limits_{j} p_{ij}(t)<1$，则由定理1.4.4的系，对一切$t>0$有$\sum\limits_{j} p_{ij}(t)<1$，从而（9）不可能成立。这样，（9）蕴含$P(t)$不中断。定理证完。

定理2. 设$P(t)\in\mathscr{P}$的密度矩阵为$Q=(q_{ij})$，预解算子为$\psi(\lambda)$，则

$$q_{ij}=\lim_{\lambda\to\infty}\lambda\{\lambda\psi_{ij}(\lambda)-\delta_{ij}\} \tag{27}$$

证。 当q_{ij}有限时，对任给$\varepsilon>0$，存在$\delta>0$使当$t<\delta$时，

$$\left|\frac{p_{ij}(t)-\delta_{ij}}{t}-q_{ij}\right|<\varepsilon.$$

于是

$$\begin{aligned}
&|\lambda\{\lambda\psi_{ij}(\lambda)-\delta_{ij}\}-q_{ij}|\\
&=\left|\lambda^2\int_0^\infty e^{-\lambda t}[P_{ij}(t)-\delta_{ij}-q_{ij}t]dt\right|\\
&\leqslant\lambda^2\int_0^\delta e^{-\lambda t}\varepsilon t\,dt+\lambda^2\int_\delta^\infty e^{-\lambda t}(2+|q_{ij}|t)dt\\
&=\varepsilon\{-e^{-\lambda\delta}(\lambda\delta+1)+1\}+2\lambda e^{-\lambda\delta}+|q_{ij}|e^{-\lambda\delta}(\lambda\delta+1).
\end{aligned}$$

当$\lambda\to\infty$时得

$$\varlimsup_{\lambda\to\infty}|\lambda\{\lambda\psi_{ij}(\lambda)-\delta_{ij}\}-q_{ij}|\leqslant\varepsilon.$$

由ε的任意性，得（27）。

当q_{ii}无限时，对任意$N>0$，存在$\delta>0$，当$t<\delta$时，

$$\frac{p_{ii}(t)-1}{t}<-N.$$

于是 $$\lambda\{\lambda\psi_{ii}(\lambda)-1\}=\lambda^2\int_0^\infty e^{-\lambda t}\{p_{ii}(t)-1\}dt$$

$$\leqslant\lambda^2\int_0^\delta e^{-\lambda t}(-Nt)dt=-N\{-e^{-\lambda\delta}(\lambda\delta-1)+1\}$$

$$\to -N,\ (\lambda\to\infty).$$

由 N的任意性，$\lim\limits_{\lambda\to\infty}\lambda\{\lambda\psi_{ii}(\lambda)-1\}=-\infty$，证毕。

当Q有限时，称条件(27)为Q条件。

定理3. $\psi(\lambda)(\lambda>0)$ 是Q过程 $P(t)\in\mathscr{P}_s(Q)$ 的预解算子的充分必要条件是范条件、预解方程和Q条件成立。

证. 这是很明显的，因为Q 条件蕴含连续性条件。

今后我们将直接称过程或Q过程$P(t)$的预解算子 $\psi(\lambda)(\lambda>0)$为过程或Q过程，并且也记为$\psi(\lambda)\in\mathscr{P}$或$\psi(\lambda)\in\mathscr{P}_s(Q)$。这样并不会引起混淆。

定理4. 设Q有限。$\psi(\lambda)\in\mathscr{P}_s(Q)$ 且满足柯氏向后方程组(3. KB)的充分必要条件是范条件、预解方程及B条件：

$$(\lambda I-Q)\psi(\lambda)=I,\ \lambda>0. \tag{28}$$

成立。B条件的元素表示形式为

$$\lambda\psi_{ij}(\lambda)-\sum_k q_{ik}\psi_{kj}(\lambda)=\delta_{ij},\ \lambda>0,\ i,\ j\in E. \tag{29}$$

证. 在(3. KB)中取拉普拉斯变换并注意初始条件（2. C）得B条件(29)。得证必要性。

充分性．由预解方程得$\psi_{ij}(\lambda)\downarrow(\lambda\uparrow)$·由(27)得

$$\psi_{ij}(\lambda)\downarrow 0,\ \lambda\uparrow\infty. \tag{30}$$

从而再由 (29) 得

$$(\lambda+q_i)\psi_{ij}(\lambda)\downarrow\delta_{ij},\ \lambda\uparrow\infty. \tag{31}$$

故 $$\lambda\psi_{ij}(\lambda)\to\delta_{ij},\ \lambda\to\infty. \tag{32}$$

即连续性条件成立。再由(29)，

$$\lambda\{\lambda\psi_{ij}(\lambda)-\delta_{ij}\}=\sum_k q_{ik}\lambda\psi_{kj}(\lambda). \tag{33}$$

由（32）及控制收敛定理，故Q条件成立。因此$\psi(\lambda)\in\mathscr{P}_s(Q)$。

依定理4.1及定理6.1，$p'_{ij}(t)$及$\sum\limits_k q_{ik}P_{kj}$（$t$）都是$t$的连续函数，而(29)说明它们有相同的拉普拉斯变换，因而它们相等，即向后方程组（3．KB）成立，证毕。

定理5. 设Q有限。$\psi(\lambda)\in\mathscr{P}_s(Q)$且满足柯氏向前方程组（3．$KF$）的充分必要条件是范条件、预解方程及$F$条件：

$$\psi(\lambda)(\lambda I-Q)=I,\ \lambda>0, \tag{34}$$

成立。F条件的元素表示形式为

$$\lambda\psi_{ij}(\lambda)-\sum_k\psi_{ik}(\lambda)q_{kj}=0,\ \lambda>0,\ i,\ j\in E. \tag{35}$$

证. 在(3．KF)两边取拉普拉斯变换并注意初始条件（2．C）得F条件(35)，必要性得证。

充分性．由(35)知(30)仍然成立．再由(35)得

$$\psi_{ij}(\lambda)(\lambda+q_j)\downarrow\delta_{ij},\ \lambda\uparrow\infty. \tag{36}$$

故(32)仍成立。再由(35)，

$$\lambda\{\lambda\psi_{ij}(\lambda)-\delta_{ij}\}=\sum_k\psi_{ik}(\lambda)q_{kj}, \tag{37}$$

故Q条件仍成立，即$\psi(\lambda)\in\mathscr{P}_s(Q)$。

由(35)，可见函数$p'_{ij}(t)$与函数$\sum\limits_k p_{ik}(t)q_{kj}$有相同的拉氏变换。因而它们必对几乎一切$t$相等，即（3．$KF$)对几乎一切$t\geqslant 0$成立。依定理7.5，向前方程组(3．$KF$)成立，证毕。

定义1 满足(28)或(35)的$\psi(\lambda)$分别称为B型的或F型的。

§9．费勒的存在定理

给定矩阵Q满足（2．6)，Q过程存在吗？或者说，$\mathscr{P}_s(Q)$是

否为空集? Feller[1]解决了这个问题，他构造了一个最小解。

定义$f_{ij}^n(t)(t\geqslant 0)$如下：

$$\left.\begin{aligned}&f_{ij}^0(t)\equiv 0,\\&f_{ij}^{n+1}(t)=\delta_{ij}e^{-q_i t}+e^{-q_i t}\int_0^t\Big[\sum_{k\neq i}q_{ik}f_{kj}^n(u)\Big]e^{q_i u}du.\end{aligned}\right\}\tag{1}$$

或等价地

$$\left.\begin{aligned}&f_{ij}^0(t)\equiv 0\\&f_{ij}^{n+1}(t)=\delta_{ij}e^{-q_i t}+e^{-q_j t}\int_0^t\Big[\sum_{k\neq j}f_{ik}^n(u)q_{kj}\Big]e^{q_j u}du.\end{aligned}\right\}\tag{2}$$

令$f_{ij}^n(t)\uparrow f_{ij}(t)$, $n\uparrow\infty$。 (3)

为了说明 (1) 与 (2) 等价，考虑它们的拉氏变换$\phi_{ij}^n(\lambda)(\lambda>0)$及$f_{ij}(t)$的拉氏变换$\phi_{ij}(\lambda)$：

$$\left.\begin{aligned}&\phi_{ij}^0(\lambda)=0,\\&\phi_{ij}^{n+1}(\lambda)=\frac{1}{\lambda+q_i}\delta_{ij}+\sum_{k\neq i}\frac{q_{ik}}{\lambda+q_i}\phi_{kj}^n(\lambda).\end{aligned}\right\}\tag{4}$$

或

$$\left.\begin{aligned}&\phi_{ij}^0(\lambda)=0,\\&\phi_{ij}^{n+1}(\lambda)=\frac{1}{\lambda+q_j}\delta_{ij}+\sum_{k\neq j}\phi_{ik}^n(\lambda)\frac{q_{kj}}{\lambda+q_j}.\end{aligned}\right\}\tag{5}$$

(3) 成为

$$\phi_{ij}^n(\lambda)\uparrow\phi_{ij}(\lambda),\quad n\uparrow\infty.\tag{6}$$

只需说明 (4) 与 (5) 等价即可。采用矩阵符号较简单。令

$\Pi=(\Pi_{ij})$：

$$\Pi_{ij}=\begin{cases}\dfrac{(1-\delta_{ij})q_{ij}}{q_i}, & \text{当}q_i>0\text{时},\\ \delta_{ij}, & \text{当}q_i=0\text{时}.\end{cases}\tag{7}$$

令$\Pi(\lambda)=\{\Pi_{ij}(\lambda)\}$：

$$\Pi_{ij}(\lambda)=\frac{q_i}{\lambda+q_i}\Pi_{ij}.\tag{8}$$

令对角形矩阵

$$\lambda I + q = diag(\lambda + q_i). \tag{9}$$

则(4)成为

$$\phi^0(\lambda) = 0,$$

$$\phi^{n+1}(\lambda) = \Pi^0(\lambda)(\lambda I + q)^{-1} + \Pi(\lambda)\phi^n(\lambda).$$

归纳便知

$$\phi^{n+1}(\lambda) = \sum_{\alpha=0}^{n} \Pi^\alpha(\lambda)(\lambda I + q)^{-1}, \quad n \geqslant 0. \tag{10}$$

从(5)得

$$\phi^0(\lambda) = 0,$$

$$\phi^{n+1}(\lambda) = \Pi^0(\lambda)(\lambda I + q)^{-1} + \phi^n(\lambda)(\lambda I + q)\Pi(\lambda)(\lambda I + q)^{-1},$$

归纳仍得(10)。于是(6)成为

$$\phi^{n+1}(\lambda) = \sum_{\alpha=0}^{n} \Pi^\alpha(\lambda)(\lambda I + q)^{-1} \uparrow \phi(\lambda), \quad \lambda \to \infty. \tag{11}$$

定理1. $f(t) \in \mathscr{P}_s(Q)$且满足柯氏向后和向前方程组．$f(t)$是最小$Q$过程，即对任意$P(t) \in \mathscr{P}_s(Q)$，有

$$P_{ij}(t) \geqslant f_{ij}(t), \quad t \geqslant 0, \quad i, \ j \in E. \tag{12}$$

定理 1 按预解算子$\phi(\lambda)$的叙述如下：

定理2. $\phi(\lambda) \in \mathscr{P}_s(Q)$，且满足向后和向前方程组．$\phi(\lambda)$是最小$Q$过程，即对任何$\psi(\lambda) \in \mathscr{P}_s(Q)$有

$$\psi_{ij}(\lambda) \geqslant \phi_{ij}(\lambda), \quad \lambda > 0, \quad i, \ j \in E. \tag{13}$$

只需证明定理 2 就够了．即证明$\phi(\lambda)$满足范条件、预解方程、B条件及F条件，并且(13)成立。

$\phi(\lambda)$的非负性是明显的．由归纳知对一切n有$\lambda \sum_j \phi^n_{ij}(\lambda) \leqslant 1$从而范条件成立。

为证对$\phi(\lambda)$的预解方程成立，只需证明对一切 n 有

$$\Pi^n(\lambda)(\lambda I + q)^{-1} - \Pi^n(\mu)(\mu I + q)^{-1}$$

$$= (\mu - \lambda) \sum_{\alpha=0}^{n} \Pi^\alpha(\lambda)(\lambda I + q)^{-1} \Pi^{n-\alpha}(\mu)(\mu I + q)^{-1}. \tag{14}$$

因为在上式中对 n 求和便得预解方程。

为证(14)；记(14)右方为A_n，则有

$$\Pi(\lambda)A_n = A_{n+1} - (\mu-\lambda)(\lambda I+q)^{-1}\Pi^{n+1}(\mu)(\mu I+q)^{-1}. \tag{15}$$

当$n=0$时(14)成立。设(14)对某n成立。将$\Pi(\lambda)$作用到(14)左边得

$$\begin{aligned}\Pi(\lambda)A_n &= \Pi^{n+1}(\lambda)(\lambda I+q)^{-1} - \Pi(\lambda)\Pi^n(\mu)(\mu I+q)^{-1}\\ &= \Pi^{n+1}(\lambda)(\lambda I+q)^{-1} - \Pi^{n+1}(\mu)(\mu I+q)^{-1}\\ &\quad - (\mu-\lambda)(\lambda I+q)^{-1}\Pi^{n+1}(\mu)(\mu I+q)^{-1},\end{aligned}$$

将上式代入(15)便得到将$n+1$代替n后的(14)式成立。于是(14)对一切n成立。

由(4)、(5)得

$$(\lambda+q_i)\phi_{ij}^{n+1}(\lambda) = \delta_{ij} + \sum_{k\neq i} q_{ik}\phi_{kj}^n(\lambda), \tag{16}$$

$$\phi_{ij}^{n+1}(\lambda)(\lambda+q_j) = \delta_{ij} + \sum_{k\neq j} \phi_{ik}^n(\lambda)q_{kj}. \tag{17}$$

令$n\to\infty$得$\phi(\lambda)$的B条件和F条件成立。

最后，设$\psi(\lambda)\in\mathscr{P}_s(Q)$。显然$\psi_{ij}(\lambda)\geqslant\phi_{ij}^0(\lambda)$。由(8.10)和(4)用归纳法易知$\psi_{ij}(\lambda)\geqslant\phi_{ij}^n(\lambda)$对一切$n$成立。从而得(13)。证毕。

§10. 最小解的性质

引理 1 设$f\geqslant0$，$g\geqslant0$。则当$n\uparrow\infty$时，

$$\xi^n \uparrow \phi(\lambda)f, \tag{1}$$

$$\eta_n \uparrow g\phi(\lambda). \tag{2}$$

其中ξ^n由下式确定：

$$\left.\begin{aligned}&\xi_i^0 = 0,\\ &\xi_i^{n+1} = \frac{f_i}{\lambda+q_i} + \sum_{k\neq i}\frac{q_{ik}}{\lambda+q_i}\xi_k^n.\end{aligned}\right\} \tag{3}$$

η^n由下式确定：

$$\left.\begin{aligned}&\eta_j^0=0,\\&\eta_j^{n+1}=\frac{g_j}{\lambda+q_j}+\sum_{k\neq j}\eta_k^n\frac{q_{kj}}{\lambda+q_j}.\end{aligned}\right\}\tag{4}$$

证 只需令$\xi^n=\phi^n(\lambda)f$，$\eta^n=g\phi^n(\lambda)$。则(1)(2)成立，并且由(9.16)(9.17)知(3)(4)成立。

引理2 设$f\geqslant 0$，而u满足

$$\left.\begin{aligned}&(\lambda+q_i)u_i=\sum_{j\neq i}q_{ij}u_j+f_i,\quad i\in E,\\&0\leqslant u_i\leqslant\infty.\end{aligned}\right\}\tag{5}$$

则$u\geqslant\phi(\lambda)f$。

设$g\geqslant 0$，而v满足

$$\left.\begin{aligned}&v_j(\lambda+q_j)=\sum_{i\neq j}v_iq_{ij}+g_j,\quad j\in E,\\&0\leqslant v_i\leqslant\infty.\end{aligned}\right\}\tag{6}$$

则$v\geqslant g\phi(\lambda)$。

证 只证前一种情况。因$u\geqslant\xi_i^0=0$，由(5)及(3)归纳得$u\geqslant\xi^n$，从而$u\geqslant\phi(\lambda)f$。证毕。

令$A(\mu,\ \lambda)=I+(\mu-\lambda)\phi(\lambda)$，$\lambda,\ \mu>0$。 (7)

引理3 $A(\mu,\ \lambda)$有界[1)]，且对任意$\lambda,\ \mu,\ \nu>0$，

$$A(\mu,\ \lambda)A(\lambda,\ \nu)=A(\mu,\ \nu).\tag{8}$$

$$\phi(\mu)A(\mu,\ \lambda)=A(\mu,\ \lambda)\phi(\mu)=\phi(\lambda).\tag{9}$$

特别地

$$A(\mu,\ \lambda)A(\lambda,\ \mu)=I\tag{10}$$

即$A(\mu,\ \lambda)$存在有界左逆$A(\lambda,\ \mu)$和有界右逆$A(\lambda,\ \mu)$。

证 利用$\phi(\lambda)$的范条件和预解方程易得$A(\mu,\ \lambda)$有界及(8)、(9)。

记方程组

$$(U_\lambda)\ \lambda u_i-\sum_j q_{ij}u_j=0,\ i\in E.\tag{11}$$

---

1) 称矩阵$A=(a_{ij})$有界，如果$\sup\limits_i\sum\limits_j|a_{ij}|<\infty$。

的解 $u\in\boldsymbol{m}$ 全体为 $\mathscr{M}_\lambda$，非负解 $u\in\boldsymbol{m}$ 的全体为 $\mathscr{M}_\lambda^+$，$\mathscr{M}_\lambda^+$ 中其界为K的全体记为 $\mathscr{M}_\lambda^+(K)$。记方程组

$$(V_\lambda)\ \lambda v_j-\sum_i v_i q_{ij}=0,\ j\in E. \tag{12}$$

的解 $v\in\boldsymbol{l}$ 全体为 $\mathscr{L}_\lambda$，非负解 $v\in\mathscr{L}_\lambda$ 全体为 $\mathscr{L}_\lambda^+$。

引理4 如 $f\in\mathscr{M}_\mu$ 或 $\mathscr{M}_\mu^+$，则 $A(\mu,\ \lambda)f\in\mathscr{M}_\lambda$ 或 $\mathscr{M}_\lambda^+$。如 $g\in\mathscr{L}_\mu$ 或 $\mathscr{L}_\mu^+$，则 $gA(\mu,\ \lambda)\in\mathscr{L}_\lambda$ 或 $\mathscr{L}_\lambda^+$。

证 设 $f\in\mathscr{M}_\mu$，则由 $\phi(\lambda)$ 的 B 条件，

$$\begin{aligned}QA(\mu,\ \lambda)f&=Qf+(\mu-\lambda)Q\phi(\lambda)f\\&=\mu f+(\mu-\lambda)[\lambda\phi(\lambda)f-f]\\&=\lambda A(\mu,\ \lambda)f\in\boldsymbol{m},\end{aligned}$$

即 $A(\mu,\ \lambda)f\in\mathscr{M}_\lambda$。

设 $f\in\mathscr{M}_\mu^+$。当 $\lambda\leqslant\mu$ 时，显然

$$A(\mu,\ \lambda)f=f+(\mu-\lambda)\phi(\lambda)f\geqslant\mathbf{0}. \tag{13}$$

当 $\lambda>\mu$ 时，有

$$(\lambda I-Q)f=(\lambda-\mu)f\geqslant\mathbf{0}.$$

由引理2，$f\geqslant\phi(\lambda)(\lambda-\mu)f$，即 $A(\mu,\ \lambda)f\geqslant\mathbf{0}$。

$gA(\mu,\ \lambda)\in\mathscr{L}_\lambda$ 或 $\mathscr{L}_\lambda^+$ 可类似证明。

定理5 $\lambda\phi(\lambda)\mathbf{1}=\mathbf{1}-\phi(\lambda)\mathbf{d}-\overline{X}(\lambda)$，即

$$\lambda\sum_j\phi_{ij}(\lambda)=1-\sum_{a\in H}\phi_{ia}(\lambda)d_a-\overline{X}_i(\lambda). \tag{14}$$

其中 H 为非保守状态集，$\boldsymbol{d}=(d_i)$ 为 Q 的非保守列矢量．$\overline{X}(\lambda)$ 是方程

$$\left.\begin{aligned}&(\lambda I-Q)u=\mathbf{0},\\&\mathbf{0}\leqslant u\leqslant\mathbf{1}.\end{aligned}\right\} \tag{15}$$

的最大解，且 $u^n=\Pi^n(\lambda)\mathbf{1}\downarrow\overline{X}(\lambda)$，$u^n$ 即

$$\left.\begin{aligned}&u_i^0\equiv0,\\&u_i^{n+1}=\sum_{k\neq i}\frac{q_{ik}}{\lambda+q_i}u_k^n.\end{aligned}\right\} \tag{16}$$

证 由 $\phi(\lambda)$ 的 B 条件

$$(\lambda I - Q)\phi(\lambda)\mathbf{d} = \mathbf{d}, \tag{17}$$

$$(\lambda I - Q)(\mathbf{1} - \lambda\phi(\lambda)\mathbf{1}) = (\lambda I - Q)\mathbf{1} - \lambda I = -Q\mathbf{1} = \mathbf{d}, \tag{18}$$

由引理2，

$$\mathbf{1} - \lambda\phi(\lambda)\mathbf{1} \geqslant \phi(\lambda)\mathbf{d},$$

从而由(17)(18)知

$$\bar{X}(\lambda) = \mathbf{1} - \lambda\phi(\lambda)\mathbf{1} - \phi(\lambda)\mathbf{d}$$

是方程(15)的解，并得到(14)。

由引理1，$\mathbf{1} - \bar{X}(\lambda) = \phi(\lambda)(\lambda\mathbf{1} + \mathbf{d})$是上升列$\xi^n$的极限，其中

$$\left.\begin{aligned} &\xi_i^0 = 0, \\ &\xi_i^{n+1} = \frac{\lambda + d_i}{\lambda + q_i} + \sum_{k \neq i} \frac{q_{ik}}{\lambda + q_i} \xi_k^n. \end{aligned}\right\} \tag{19}$$

因而$\bar{X}(\lambda)$是下降列$u^n = 1 - \xi^n$的极限。由上式得(16)。

最后，设u是方程(15)的解，则由$u \leqslant \mathbf{1}$有$u = \Pi(\lambda)u \leqslant \Pi(\lambda)\mathbf{1}$，故$u \leqslant \Pi^n(\lambda)\mathbf{1}$，于是$u \leqslant \lim\limits_{n\to\infty} \Pi^n(\lambda)\mathbf{1} = \bar{X}(\lambda)$，即$\bar{X}(\lambda)$是方程(15)的最大解，证毕。

引理6 设$u \in \mathscr{M}_1$且$|u| \leqslant 1$，则

$$-\bar{X}(\lambda) \leqslant u \leqslant \bar{X}(\lambda). \tag{20}$$

证 右方不等式已在定理5中证明，左方不等式可类似证明。

引理7 设$\bar{X}(\lambda) \neq \mathbf{0}$。则

$$\sup_i \bar{X}_i(\lambda) = 1, \quad \inf_i \sum_{a \in H} \phi_{ia}(\lambda) d_a = 0, \tag{21}$$

$$\inf_i [\phi(\lambda) u]_i = 0, \quad u \in \boldsymbol{m}. \tag{22}$$

证 令$\sup\limits_i \bar{X}_i(\lambda) = a$，则$0 < a \leqslant 1$。因$a^{-1}\bar{X}(\lambda) \in \mathscr{M}_1^+(1)$，由$\bar{X}(\lambda)$的最大性，$a^{-1}\bar{X}(\lambda) \leqslant \bar{X}(\lambda)$，故$1 \leqslant a$，所以$a = 1$，即$\sup\limits_i \bar{X}_i(\lambda) = 1$。由此及(14)得$\inf\limits_i \sum\limits_{a \in H} \phi_{ia}(\lambda) d_a = 0$及$\inf\limits_i [\lambda\phi(\lambda)\mathbf{1}]_i = 0$，从而得$\inf\limits_i [\phi(\lambda)u]_i = 0$对$u \in \boldsymbol{m}$。证完。

§11. 流出族和流入族

定义1 称$(\xi(\lambda), \lambda>0)$为列协调族，也称流出族，如果$0\leqslant \xi(\lambda)\in \mathfrak{m}$且

$$\xi(\mu)=A(\lambda, \mu)\xi(\lambda), \quad \lambda, \mu>0. \tag{1}$$

满足$\xi(\lambda)\in \mathscr{M}_1^+$的列协调族$(\xi(\lambda), \lambda>0)$称为列协调解族，也称调和流出族。

称$(\eta(\lambda), \lambda>0)$为行协调族，也称流入族，如果$0\leqslant \eta(\lambda)\in l$，且

$$\eta(\mu)=\eta(\lambda)A(\lambda, \mu), \quad \lambda, \mu>0. \tag{2}$$

满足$\eta(\lambda)\in \mathscr{S}_1^+$的行协调族$(\eta(\lambda), \lambda>0)$称为行协调解族，也称调和流入族。

显然，对协调族$\xi(\lambda)$或$\eta(\lambda)$，有

$$\xi(\lambda)\downarrow, \quad \eta(\lambda)\downarrow, \quad (\lambda\uparrow\infty). \tag{3}$$

而且$\xi(\lambda)=0$或$\eta(\lambda)=0$对某个λ成立，则对一切$\lambda>0$成立。

定义2 称

$$\xi=\lim_{\lambda\downarrow 0}\xi(\lambda) \tag{4}$$

为列协调族$\xi(\lambda)(\lambda>0)$的标准映象。称

$$\eta=\lim_{\lambda\downarrow 0}\eta(\lambda) \tag{5}$$

为行协调族$\eta(\lambda)(\lambda>0)$的标准映象。

引理1 设ξ为列协调族$\xi(\lambda)$的标准映象，则[1)]

$$\xi=\xi(\mu)+\mu\Gamma\xi(\mu), \quad \mu>0. \tag{6}$$

$$\xi=\xi(\lambda)+\lambda\phi(\lambda)\xi, \quad \lambda>0. \tag{7}$$

其中
$$\Gamma=\lim_{\lambda\downarrow 0}\phi(\lambda)=\sum_{n=0}^{\infty}\Pi^n q^{-1}. \tag{8}$$

---

1) 约定$0\cdot\infty=\infty\cdot 0=0$，$\frac{1}{0}=\infty$。

设η为行协调族$\eta(\lambda)$的标准映象，则

$$\eta=\eta(\mu)+\mu\eta(\mu)\Gamma,\quad \mu>0. \tag{9}$$

$$\eta=\eta(\lambda)+\lambda\eta\phi(\lambda),\quad \lambda>0. \tag{10}$$

证 因

$$\xi(\lambda)=\xi(\mu)+(\mu-\lambda)\phi(\lambda)\xi(\mu), \tag{11}$$

$$\xi(\mu)=\xi(\lambda)+(\lambda-\mu)\phi(\lambda)\xi(\mu). \tag{12}$$

如果$\sum_j \Gamma_{ij}\xi_j(\mu)<\infty$，在(11)中令$\lambda\downarrow 0$得(6)。如果

$\sum_j \Gamma_{ij}\xi_j(\mu)=\infty$，在(11)中令$\lambda\downarrow 0$，由法都引理，

$$\xi_i\geqslant\xi_i(\mu)+\mu\sum_j \Gamma_{ij}\xi_j(\mu)=\infty.$$

因而(6)仍然成立。按照同样考虑在(12)中令$\mu\downarrow 0$得(7)。(9)和(10)可类似地证明。证毕。

系 指定i，$\xi_i<\infty$当且仅当对某个（从而一切）$\lambda>0$有

$$[\lambda\phi(\lambda)\xi]_i=[\lambda\Gamma\xi(\lambda)]_i<\infty,$$

此时

$$[\lambda\phi(\lambda)\xi]_i=\xi_i-\xi_i(\lambda),\quad \lambda>0.$$

同样地，指定j，$\eta_j<\infty$当且仅当对某个（从而一切）$\lambda>0$有

$$[\lambda\eta\phi(\lambda)]_j=[\lambda\eta(\lambda)\Gamma]_j<\infty,$$

此时

$$[\lambda\eta\phi(\lambda)]_j=\eta_j-\eta_j(\lambda),\quad \lambda>0.$$

引理2 设$X^0=\lim_{\lambda\downarrow 0}\lambda\phi(\lambda)\mathbf{1}$，则

$$\sum_{a\in H}X^a+\overline{X}+X^0=\mathbf{1}, \tag{13}$$

其中 $$X_i^a=\Gamma_{ia}d_a,\quad a\in H, \tag{14}$$

是列协调族

$$X_i^a(\lambda)=\phi_{ia}(\lambda)d_a,\quad a\in H,\quad \lambda>0, \tag{15}$$

的标准映象，因而

$$\lambda\phi(\lambda)X^a=X^a-X^a(\lambda),\quad a\in H. \tag{16}$$

$\overline{X}$是列协调族$\overline{X}(\lambda)$的标准映象，因而

$$\lambda\phi(\lambda)\overline{X}=\overline{X}-\overline{X}(\lambda),\quad \lambda>0。\tag{17}$$

$\overline{X}$是方程

$$\left.\begin{array}{l}\Pi u=u,\\ 0\leqslant u\leqslant 1。\end{array}\right\}\tag{18}$$

的解，而

$$\lambda\phi(\lambda)X^0=X^0,\tag{19}$$

且X^0是方程(18)满足条件$\lambda\phi(\lambda)u=u$的最大解。

证 由$\phi(\lambda)$的预解方程知$X^a(\lambda)(a\in H)$是列协调族，其标准映象

$$X_i^a=\lim_{\lambda\downarrow 0}X_i^a(\lambda)=\lim_{\lambda\downarrow 0}\phi_{ia}(\lambda)d_a=\Gamma_{ia}d_a。\tag{20}$$

由(10.14)，$\sum\limits_{a\in H}X^a(\lambda)\leqslant 1$，故$\sum\limits_{a\in H}X^a\leqslant 1$。因而由(7)得(16)。

由(10.14)及(16)得

$$\lambda\phi(\lambda)\Big(1-\sum_{a\in H}X^a\Big)=1-\sum_{a\in H}X^a-\overline{X}(\lambda)。\tag{21}$$

由此及$\phi(\lambda)$的预解方程知$\overline{X}(\lambda)(\lambda>0)$是列协调族，且由于$\overline{X}(\lambda)\leqslant 1$，故其标准映象$\leqslant 1$，因而由(7)得(17)。又因为

$$\left.\begin{array}{l}(\lambda I-Q)\overline{X}(\lambda)=0,\\ 0\leqslant\overline{X}(\lambda)\leqslant 1。\end{array}\right\}$$

故令$\lambda\downarrow 0$知$\overline{X}$是方程(18)的解。

由(10.14)，$\sum\limits_{a\in H}X^a(\lambda)+\overline{X}(\lambda)\leqslant 1$，令$\lambda\downarrow 0$得$\lambda\phi(\lambda)1\downarrow X^0$，且

$$X^0=1-\sum_{a\in H}X^a-\overline{X}\geqslant 0,$$

由此得(13)。由(17)及(21)得(19)。由$\phi(\lambda)$的B条件有

$$(\lambda I-Q)\lambda\phi(\lambda)1=\lambda\tag{22}$$

及$\lambda\phi(\lambda)1\downarrow X^0(\lambda\downarrow 0)$。在上式中令$\lambda\downarrow 0$得$QX^0=0$，即$X^0$满足(18)。设$u$满足(18)且$\lambda\phi(\lambda)u=u$，则$u=\lambda\phi(\lambda)u\leqslant\lambda\phi(\lambda)1$，故$u\leqslant X^0$，证毕。

定义3 称$\bar{X}$为矩阵Q的最大流出解，称X^0为矩阵Q的最大消极解。

引理3 (i) $\eta(\lambda)(\lambda>0)$ 为行协调族的充要条件是有下列的Riesz分解：

$$\eta(\lambda)=\alpha\phi(\lambda)+\bar{\eta}(\lambda)。\tag{23}$$

其中行矢量$\alpha\geqslant\mathbf{0}$且对某个（从而一切）$\lambda>0$使$\alpha\phi(\lambda)\in\boldsymbol{l}$即

$$\left[\alpha,\ \mathbf{1}-\sum_{a\in H}X^a(\lambda)-\bar{X}(\lambda)\right]<\infty,\tag{24}$$

而$\bar{\eta}(\lambda)\in\mathscr{L}_1^+$是行协调解族。$\alpha$因而$\bar{\eta}(\lambda)$由$\eta(\lambda)$唯一决定：

$$\eta(\lambda)(\lambda I-Q)=\alpha,\tag{25}$$

$$\eta(\lambda)\downarrow\mathbf{0},\ \lambda\eta(\lambda)\to\alpha,\ (\lambda\uparrow\infty)。\tag{26}$$

(ii) $\xi(\lambda)(\lambda>0)$为列协调解族的充要条件是有下列Riesz分解：

$$\xi(\lambda)=\phi(\lambda)\beta+\bar{\xi}(\lambda)。\tag{27}$$

其中列矢量$\beta\geqslant\mathbf{0}$且对某个（从而一切）$\lambda>0$使$\phi(\lambda)\beta\in\boldsymbol{m}$，而$\bar{\xi}(\lambda)\in\mathscr{M}_1^+$是列协调族。$\beta$因而$\bar{\xi}(\lambda)$由$\xi(\lambda)$唯一决定：

$$(\lambda I-Q)\xi(\lambda)=\beta,\tag{28}$$

$$\xi(\lambda)\downarrow\mathbf{0},\ \lambda\xi(\lambda)\to\beta,\ (\lambda\uparrow\infty)。\tag{29}$$

证 证(i)。往证必要性。因为$(\eta(\lambda),\ \lambda>0)$是行协调族，故对任意$\nu>0$，$\lambda>0$，有

$$\eta(\nu)+(\nu-\lambda)\eta(\nu)\phi(\lambda)=\eta(\lambda)\geqslant\mathbf{0},$$

$$\eta(\nu)\geqslant(\lambda-\nu)\eta(\nu)\phi(\lambda),$$

$$\eta_j(\nu)\geqslant(\lambda-\nu)\eta_j(\nu)\lambda^{-1}+(\lambda-\nu)\sum_i\eta_i(\nu)[\phi_{ij}(\lambda)$$

$$-\lambda^{-1}\delta_{ij}],$$

$$\nu\eta_j(\nu)\geqslant(1-\nu\lambda^{-1})\sum_i\eta_i(\nu)\lambda[\lambda\phi_{ij}(\lambda)-\delta_{ij}]。$$

在上式右方的求和中，对$i\neq j$的项均非负，故依$\phi(\lambda)$的Q条件及法都引理，令$\lambda\to\infty$得

$$\nu\eta_j(\nu)\geqslant\sum_i\eta_i(\nu)q_{ij}。$$

于是存在有限值的非负行矢量$\alpha(\nu)$使

$$\eta(\nu)(\nu I-Q)=\alpha(\nu). \tag{30}$$

依引理10.2，$\eta(\nu)\geqslant\alpha(\nu)\phi(\nu)$，故$\alpha(\nu)\phi(\nu)\in\boldsymbol{l}$对一切$\nu>0$成立。由于$\phi(\lambda)$的$F$条件，对任意非负行矢量$\alpha$，只要$\alpha\phi(\nu)\in\boldsymbol{l}$，就有

$$\alpha\phi(\nu)(\nu I-Q)=\alpha. \tag{31}$$

于是

$$\eta(\nu)=\alpha(\nu)\phi(\nu)+\tilde{\eta}(\nu), \tag{32}$$

这里$\tilde{\eta}(\nu)\in\mathscr{L}^+_\nu$。右乘$A(\nu,\lambda)$，注意(10.9)及(2)，得

$$\eta(\lambda)=\alpha(\nu)\phi(\lambda)+\tilde{\eta}(\nu)A(\nu,\lambda). \tag{33}$$

依引理10.4，$\tilde{\eta}(\nu)A(\nu,\lambda)\in\mathscr{L}^+_\lambda$。将(33)代入(30)并注意(31)得$\alpha(\nu)=\alpha(\lambda)$，即$\alpha(\lambda)=\alpha$与$\lambda$无关，从而$\tilde{\eta}(\nu)A(\nu,\lambda)=\tilde{\eta}(\lambda)$。这样，(32)成为(23)，其中非负行矢量$\alpha$使$\alpha\phi(\lambda)\in\boldsymbol{l}$对一切$\lambda>0$成立，$(\tilde{\eta}(\lambda),\lambda>0)$是行协调解族。由(3)及(25)得(26)第一式。由此及控制收敛定理，在(25)中令$\lambda\to\infty$得(26)第二式.必要性证完。充分性是明显的。

往证(ii)。设$(\xi(\lambda),\lambda>0)$是列协调族。注意(2.6)，对任意$\nu>0$，列矢量

$$\beta(\nu)\equiv(\nu I-Q)\xi(\nu) \tag{34}$$

取有限的值。由于(2.6)及$\phi(\lambda)$的B条件，

$$\begin{aligned}(\nu I-Q)\xi(\nu)&=(\nu I-Q)\{[I+(\lambda-\nu)\phi(\nu)]\xi(\lambda)\}\\&=\{(\nu I-Q)[I+(\lambda-\nu)\phi(\nu)]\}\xi(\lambda)\\&=[\nu I-Q+(\lambda-\nu)I]\xi(\lambda)\\&=(\lambda I-Q)\xi(\lambda).\end{aligned}$$

故$\beta(\lambda)=\beta$与$\lambda>0$无关，因而(28)成立。由(3)及(28)得(29)第一式。由此及控制收敛定理，在(28)中令$\lambda\to\infty$得(29)第二式.因$\xi(\lambda)\geqslant0$，故(29)中第二式的极限$\beta\geqslant0$。依引理10.2，从(28)得$\xi(\nu)\geqslant\phi(\nu)\beta$，从而$\phi(\nu)\beta\in\boldsymbol{m}$对一切$\nu>0$成立。由$\phi(\lambda)$的$B$条件，对任意非负列矢量$\beta$，只要$\phi(\nu)\beta\in\boldsymbol{m}$，就有

$$(\nu I-Q)\phi(\nu)\beta=\beta. \tag{35}$$

于是由上式，(28)及引理10.2，

$$\xi(\nu)=\phi(\nu)\beta+\bar{\xi}(\nu), \tag{36}$$

其中$\bar{\xi}(\nu)\in\mathscr{M}_\nu^+$。左乘上式$A(\nu,\ \lambda)$并注意(10.9)及(1)，得

$$\xi(\lambda)=\phi(\lambda)\beta+A(\nu,\ \lambda)\bar{\xi}(\nu),$$

故$\bar{\xi}(\lambda)=A(\nu,\ \lambda)\bar{\xi}(\nu)$，从而$(\bar{\xi}(\lambda),\ \lambda>0)$是列协调族。必要性证完。充分性是明显的。证完。

系 当$\lambda\uparrow\infty$时，

$$X^a(\lambda)\downarrow 0,\quad \lambda X_i^a(\lambda)\to\delta_{ia}d_a,\quad a\in H, \tag{37}$$

$$\bar{X}(\lambda)\downarrow 0,\quad \lambda\bar{X}(\lambda)\to 0. \tag{38}$$

引理4 设$(\eta(\lambda),\ \lambda>0)$是行协调族，则

$$\sigma^0=\lambda[\eta(\lambda),X^0]<\infty,\text{与}\lambda>0\text{无关}. \tag{39}$$

如果$(\xi(\lambda),\ \lambda>0)$是列协调族，并且其标准映象$\xi\in\boldsymbol{m}$。则

$$(\lambda-\mu)[\eta(\lambda),\ \xi(\mu)]=\lambda[\eta(\lambda),\ \xi]-\mu[\eta(\mu),\ \xi], \tag{40}$$

$$\lambda[\eta(\lambda),\ \xi]\uparrow V\leqslant\infty. \tag{41}$$

特别地，当$a\in H$时

$$V_\lambda^a=\lambda[\eta(\lambda),\ X^a]\uparrow V^a<\infty, \tag{42}$$

其中

$$V^a=V_\mu^a+\eta_a(\mu)d_a,\text{与}\mu\text{无关}. \tag{43}$$

记行协调族$(\eta(\lambda),\ \lambda>0)$的标准映象为$\eta$。则

$$[\eta(\lambda),\ \xi]=[\eta,\ \xi(\lambda)]. \tag{44}$$

如果$[\eta,\ X^0]<\infty$，则

$$[\eta,\ X^0]=0. \tag{45}$$

证 由(2)(7)及$\xi\in\boldsymbol{m}$，

$$\begin{aligned}
&\lambda[\eta(\lambda),\ \xi]=\lambda[\eta(\mu)A(\mu,\ \lambda),\ \xi]\\
&=\lambda[\eta(\mu),\ A(\mu,\ \lambda)\xi]\\
&=\lambda[\eta(\mu),\ \xi]+(\mu-\lambda)[\eta(\mu),\ \lambda\phi(\lambda)\xi]\\
&=\lambda[\eta(\mu),\ \xi]+(\mu-\lambda)[\eta(\mu),\ \xi-\xi(\lambda)]\\
&=\mu[\eta(\mu),\ \xi]+(\lambda-\mu)[\eta(\mu),\ \xi(\lambda)].
\end{aligned} \tag{46}$$

由此得(40)，从而得(41)。类似地，利用(19)可得σ^0与λ无关。

当$a\in H$，$\xi(\lambda)=X^a(\lambda)$时，(40)成为

$$V_\lambda^a=V_\mu^a+(\lambda-\mu)[\eta(\mu),\ X^a(\lambda)]. \tag{47}$$

因为$\lambda X^a(\lambda)\leqslant d_a$，由(37)及控制收敛定理，

$$[\eta(\mu),\ \lambda X^a(\lambda)]\to\eta_a(\mu)d_a,\ \lambda\uparrow\infty,\ a\subset H. \tag{48}$$

因而在(47)中令$\lambda\to\infty$得(42)和(43)。

当$\mu\to 0$时，

$$(\lambda-\mu)[\eta(\lambda),\ \xi(\mu)]=\lambda[\eta(\lambda),\ \xi(\mu)]-\mu[\eta(\lambda),\xi(\mu)]$$
$$\to\lambda[\eta(\lambda),\ \xi].$$

由此，在(40)中令$\mu\to 0$得

$$\mu[\eta(\mu),\ \xi]\to 0,\ 如\mu\to 0. \tag{48'}$$

从而在(40)中令$\lambda\to 0$得(44)。

由(39)，如$[\eta,\ X^0]<\infty$，则

$$\sigma^0\leqslant\lambda[\eta,\ X^0]\to 0,\ 当\lambda\to 0时。$$

故$\sigma^0=0$，$[\eta(\lambda),\ X^0]=0$。令$\lambda\to 0$得(45)。证完。

引理5 如果Q非保守，则

$$\lambda\sum_{a\subset H}X^a(\lambda)\to\mathbf{d},\ \lambda\to\infty. \tag{49}$$

证 因$Z(\lambda)=\mathbf{1}-\lambda\phi(\lambda)\mathbf{1}=\sum_{a\subset H}X^a(\lambda)+X(\lambda)$是列协调族，且$(\lambda I-Q)Z(\lambda)=\mathbf{d}$。故由引理3，$\lambda Z(\lambda)\to\mathbf{d}$。注意(38)便得(49)，证毕。

引理6 如果行矢量a使$a\phi(\lambda)=\mathbf{0}$（即对每个$j$，$\sum_i a_i\phi_{ij}(\lambda)=0$，且级数绝对收敛），则$a=\mathbf{0}$。

如果列矢量β使$\phi(\lambda)\beta=\mathbf{0}$（即对每个$i$，$\sum_j\phi_{ij}(\lambda)\beta_j=0$，且级数绝对收敛），则$\beta=\mathbf{0}$。

证 因$\phi(\lambda)$满足向后方程组，故

$$\lambda\phi_{ij}(\lambda)=\delta_{ij}+\sum_k q_{ik}\phi_{kj}(\lambda). \tag{50}$$

固定i，两边乘β_j并对j求和得

$$0=\beta_i+\sum_j\sum_k q_{ik}\phi_{kj}(\lambda)\beta_j$$

$$= \beta_i + \sum_k q_{ik}\Big(\sum_j \phi_{kj}(\lambda)\beta_j \Big) = \beta_i,$$

求和号可以交换是由于

$$\sum_j \Big(\sum_k |q_{ik}| \phi_{kj}(\lambda) |\beta_j| = \Big) \sum_j |\beta_j| \Big(\sum_k |q_{ik}| \phi_{kj}(\lambda) \Big)$$

$$= \sum_j |\beta_j| \Big(\sum_k q_{ik}\phi_{kj}(\lambda) + 2q_i\phi_{ij}(\lambda) \Big)$$

$$= \sum_j |\beta_j| \{ \lambda\phi_{ij}(\lambda) - \delta_{ij} + 2q_i\phi_{ij}(\lambda) \}$$

$$\leqslant (\lambda + 2q_i) \sum_j \phi_{ij}(\lambda) |\beta_j| < \infty.$$

应用$\phi(\lambda)$的F条件

$$\lambda\phi_{ij}(\lambda) = \delta_{ij} + \sum_k \phi_{ik}(\lambda) q_{kj}, \tag{51}$$

类似可证$\alpha = 0$，证毕.

§12. Q过程的一般形式

设$\psi(\lambda) \in \mathscr{P}_s(Q)$，则$\psi(\lambda) - \phi(\lambda) \geqslant 0$. 由于$\psi(\lambda)$满足向后不等式组(8.10)以及(8.12)，而$\phi(\lambda)$满足$B$条件，故$(\lambda I - Q)[\psi(\lambda) - \phi(\lambda)] \geqslant 0$，而且当$i \in E - H$时等号成立。更精确些，对固定$j$，$u_i = \psi_{ij}(\lambda) - \phi_{ij}(\lambda)$满足

$$\lambda u_i - \sum_k q_{ik} u_k = d_i F_j^i(\lambda), \tag{1}$$

其中$F_j^i(\lambda) \geqslant 0$，$d$为非保守量. 由引理10.2，

$$B_{ij}(\lambda) = \psi_{ij}(\lambda) - \phi_{ij}(\lambda) - \sum_{a \in H} \phi_{ia}(\lambda) d_a F_j^a(\lambda) \geqslant 0.$$

当j固定时，$B_{\cdot j}(\lambda) \in \mathscr{M}_\lambda^+\Big(\frac{1}{\lambda}\Big)$. 这样我们得到下面的定理的前一部分.

定理1 任意Q过程$\psi(\lambda) \in \mathscr{P}_s(Q)$必具有下面形式：

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+\sum_{a\in H}X_i^a(\lambda)F_j^a(\lambda)+B_{ij}(\lambda)。\tag{2}$$

其中$X_i^a(\lambda)=\phi_{ia}(\lambda)d_a(a\in H)$。当$i$固定时，$0\leqslant B_{i\cdot}(\lambda)\in\boldsymbol{l}$，当$j$固定时，$B_{\cdot j}(\lambda)\in\mathscr{M}_\lambda^+\left(\frac{1}{\lambda}\right)$，而

$$F^a(\lambda)\geqslant 0,\ \lambda[F^a(\lambda),\ 1]\leqslant 1.\ a\in H。\tag{3}$$

如果 $\psi(\lambda)$ 满足向后方程组，且 Q 非保守，则 $F^a(\lambda)=0$ $(a\in H)$。

如果$\psi(\lambda)$满足向前方程组，则当i固定时，$B_{i\cdot}(\lambda)\in\mathscr{L}_\lambda^+$。而且如果$Q$非保守，则$F^a(\lambda)\in\mathscr{L}_\lambda^+$ $(a\in H)$。

证 因i固定时，$\psi_{i\cdot}(\lambda)\in\boldsymbol{l}$，故$F^a(\lambda)\in\boldsymbol{l}$ $(a\in H)$，$B_{i\cdot}(\lambda)\in\boldsymbol{l}$。

设 $\psi(\lambda)$ 满足向后方程组，且 Q 非保守。由定理 8.4，$\sum_{a\in H}X_i^a(\lambda)F_j^a(\lambda)=\sum_{a\in H}\phi_{ia}(\lambda)d_aF_j^a(\lambda)=0$。由引理11.6，这等价于$F^a(\lambda)=0$ $(a\in H)$。

如果$\psi(\lambda)$满足向前方程组，则由于$\psi(\lambda)$和 $\phi(\lambda)$ 都满足F条件，故若记$G(\lambda)=\{d_iF_j^i(\lambda)\}(i,\ j\in E)$，则

$$\{\phi(\lambda)G(\lambda)+B(\lambda)\}(\lambda I-Q)=0,$$

左乘$\lambda I-Q$得

$$G(\lambda)(\lambda I-Q)=0,$$

从而$B(\lambda)(\lambda I-Q)=0$，即$i$固定时，$B_{i\cdot}(\lambda)\in\mathscr{L}_\lambda^+$，$F^a(\lambda)\in\mathscr{L}_\lambda^+$ $(a\in H)$。

往证(3)。在第六章第18节也将证明(3)。这里给出另一证明。

在(2)两边作用$\lambda I-Q$后并对j求和得

$$(\lambda I-Q)(\lambda\psi(\lambda)1)_i=\lambda+\sum_{a\in H}\delta_{ia}d_a\lambda[F^a(\lambda),\ 1]。\tag{4}$$

当$i=a$时，

$$(\lambda I-Q)(\lambda\psi(\lambda)1)_a=\lambda+d_a\lambda[F^a(\lambda),\ 1]。\tag{5}$$

但在(7.3)两边取拉普拉斯变换后可得

$$(\lambda I-Q)\lambda\psi(\lambda)\mathbf{1}\leqslant\lambda\mathbf{1}+\mathbf{d}. \tag{6}$$

由(5)、(6)得(3)，证毕。

定理2 每个Q过程$\psi(\lambda)\in\mathscr{T}_s(Q)$必具有下面形式：

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+\sum_k H_{ik}(\lambda)\phi_{kj}(\lambda)+C_{ij}(\lambda). \tag{7}$$

其中$H(\lambda)=\{H_{ik}(\lambda)\}\geqslant 0$，$C(\lambda)=\{C_{ij}(\lambda)\}\geqslant 0$。当$i$固定时，$C_{i\cdot}(\lambda)\in\mathscr{L}_\lambda^+$。当$j$固定时，$C_{\cdot j}(\lambda)\in\boldsymbol{m}$，$H_{\cdot j}(\lambda)\in\boldsymbol{m}$。

$\psi(\lambda)$满足向后方程组的充要条件是：$H_{\cdot j}(\lambda)\in\mathscr{M}_\lambda^+$，$C_{\cdot j}(\lambda)\in\mathscr{M}_\lambda^+\left(\frac{1}{\lambda}\right)$。

$\psi(\lambda)$满足向前方程组的充要条件是：$H(\lambda)=0$。

证 由于$\psi(\lambda)$的向前不等式组(8.11)及$\phi(\lambda)$的F条件，故

$$[\psi(\lambda)-\phi(\lambda)](\lambda I-Q)=H(\lambda)\geqslant 0.$$

故由引理10.2，

$$C(\lambda)\equiv\psi(\lambda)-\phi(\lambda)-H(\lambda)\phi(\lambda)\geqslant 0.$$

且$C_{i\cdot}(\lambda)\in\mathscr{L}_\lambda^+$，从而得(7)。由(7)得$C_{\cdot j}(\lambda)\in\boldsymbol{m}$。因

$$H_{\cdot j}(\lambda)\lambda\phi_{jj}(\lambda)\leqslant\lambda\psi_{\cdot j}(\lambda)\leqslant 1,$$

故$H_{\cdot j}(\lambda)\in\boldsymbol{m}$。

$\psi(\lambda)$的B条件等价于

$$(\lambda I-Q)[H(\lambda)\phi(\lambda)+C(\lambda)]=0.$$

右乘上式$(\lambda I-Q)$得$(\lambda I-Q)H(\lambda)=0$。从而由上式得$(\lambda I-Q)C(\lambda)=0$。因此$B$条件等价于$C_{\cdot j}(\lambda)\in\mathscr{M}_\lambda^+\left(\frac{1}{\lambda}\right)$,$H_{\cdot j}(\lambda)\in\mathscr{M}_\lambda^+$。

$\psi(\lambda)$的F条件等价于

$$[H(\lambda)\phi(\lambda)+C(\lambda)](\lambda I-Q)=0.$$

即$H(\lambda)=0$。证完。

第二章　简单情形的Q过程的构造

§1. 引　言

构造Q过程的许多文献，如Reuter[2]，Feller[10]，Doob[1]，Chung[2]，Williams[1]和杨向群[1]中都假定Q保守，因为此时任何Q过程都满足向后方程组。对于非保守的Q，只有少量文献例如Reuter[3]，Feller[5]，和杨向群[2]中有所涉及。如果只构造满足向后方程组的Q过程，原则上可以按定理1.7.7化为保守的情形，但不是直接的构造。而且这种做法不适用于满足向前方程组的Q过程。

本章我们考虑简单情形的Q过程的构造，不必假定Q保守。当Q保守单流出时，Reuter[2]中已构造了全部Q过程。对于非保守的Q，Reuter[3]中构造了一类不满足向后方程组的不中断的Q过程。§2中对单流出时直接构造了满足向后方程组的全部Q过程。§3中对非保守的Q构造了一类包含Reuter[3]中结果的过程。特别，当Q单非保守零流出时，我们构造了全部Q过程。§4中对最小解中断且单流入时，构造了满足向前方程组的全部Q过程。本章结果见杨向群[12]。

§2. 单流出时满足向后方程组的Q过程的构造

由引理1.10.4，方程组

$$\left.\begin{array}{l}(\lambda I-Q)u=0,\\ 0\leqslant u\in \boldsymbol{m}.\end{array}\right\}\qquad(1)$$

的解空间$\mathscr{M}_\lambda^+$的维数m^+与λ无关。当 $m^+=0$即 $\mathscr{M}_\lambda^+$ 只含零元时，称Q为零流出的。当$m^+=1$时，称 Q 为单流出的。当m^+有限时，称Q为有限流出的。

假定 Q 单流出。由引理1.10.6，$\bar{X}(\lambda)\neq 0$，由定理1.10.5，最小解$\phi(\lambda)$ 中断。由定理1.12.1，每个满足向后方程组的 Q 过程$\psi(\lambda)$具有下列形式：

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+\bar{X}_i(\lambda)F_j(\lambda).\qquad(2)$$

我们将在$m^+>0$的条件下，决定$F(\lambda)$使$\psi(\lambda)\in\mathscr{F}_\lambda(Q)$的条件，即$\psi(\lambda)$满足范条件、预解方程。因为对于由(2)确定的$\psi(\lambda)$，$B$条件是恒满足的。

定理1 设$m^+>0$。为使(2)确定的$\psi(\lambda)$是Q过程的充分必要条件是：或者$\psi(\lambda)=\phi(\lambda)$，或者$\psi(\lambda)$ 可如下得到：取行矢量$\alpha\geqslant 0$使$\alpha\phi(\lambda)\in \boldsymbol{l}$，取行协调族$\bar{\eta}(\lambda)\in\mathscr{L}_\lambda^+$，并且

$$\eta(\lambda)=\alpha\phi(\lambda)+\bar{\eta}(\lambda)\neq \mathbf{0}.\qquad(3)$$

取常数c满足

$$[\alpha,\ X^0]+\bar{\sigma}^0+\sum_{a\in H}([\alpha,\ X^a]+\bar{V}^a)\leqslant c.\qquad(4)$$

其中X^0，$\bar{X}$，$X^a(a\in H)$由引理1.11.2确定，而

$$\bar{\sigma}^0=\lambda[\bar{\eta}(\lambda),\ X^0]<\infty\text{与}\lambda\text{无关}.\qquad(5)$$

$$\bar{V}_\lambda^a=\lambda[\bar{\eta}(\lambda),\ X^a]\uparrow\bar{V}^a<\infty,\ \lambda\uparrow\infty,\ a\in H.\qquad(6)$$

$$\bar{V}^a=\bar{V}_\lambda^a+\bar{\eta}_a(\lambda)d_a\text{与}\lambda\text{无关}.\qquad(7)$$

最后令$\psi_{ij}(\lambda)$

$$=\phi_{ij}(\lambda)+\bar{X}_i(\lambda)\frac{\sum\limits_k\alpha_k\phi_{kj}(\lambda)+\bar{\eta}_j(\lambda)}{c+[\alpha,\bar{X}-\bar{X}(\lambda)]+\lambda[\bar{\eta}(\lambda),\bar{X}]}.\qquad(8)$$

过程$\psi(\lambda)$不中断的充要条件是：

$$Q\text{保守并且}[\alpha,\ X^0]+\bar{\sigma}^0=c.\qquad(9)$$

过程$\psi(\lambda)$满足向后方程组. 过程满足向前方程组的充要条件是$\alpha=0$。

当$m^+=1$时，上面的过程已穷尽一切满足向后方程组的 Q过程。当Q保守且$m^+=1$时，上面的过程已穷尽一切Q过程。

证 由 $\phi(\lambda)$ 的最小性，$\psi(\lambda)\geqslant 0$ 等价于 $F(\lambda)\geqslant\mathbf{0}$。注意(1.10.14)，范条件等价于

$$F(\lambda)\geqslant\mathbf{0},\ \overline{X}(\lambda)\lambda[F(\lambda),\mathbf{1}]\leqslant\overline{X}(\lambda)+\sum_{a\in H}X^a(\lambda). \tag{10}$$

由于(1.10.14)，

$$\sum_{a\in H}X^a(\lambda)=\phi(\lambda)\mathbf{d}=\sum_{n=0}^{\infty}\prod^n(\lambda)(\lambda+q)^{-1}\mathbf{d}\leqslant\mathbf{1}. \tag{11}$$

故
$$\prod^n(\lambda)\Big(\sum_{a\in H}X^a(\lambda)\Big)=\sum_{k=n}^{\infty}\prod^k(\lambda)(\lambda I+q)^{-1}\mathbf{d}\to\mathbf{0},$$

$$n\to\infty. \tag{12}$$

在(10)中左乘$\prod^n(\lambda)$并取极限得

$$\overline{X}(\lambda)\lambda[F(\lambda),\ \mathbf{1}]\leqslant\overline{X}(\lambda).$$

故$\lambda[F(\lambda),\ \mathbf{1}]\leqslant 1$。因此范条件等价于

$$F(\lambda)\geqslant 0,\ \lambda[F(\lambda),\ \mathbf{1}]\leqslant 1. \tag{13}$$

$\psi(\lambda)$不中断的充要条件是

$$Q\text{保守并且}\lambda[F(\lambda),\ \mathbf{1}]=1. \tag{14}$$

由于$\phi(\lambda)$满足预解方程，将$\psi(\lambda)$代入预解方程中，注意$\overline{X}(\lambda)$是列协调族，得$\psi(\lambda)$的预解方程等价于

$$F(\lambda)A(\lambda,\ \mu)=\{1+(\mu-\lambda)[F(\lambda),\ \overline{X}(\mu)]\}F(\mu). \tag{15}$$

或由于(1.10.10)，将$A(\mu,\ \lambda)$右乘上式后，(15)等价于

$$F(\lambda)=\{1+(\mu-\lambda)[F(\lambda),\ \overline{X}(\mu)]\}F(\mu)A(\mu,\ \lambda). \tag{16}$$

如果对某$\mu>0$，$F(\mu)=\mathbf{0}$。则由上式知对一切 $\lambda>0$，$F(\lambda)=\mathbf{0}$。因而$\psi(\lambda)=\phi(\lambda)$。

否则，对一切$\mu>0$，$F(\mu)\neq 0$。由于$\lambda[F(\lambda),\bar{X}(\mu)]\leqslant\lambda[F(\lambda),1]\leqslant 1$，故 $1+(\mu-\lambda)[F(\lambda),\bar{X}(\mu)]>0$。由（16）可见$F(\mu)A(\mu,\lambda)\geqslant 0$。因此，如果固定一个$\mu>0$，则$\eta(\lambda)=F(\mu)A(\mu,\lambda)$是行协调族。这样，(16)等价于

$$F(\lambda)=m_\lambda\eta(\lambda),\quad m_\lambda>0,\quad \eta(\lambda)\neq 0. \tag{17}$$

其中数量m_λ满足

$$m_\lambda=m_\mu+(\mu-\lambda)m_\lambda[\eta(\lambda),\bar{X}(\mu)]m_\mu, \tag{18}$$

而$\eta(\lambda)$为非零行协调族。按引理1.11.3，$\eta(\lambda)$有Riesz表现(3)。

在(18)两边除以$m_\lambda m_\mu$得

$$m_\mu^{-1}=m_\lambda^{-1}+(\mu-\lambda)[\eta(\lambda),\bar{X}(\mu)]. \tag{19}$$

但由(1.11.40)，

$$(\mu-\lambda)[\eta(\lambda),\bar{X}(\mu)]=\mu[\eta(\mu),\bar{X}]-\lambda[\eta(\lambda),\bar{X}]. \tag{20}$$

因而(19)成为

$$m_\lambda^{-1}-\lambda[\eta(\lambda),\bar{X}]=c\qquad(\text{常数}). \tag{21}$$

因此由(1.11.17)，

$$m_\lambda=\frac{1}{c+\lambda[\eta(\lambda),\bar{X}]}=\frac{1}{c+\lambda[\alpha\phi(\lambda),\bar{X}]+\lambda[\bar{\eta}(\lambda),\bar{X}]}$$

$$=\frac{1}{c+[\alpha,\bar{X}-\bar{X}(\lambda)]+\lambda[\bar{\eta}(\lambda),\bar{X}]}. \tag{22}$$

而且由（18）—(22)的每一步推导都是可逆的。

将(17)(22)代入(13)得

$$\lambda[\eta(\lambda),1-\bar{X}]\leqslant c. \tag{23}$$

但由(1.11.13)，(1.11.16)—(1.11.19)及引理1.11.4，

$$\lambda[\eta(\lambda),1-\bar{X}]=\lambda[\alpha\phi(\lambda),1-\bar{X}]+\lambda[\bar{\eta}(\lambda),1-\bar{X}]$$

$$=\lambda[\alpha\phi(\lambda),X^0+\sum_{a\in H}X^a]+\lambda[\bar{\eta}(\lambda),X^0+\sum_{a\in H}X^a]$$

$$=[\alpha,\lambda\phi(\lambda)(X^0+\sum_{a\in H}X^a)]+\bar{V}^0+\sum_{a\in H}\bar{V}_\lambda^a$$

$$=[\alpha,X^0]+\bar{\sigma}^0+\sum_{a\in H}([\alpha,X^a-X^a(\lambda)]+\bar{V}^a_\lambda)$$

$$\uparrow[\alpha,X^0]+\bar{\sigma}^0+\sum_{a\in H}([\alpha,X^a]+\bar{V}^a),\lambda\uparrow\infty。\qquad(24)$$

最后一步是由于(1.11.37)。因此范条件(23)成为(4)。而(14)成为(9)。将$(\lambda I-Q)$右乘(8)得F条件成立的充要条件为$\alpha=\mathbf{0}$。定理其他结论明显，证毕。

在定理1的证明中，如果在(2)中用方程 (1.10.15)的非零列协调族解$\bar{\xi}(\lambda)$代替$\bar{X}(\lambda)$，稍作修改，定理仍然有效。因此下面有定理2。

定理2 设$m^+>0$。$\bar{\xi}(\lambda)\in\mathscr{M}^+_\lambda(1)$是非零列协调族，$\bar{\xi}$为$\bar{\xi}(\lambda)$的标准映象，$\sup_i\bar{\xi}_i(\lambda)=1$。为使

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+\bar{\xi}_i(\lambda)F_j(\lambda)\qquad(25)$$

确定的 $\psi(\lambda)\in\mathscr{F}_s(Q)$ 的充要条件是：或者 $\psi(\lambda)=\phi(\lambda)$，或者 $\psi(\lambda)$ 可如下得到：取行矢量$\alpha\geqslant\mathbf{0}$使$\alpha\phi(\lambda)\in\mathbf{l}$，取行协调族 $\bar{\eta}(\lambda)\in\mathscr{L}^+_\lambda$，使(3)成立且满足

$$[\alpha,\bar{X}-\bar{\xi}]<\infty,W_\lambda\equiv\lambda[\bar{\eta}(\lambda),\bar{X}-\bar{\xi}]\uparrow W<\infty,$$
$$\lambda\uparrow\infty。\qquad(26)$$

取常数c满足

$$[\alpha,\ X^0]+\bar{\sigma}^0+[\alpha,\ \bar{X}-\bar{\xi}]+W+\sum_{a\in H}([\alpha,\ X^a]+\bar{V}^a)$$
$$\leqslant c\qquad(27)$$

其中记号同定理1。最后令

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+\bar{\xi}_i(\lambda)\frac{\sum_k\alpha_k\phi_{kj}(\lambda)+\bar{\eta}_j(\lambda)}{c+[\alpha,\bar{\xi}-\bar{\xi}(\lambda)]+\lambda[\bar{\eta}(\lambda),\bar{\xi}]}.$$
$$(28)$$

过程$\psi(\lambda)$满足向后方程组。过程不中断的充要条件是

Q保守，$\bar{\xi}(\lambda)=\bar{X}(\lambda)$，$[\alpha,\ X^0]+\bar{\sigma}^0=c\qquad(29)$

§3. 单非保守零流出时Q过程的构造

假定Q非保守，并且

$$Z(\lambda)=\mathbf{1}-\lambda\phi(\lambda)\mathbf{1}=\sum_{a\in H}X^a(\lambda)+\underline{X}(\lambda). \tag{1}$$

显然，$Z(\lambda)$是非零列协调族，其标准映象为

$$Z=\sum_{a\in H}X^a+\underline{X}=\mathbf{1}-X^0. \tag{2}$$

我们来决定$F(\lambda)$使

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+Z_i(\lambda)F_j(\lambda) \tag{3}$$

确定的$\psi(\lambda)\in\mathscr{F}_*(Q)$的充要条件。由于$Z(\lambda)\,\overline{\in}\,\mathscr{M}^+_1$，故$B$条件不成立。因而我们需要考虑范条件、预解方程和$Q$条件。

定理1 为使(3)确定的$\psi(\lambda)$是非最小Q过程的充要条件是$\psi(\lambda)$可以如下得到：取行矢量$\alpha\geqslant 0$使$\alpha\phi(\lambda)\in\mathbf{l}$，取行协调族$\overline{\eta}(\lambda)\in\mathscr{L}^+_1$，并且(2.3)成立，还应满足

$$[\alpha,\ Z]+U=\infty,\ \text{如果}\alpha\neq 0. \tag{4}$$

或等价地

$$[\alpha,\ \mathbf{1}]+Y=\infty,\ \text{如果}\alpha\neq 0. \tag{5}$$

其中 $$U_\lambda=\lambda[\overline{\eta}(\lambda),\ Z]\uparrow U,\ \lambda\uparrow\infty. \tag{6}$$

$$Y_\lambda=\lambda[\overline{\eta}(\lambda),\ \mathbf{1}]\uparrow Y,\ \lambda\uparrow\infty. \tag{7}$$

取常数c满足

$$[\alpha,\ X^0]+\sigma^0\leqslant c. \tag{8}$$

其中σ^0如(2.5)，最后令

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+Z_i(\lambda)\frac{\sum_h\alpha_h\phi_{hj}(\lambda)+\overline{\eta}_j(\lambda)}{c+[\alpha,Z-Z(\lambda)]+\lambda[\overline{\eta}(\lambda),Z]}. \tag{9}$$

过程$\psi(\lambda)$不满足向后方程组，过程满足向前方程组当且仅当

$\alpha=0$。过程$\psi(\lambda)$不中断的充要条件是

$$[\alpha,\ X^0]+\bar{\sigma}^0=c。\tag{10}$$

证 仿定理2.1，为使范条件及预解方程成立，必须而且只需(2.17)成立，其中$\eta(\lambda)$有表现(2.3)，m_i由(2.22)(用Z,$Z(\lambda)$代替其中的X，$X(\lambda)$)确定．代入范条件(2.13)得

$$\lambda[\eta(\lambda),\ 1-Z]\leqslant c \tag{11}$$

但 $\lambda[\eta(\lambda),\ 1-Z]=\lambda[\eta(\lambda),\ X^0]=[\alpha,\ X^0]+\bar{\sigma}^0$。

$$\tag{12}$$

故(11)成为(8)。

为使(9)中的$\psi(\lambda)$是Q过程，尚需验证Q条件，即

$$\lim_{\lambda\to\infty}\lambda Z_i(\lambda)\frac{\lambda\eta_j(\lambda)}{c+\lambda[\eta(\lambda),\ Z]}=0。\tag{13}$$

注意(1.11.38)，从(2)及引理1.11.5得

$$\lambda Z_i(\lambda)\to d_i(\lambda\to\infty)。\tag{14}$$

由引理1.11.3，$\lambda\eta(\lambda)\to\alpha$。又由于$Z_i(\lambda)\downarrow 0$，故

$$\begin{aligned}&\lambda[\eta(\lambda),\ Z]=[\alpha,\ Z-Z(\lambda)]+\lambda[\bar{\eta}(\lambda),\ Z]\\&\uparrow[\alpha,\ Z]+U,\qquad \lambda\uparrow\infty。\end{aligned}\tag{15}$$

于是(13)成为

$$d_i\frac{\alpha_j}{c+[\alpha,\ Z]+U}=0。\tag{16}$$

而这等价于(4)。由$Z=1-X^0$，而由(1.11.39)，

$$\sigma^0=\lambda[\eta(\lambda),\ X^0]=[\alpha,\ X^0]+\overline{\sigma^0}<\infty,\tag{17}$$

故(4)与(5)等价，证毕。

由引理1.10.4，方程

$$\left.\begin{aligned}&v(\lambda I-Q)=0,\\&0\leqslant v\in \boldsymbol{l}。\end{aligned}\right\}\tag{18}$$

的解空间$\mathscr{L}_\lambda^+$的维数n^+与λ无关。当$n^+=0$时，称矩阵Q是零流入的。当$n^+=1$时，称矩阵Q是单流入的，当n^+有限时，称矩阵Q是有限流入的。

如果Q非保守且非零流入，则在定理1中可取$\alpha=0$和非零行协调族$\bar{\eta}(\lambda)\in\mathscr{L}_\lambda^+$。因而定理1中的非最小$Q$过程$\psi(\lambda)$是存在的。

如果Q非保守且零流入。则为使定理1中的非最小Q过程$\psi(\lambda)$存在，必须而且只需：存在$\alpha\geqslant 0$使$\alpha\phi(\lambda)\in\boldsymbol{l}$，且$[\alpha, 1]=\infty$。而这条件可由下面的引理（见侯振挺的书[1，引理12．2．4]）给出。

引理2 存在$\alpha\geqslant 0$使$\alpha\phi(\lambda)\in\boldsymbol{l}$而$[\alpha, 1]=\infty$的充要条件是对某个(从而一切)$\lambda>0$，

$$\inf_i \lambda\sum_j \phi_{ij}(\lambda)=0. \tag{19}$$

证 设存在$\alpha\geqslant 0$使对某个$\lambda>0$有$\alpha\phi(\lambda)\in\boldsymbol{l}$且$[\alpha, 1]=\infty$。由$\phi(\lambda)$的预解方程及范条件知$\alpha\phi(\lambda)\in\boldsymbol{l}$对一切$\lambda>0$成立。由于

$$\begin{aligned}\lambda[\alpha\phi(\lambda), 1]&=[\alpha, \lambda\phi(\lambda)1]\\&\geqslant[\alpha, 1]\inf_i \lambda\sum_j\phi_{ij}(\lambda),\end{aligned}$$

故(19)对一切$\lambda>0$成立。

设(19)成立。因为$\lambda\phi_{ii}(\lambda)>0$，故$\lambda\sum_j\phi_{ij}(\lambda)>0$。于是从(19)推出，存在互不相同的$i_k\in E$，$k=1, 2, 3, \cdots$使

$$\lambda\sum_j\phi_{i_k j}(\lambda)<\frac{1}{2^k}.$$

取

$$\alpha_j=\begin{cases}1, & \text{如}j\in\{i_1, i_2, \cdots\},\\ 0, & \text{如}j\bar{\in}\{i_1, i_2, \cdots\}.\end{cases}$$

则

$$\lambda[\alpha\phi(\lambda), 1]=[\alpha, \lambda\phi(\lambda)1]<\sum_{k=1}^{\infty}\frac{1}{2^k}=1,$$

$$[\alpha,1]=\sum_{k=1}^{\infty}\alpha_{i_k}=\sum_{k=1}^{\infty}1=\infty.$$

引理证完。

条件(19)等价于

$$\sup_i Z_i(\lambda)=1. \tag{20}$$

如果$\overline{X}(\lambda)\neq 0$，则由引理1.10.7，

$$\sup_i \overline{X}_i(\lambda)=1. \tag{21}$$

因而(20)是满足的。

如果$\overline{X}(\lambda)=0$，则(20)成为

$$\sup_i \sum_{a\in H} X_i^a(\lambda)=1 . \tag{22}$$

如果Q单非保守，即H只含一个状态a，则(22)成为

$$\sup_i X_i^a(\lambda)=1. \tag{23}$$

或等价地

$$\sup_i \phi_{ia}(\lambda)=\frac{1}{d_a}. \tag{24}$$

定理3 设Q零流出，且只在一个状态a非保守。此时

$$Z_i(\lambda)=\phi_{ia}(\lambda)d_a,\ \ Z_i=\Gamma_{ia}d_a=1-X_i^0. \tag{25}$$

其中Γ由(1.10.8)确定。如果Q零流入，且

$$\sup_i \phi_{ia}(\lambda)<\frac{1}{d_a},$$

则Q过程唯一。

如果Q非零流入，或者Q零流入且(24)成立。则Q过程不唯一。此时每个非最小Q过程都可按定理1方式得到。

§4. 单流入时满足向前方程组的Q过程的构造

本节不要求Q保守，但假定最小解中断，而且$n^+>0$。此时可以选取非零的行协调族$\overline{\eta}(\lambda)\in\mathscr{L}_1^+$.如果$Q$过程$\psi(\lambda)$具有形式：

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+F_i(\lambda)\overline{\eta_j}(\lambda), \tag{1}$$

则$\psi(\lambda)$满足向前方程组。反之，如果$n^+=1$，则任意满足向前方程组的Q过程$\psi(\lambda)$必定具有形式(1)。我们将在$n^+>0$的条件下，

确定$F(\lambda)$使(1)中的$\psi(\lambda)$是Q过程。由于F条件恒满足，因此我们只需考察范条件和预解方程。

定理1 设最小解$\phi(\lambda)$中断，$n^+>0$。为使(1)确定的$\psi(\lambda)$是Q过程的充要条件是：或者$\psi(\lambda)=\phi(\lambda)$，或者$\psi(\lambda)$可以如下得到：取常数$\delta\geqslant 0$及列协调解族$\bar{\xi}(\lambda)\in\mathscr{M}_1^+(1)$，其标准映象为$\bar{\xi}$。如果$\delta>0$，则还要求满足$\sup_i\bar{\xi}_i(\lambda)=1$。如果$Q$非保守，则对每个$a\in H$，取数量$\beta^a\geqslant 0$使$\sum_{a\in H}\beta^a X^a(\lambda)\in\boldsymbol{m}$，而且还满足

$$\xi(\lambda)=\sum_{a\in H}\beta^a X^a(\lambda)+\delta\bar{\xi}(\lambda)\neq\mathbf{0}. \tag{2}$$

及

$$\left.\begin{aligned}&\lambda[\bar{\eta}(\lambda),\ \xi]<\infty,\\&W_\lambda=\lambda[\bar{\eta}(\lambda),\ \overline{X}-k\delta\bar{\xi}]\uparrow W<\infty,\ \lambda\uparrow\infty.\end{aligned}\right\} \tag{3}$$

其中

$$\left.\begin{aligned}&\xi=\sum_{a\in H}\beta^a X^a+\delta\bar{\xi}\neq\mathbf{0},\\&k=\inf\left\{\frac{1}{\delta},\ \frac{1}{\beta^a},\ a\in H\right\}^{1)}\end{aligned}\right\} \tag{4}$$

取常数c满足

$$\bar{\sigma}^0+W+\sum_{a\in H}(1-k\beta^a)\bar{V}^a\leqslant kc. \tag{5}$$

其中$\bar{\sigma}^0$，$\bar{V}^a$由(2.5)(2.6)确定。最后令

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+\frac{\left(\sum_{a\in H}\beta^a X_i^a(\lambda)+\delta\bar{\xi}_i(\lambda)\right)\bar{\eta}_j(\lambda)}{c+\sum_{a\in H}\beta^a\lambda[\bar{\eta}(\lambda),\ X^a]+\delta\lambda[\bar{\eta}(\lambda),\ \bar{\xi}]}. \tag{6}$$

过程$\psi(\lambda)$不中断的充要条件是

$$\bar{\xi}=\overline{X},\ \beta^a=\delta(a\in H),\ \delta^{-1}c=\bar{\sigma}^0. \tag{7}$$

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1) 约定$\frac{1}{0}=\infty$。

过程$\psi(\lambda)$满足向前方程组．过程$\psi(\lambda)$满足向后方程组的充要条件是$\beta^a=0(a\in H)$。当$n^+=1$时，上面得到的过程已穷尽了满足向前方程组的全部Q过程。

证 由(1.10.14)，范条件等价于

$$0\leqslant F(\lambda),\ F(\lambda)\lambda[\bar{\eta}(\lambda),\ 1]\leqslant\sum_{a\in H}X^a(\lambda)+\bar{X}(\lambda). \tag{8}$$

类似(2.15)，预解方程等价于

$$A(\mu,\ \lambda)F(\mu)=F(\lambda)+(\lambda-\mu)F(\lambda)[\bar{\eta}(\lambda),\ F(\mu)], \tag{9}$$

或等价地

$$F(\mu)=\{1+(\lambda-\mu)[\bar{\eta}(\lambda),\ F(\mu)]\}A(\lambda,\ \mu)F(\lambda), \tag{10}$$

故对某个$\lambda>0$有$F(\lambda)=0$则对一切$\lambda>0$有$F(\lambda)=0$，因而$\psi(\lambda)=\phi(\lambda)$。否则(10)等价于

$$F(\lambda)=m_\lambda\xi(\lambda),\ m_\lambda>0,\ \xi(\lambda)\neq 0. \tag{11}$$

其中数量$m_\lambda>0$满足

$$m_\mu=m_\lambda\{1+(\lambda-\mu)[\bar{\eta}(\lambda),\ \xi(\mu)]m_\mu\}. \tag{12}$$

而$\xi(\lambda)$是非零列协调族．依引理1.11.3，$\xi(\lambda)$有*Riesz*表现

$$\xi(\lambda)=\phi(\lambda)\beta+\tilde{\xi}(\lambda)\neq 0. \tag{13}$$

其中列矢量$\beta\geqslant 0$使$\phi(\lambda)\beta\in\boldsymbol{m}$，列协调族$\tilde{\xi}(\lambda)\in\mathscr{M}^+_\lambda$．利用引理1.10.4和1.10.7，易证$\delta=\sup_i\tilde{\xi}_i(\lambda)$与$\lambda>0$无关，如$\delta=0$，令$\bar{\xi}(\lambda)=0$；如$\delta>0$，令$\bar{\xi}(\lambda)=\delta^{-1}\tilde{\xi}(\lambda)$。则$(\bar{\xi}(\lambda),\lambda>0)$是列协调解族，设其标准映象为$\bar{\xi}$，于是

$$\sup_i\bar{\xi}(\lambda)=1,\ 如\delta>0. \tag{14}$$

比较(1.12.2)便得$\beta_j=0(j\in E-H)$。故令$\beta^a=\dfrac{\beta_a}{d_a}(a\in H)$后，(13)成为

$$\xi(\lambda)=\sum_{a\subset H}\beta^a X^a(\lambda)+\delta\xi(\lambda)\neq 0. \tag{15}$$

其标准映象为

$$\xi=\sum_{a\subset H}\beta^a X^a+\delta\xi\neq 0. \tag{16}$$

往证$\lambda[\bar{\eta}(\lambda), \xi]<\infty$。不然，在(12)两边除以$m_\lambda m_\mu$后得

$$m_\lambda^{-1}=m_\mu^{-1}+(\lambda-\mu)[\bar{\eta}(\lambda), \xi(\mu)],$$

令$\mu\downarrow 0$得

$$m_\lambda^{-1}\geqslant\lim_{\mu\downarrow 0}(\lambda-\mu)[\bar{\eta}(\lambda), \xi(\mu)]=\lambda[\bar{\eta}(\lambda), \xi]=\infty,$$

从而$m_\lambda=0$，此与(11)矛盾。

利用(1.11.40)，

$$(\mu-\lambda)[\bar{\eta}(\lambda), \xi(\mu)=\mu[\bar{\eta}(\mu), \xi]-\lambda[\bar{\eta}(\lambda), \xi]. \tag{17}$$

故在(12)中除以$m_\lambda m_\mu$后得：存在常数c使

$$m_\lambda=(c+\lambda[\bar{\eta}(\lambda), \xi])^{-1}. \tag{18}$$

将(11)(15)代入(1)后与(1.12.2)比较得 $F^a(\lambda)=\beta^a m_\lambda\bar{\eta}(\lambda)$。注意(1.12.3)，有

$$\beta^a m_\lambda\lambda[\bar{\eta}(\lambda), 1]\leqslant 1, \quad a\in H. \tag{19}$$

将(15)(11)代入范条件(8)得

$$\Big(\sum_{a\in H}\beta^a X^a(\lambda)+\delta\xi(\lambda)\Big)m_\lambda\lambda[\bar{\eta}(\lambda), 1]\leqslant\sum_{a\in H}X^a(\lambda)+\bar{X}(\lambda). \tag{20}$$

在上式中作用$\Pi^n(\lambda)$并令$n\to\infty$，注意(2.12)得

$$\delta\xi(\lambda)m_\lambda\lambda[\vec{\eta}(\lambda), 1]\leqslant\bar{X}(\lambda). \tag{21}$$

于是可见，范条件(20)等价于(19)(21)。

由(14)及

$$\xi(\lambda)\leqslant\bar{X}(\lambda) \tag{22}$$

得(21)等价于

$$\delta m_{\lambda}\lambda[\bar{\eta}(\lambda),\ 1]\leqslant 1. \tag{23}$$

这样，范条件(21)(19)成为

$$m_{\lambda}\lambda[\bar{\eta}(\lambda),\ 1]\leqslant k, \tag{24}$$

由于(16)(4)，$0<k<\infty$。将(15)(18)代入(24)得

$$\lambda[\bar{\eta}(\lambda), X^0]+\sum_{a\in H}\lambda[\bar{\eta}(\lambda),\ X^a]+\lambda[\bar{\eta}(\lambda), \bar{X}]$$

$$\leqslant k\dot{c}+k\sum_{a\in H}\beta^a\lambda[\bar{\eta}(\lambda),\ X^a]+k\delta\lambda[\bar{\eta}(\lambda),\ \bar{\xi}].$$

即 $$\bar{\sigma}^0+\bar{W}_{\lambda}+\sum_{a\in H}(1-k\beta^a)\bar{V}^a_{\lambda}\leqslant kc. \tag{25}$$

令$\lambda\uparrow\infty$得范条件等价于(5)。

如果$\psi(\lambda)$不中断，必须而且只需(8)第二式成立等号，即(21)(19)成立等号。从而$\beta^a=\delta(a\in H)$，$\xi(\lambda)=\bar{X}(\lambda)$，即$\xi=\bar{X}$。因而$k=\delta^{-1}$。(5)中等号成为$\bar{\sigma}^0=\sigma^{-1}C$。于是$\psi(\lambda)$不中断的充要条件是(7)。定理其余结论明显，证毕。

第三章　唯一性问题

§1. 引　言

给定Q满足(1.2.6)，Q过程总是存在的。本章讨论唯一性问题。§2中给出了满足向后方程组的Q过程唯一的充要条件。§3中给出了满足向前方程组的Q过程唯一的充要条件。这两节的内容取自*Reuter*[1]。§4中侯振挺—芦脱(*Reuter*)定理，即Q过程的唯一性准则的证明，取自*Reuter*[4]，但现在作了简化和改进。因此，构造论中的唯一性问题得以完满解决。

§2. 唯一性定理：向后方程组

考虑满足向后方程组的Q过程$\psi(\lambda)$的唯一性问题时，方程

$$(U_\lambda)\qquad \lambda u - Qu = 0. \tag{1}$$

将起很大的作用。回忆方程(U_λ)的解$u\in\mathbf{m}$的全体记为$\mathscr{M}_\lambda$，非负解$u\in\mathbf{m}$全体记为$\mathscr{M}_\lambda^+$，$\mathscr{M}_\lambda^+$中界为K的全体记为$\mathscr{M}_\lambda^+(K)$。$\mathscr{M}_\lambda^+$的维数记为m^+。

定理1　下列条件等价：

(i) 满足向后方程组的Q过程唯一。

(ii) 对某$\lambda>0$(从而一切$\lambda>0$)，$\mathscr{M}_\lambda$仅由零解组成。

(iii) 对某$\lambda>0$(从而一切$\lambda>0$)，$\mathscr{M}_\lambda^+$仅由零解组成。

如果上面的一个条件不成立，则有无穷多个Q过程满足向后方程组。如果Q还保守，则有无穷多个不中断的Q过程满足向后

方程组；如果Q非保守，则一切满足向后方程组的Q过程都是中断的。

证．(i)⇒(ii)．设(ii)不成立．由引理1.10.6，$\bar{X}(\lambda)\neq 0$．在(2.2.8)中取$\bar{\eta}(\lambda)=0$，$\alpha\geqslant 0$，$[\alpha,1]=1$，$c=[\alpha,X^0]$而得的过程$\psi(\lambda)$满足向后方程组．如果Q保守，这样的$\psi(\lambda)$还是不中断的．但α的选法有无穷多种，因而与(i)冲突．故(ii)成立．

(ii)⇒(iii)不待证．往证(iii)⇒(i)．设(iii)成立．因$\bar{X}(\lambda)\in\mathscr{M}_\lambda^+(1)$，故$\bar{X}(\lambda)=0$．由于$\bar{X}(\lambda)$是列协调族，故对一切$\lambda>0$有$\bar{X}(\lambda)=0$．如果$\psi(\lambda)$满足向后方程组，依定理1.12.1，(1.12.2)中的$F^a(\lambda)=0\,(a\in H)$．而当i固定时$u_i=\lambda B_{ij}(\lambda)\in\mathscr{M}_\lambda^+(1)$．由$\bar{X}(\lambda)$的最大性$\lambda B_{ij}(\lambda)\leqslant\bar{X}_i(\lambda)$，故$B_{ij}(\lambda)=0$．从而(1.12.2)成为$\psi(\lambda)=\phi(\lambda)$．即得(i)．

如果有不中断的Q过程$\psi(\lambda)$满足向后方程组，则在$\psi(\lambda)$的B条件中对j求和得Q保守，证毕．

§3．唯一性定理：向前方程组

考虑满足向前方程组的Q过程的唯一性问题时，方程组

$$(V_\lambda)\qquad \lambda v-vQ=0 \tag{1}$$

将起很大的作用．回忆方程(V_λ)的解$v\in\boldsymbol{l}$的全体记为$\mathscr{L}_\lambda$，非负解$v\in\boldsymbol{l}$的全体记为$\mathscr{L}_\lambda^+$．$\mathscr{L}_\lambda^+$的维数记为n^+．

定理1 (i)如果最小解不中断，或者最小解中断但$n^+=0$，则满足向前方程组的Q过程唯一．

(ii) 如果最小解中断且$n^+=1$，则有无穷多个Q过程满足向前方程组，其中只有一个是不中断的．

(iii) 如果最小解中断，且$n^+>1$，则有无穷多个Q过程满足向前方程组，其中有无穷多个是不中断的．

证 (i) 最小解不中断时，唯一性显然．设最小解中断且$n^+=0$．如果$\psi(\lambda)$是满足向前方程组的Q过程，由定理1.12.1，(1.12.2)中的$F^a(\lambda)\in\mathscr{L}_\lambda^+\,(a\in H)$，$u_j=B_{ij}(\lambda)\in\mathscr{L}_\lambda^+$，故$F^a(\lambda)=0$

$(a\in H)$，$B_{ij}(\lambda)=0$。(1.12.2)成为 $\psi(\lambda)=\phi(\lambda)$。即满足向前方程组的$Q$过程唯一。

(ii) 定理2.4.1已给出构造性证明。

(iii) 前一部分已由定理2.4.1回答。因为$n^+>1$，故可以选行协调族$\bar{\eta}^a(\lambda)\in\mathscr{L}_1^+(a=1,2)$，使$\bar{\eta}^1(\lambda)$，$\bar{\eta}^2(\lambda)$线性独立。任取常数$p^a\geqslant 0(a=1,2)$，使

$$\bar{\eta}(\lambda)=p^1\ \bar{\eta}^1(\lambda)+\ p^2\ \bar{\eta}^2(\lambda)\neq 0\ .$$

对于$\bar{\eta}(\lambda)$，按定理2.4.1，存在一个不中断的Q过程满足向前方程组。但可以有无穷多种方式选取$p^a(a=1,2)$而使$\bar{\eta}(\lambda)$不相同(常数因子不考虑)，因而存在无穷多个不中断的Q过程满足向前方程组，证毕。

§4. 唯一性准则：侯振挺—芦脱(Reuter)定理

Q过程的唯一性准则由侯振挺[1]给出，并已总结于他的书[1]中。他的[1，2]中还讨论了各种情况的组合的存在和唯一性问题，即所谓定性理论。*Reuter*[4]对侯振挺[1]的证明做了简化，这里采用*Reuter*的简化证明并作了进一步的改进。

定理1 设给定矩阵Q满足(1.2.6)。则Q过程唯一的充要条件是最小Q过程$\phi(\lambda)$不中断，或者最小Q过程中断并满足下面两个条件：

(i) $$\inf_{i}\lambda\sum_{j}\phi_{ij}(\lambda)=\eta_\lambda>0,\ \lambda>0\ . \tag{1}$$

(ii) $n^+=0$，即方程(V_λ)没有非零非负解$v\in l$。

我们指出：条件(i)蕴含$m^+=0$。

实际上，由(1.10.14)，(1)成为

$$\sup_i\left\{\sum_{a\in H}X_i^a(\lambda)+\bar{X}_i(\lambda)\right\}<1,\lambda>0. \tag{2}$$

因为$\bar{X}(\lambda)\neq 0$时，(1.10.21)成立。由(2)必有$\bar{X}(\lambda)=0$，从而

$m^+ = 0$.

这样，条件(i)等价于下面两个条件：

(i$_1$) $m^+ = 0$.

(ii$_2$) $\sup\limits_i \sum\limits_{a\in H} \phi_{ia}(\lambda) d_a < 1$，$\lambda > 0$.

定理的证明 必要性．设最小Q过程中断且Q过程唯一，从而F型Q过程唯一．依定理3.1，$n^+ = 0$．今谬设

$$\inf_i \lambda \sum_j \phi_{ij}(\lambda) = 0. \tag{3}$$

依引理3.2，存在行矢量$\alpha \geqslant 0$使$[\alpha, 1] = \infty$且$\alpha\phi(\lambda) \in \boldsymbol{l}$，对此$\alpha$，依定理2.3.1，

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + z_i(\lambda) \frac{\sum\limits_k \alpha_k \phi_{kj}(\lambda)}{c + [\alpha, z - z(\lambda)]} \tag{4}$$

是Q过程，其中 $z(\lambda) = 1 - \lambda\phi(\lambda)1 \neq 0$，$c \geqslant [\alpha, X^0]$．因为$c$的选取可以不唯一，故$Q$过程不唯一．这与必要性假设相冲突．于是(i) 成立．

充分性 设(i) (ii)成立，且$\psi(\lambda)$是Q过程，由(i$_1$)，故定理1.12.1中的，$B(\lambda) = 0$，从而(1.12.2)化为

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \sum_{a\in H} X_i^a(\lambda) F_j^a(\lambda). \tag{5}$$

如果H为空集，由上式知 $\psi(\lambda) = \phi(\lambda)$，因而$Q$过程唯一．下设$H \neq \phi$．将 (5) 式代入$\psi(\lambda)$的预解方程，注意$\phi(\lambda)$满足预解方程，以及由于引理1.11.6，$X^a(\lambda)$，$a \in H$线性独立，我们得

$$F^a(\lambda) A(\lambda, \mu) = F^a(\mathscr{U}) + (\mathscr{U} - \lambda) \sum_{b\in H} [F^a(\lambda), X^b(\mu)] F^b(\mu),$$

$$a \in H. \tag{6}$$

因为$F^a(\lambda) \geqslant 0$，$\lambda[F^a(\lambda), 1] \leqslant 1$．由上式可见，对任意$\lambda$，$\mu > 0$，$F^a(\lambda) A(\lambda, \mu) \in \boldsymbol{l}$．于是固定$a$及$\lambda > 0$，由(1.10.8)，$\eta(\mu) = F^a(\lambda) A(\lambda, \mu)$ $(\mu > 0)$ 是行协调族．依引理1.11.3，

$$\eta(\mu) = \alpha\phi(\mu) + \bar{\eta}(\mu),$$

其中$a\geqslant 0$与μ无关，使$a\phi(\mu)\in \boldsymbol{I}$，（$\bar{\eta}(\mu)$，$\mu>0$）为行协调解族。由(ii)，$n^{+}=0$，故$\bar{\eta}(\mu)=0$。又由于$a$与$a$及$\lambda$有关，故$a=a^{a}(\lambda)$，$\eta(\mu)=a^{a}(\lambda)\phi(\mu)$，即

$$F^{a}(\lambda)A(\lambda,\ \mu)=a^{a}(\lambda)\phi(\mu). \tag{7}$$

特别地，当$\mu=\lambda$时，

$$F^{a}(\lambda)=a^{a}(\lambda)\phi(\lambda). \tag{8}$$

由条件(i)，

$$\begin{aligned}1&\geqslant\lambda[F^{a}(\lambda),\ \mathbf{1}]=\lambda[a^{a}(\lambda)\phi(\lambda),\ \mathbf{1}]\\&=[a^{a}(\lambda),\ \lambda\phi(\lambda)\mathbf{1}]\geqslant\eta_{\lambda}[a^{a}(\lambda),\ \mathbf{1}],\end{aligned}$$

故

$$[a^{a}(\lambda),\ \mathbf{1}]\leqslant\frac{1}{\eta_{\lambda}}. \tag{9}$$

将(8) 代入(6)，注意 (1.10.8)及引理1.11.6，我们有

$$a^{a}(\lambda)=a^{a}(\mu)+(\mu-\lambda)\sum_{b\in H}[a^{a}(\lambda),\ \phi(\lambda)x^{b}(\mu)]a^{b}(\mu. \tag{10}$$

或由于 $(X^{a}(\lambda),\ \lambda>0)$ 的列协调性有

$$a^{a}(\lambda)=a^{a}(\mu)+\sum_{b\in H}[a^{a}(\lambda),X^{b}(\lambda)-x^{b}(\mu)]a^{b}(\mu). \tag{11}$$

由(10)，$a^{a}(\lambda)$随λ增加而不增。往证

$$a^{a}(\lambda)\downarrow 0,\ \lambda\uparrow\infty. \tag{12}$$

实际上，由于$\psi(\lambda)$和$\phi(\lambda)$均满足Q条件，由(5)及(8)得

$$\lim_{\lambda\to\infty}\sum_{a\in H}\lambda X_{i}^{a}(\lambda)[\lambda a^{a}(\lambda)\phi(\lambda)]_{j}=0,$$

从而

$$\lim_{\lambda\to\infty}\lambda X_{i}^{a}(\lambda)a_{j}^{a}(\lambda)\lambda\phi_{jj}(\lambda)=0,\quad a\in H.$$

由$\phi(\lambda)$的连续性条件，即

$$\delta_{ia}d_{a}\lim_{\lambda\to\infty}a_{j}^{a}(\lambda)\delta_{jj}=0.$$

取$i=a$得证(12)。

因为当$\lambda>\mu$时，由(9)，

$$\sum_{b\in H}[\alpha^a(\lambda),\ X^b(\lambda)]\alpha^b(\mu)\leqslant\sum_{b\in H}[\alpha^a(\lambda),\ X^b(\mu)]\alpha^b(\mu)$$

$$\leqslant\sum_{b\in H}[\alpha^a(\mu),\ X^b(\mu)]\alpha^b(\mu)\leqslant\sum_{b\in H}[\alpha^a(\mu),\ X^b(\mu)]\frac{1}{\eta_\mu}$$

$$\leqslant[\alpha^a(\mu),\ \sum_{b\in H}X^b(\mu)]\frac{1}{\eta_\mu}\leqslant[\alpha^a(\mu),\ 1]\frac{1}{\eta_\mu}\leqslant\frac{1}{\eta_\mu^2}<\infty.$$

因此(11) 可写成

$$\alpha^a(\lambda)+\sum_{b\in H}[\alpha^a(\lambda),\ X^b(\mu)]\alpha^b(\mu)$$

$$=\alpha^a(\mu)+\sum_{b\in H}[\alpha^a(\lambda),\ X^b(\lambda)]\alpha^b(\mu),$$

并且当$\lambda\longrightarrow\infty$时，可用控制收敛定理得

$$0+\sum_{b\in H}[0,\ x^b(\mu)]\alpha^b(\mu)=\alpha^a(\mu)+\sum_{b\in H}[0,\ 0]\alpha^b(\mu),$$

所以$\alpha^a(\mu)=0(a\in H,\ \mu>0)$。这样，$F^a(\lambda)=0(a\in H,\ \lambda>0)$，从而$\psi(\lambda)=\phi(\lambda)$，因而$Q$过程是唯一的。定理证完。

第二篇　生灭过程构造论

为了进一步研究过程的构造论，研究两种特殊的过程即双边生灭过程和单边生灭过程的构造，是极其有益的。我们考虑的双边生灭过程保守，单边生灭过程可能有一个非保守状态。对于双边生灭过程，其解空间$\mathscr{A}_1^+$的维数 $m^+\leqslant 2$，对于单边生灭过程，其$m^+\leqslant 1$，且可能有一个非保守状态，因而较简单而富有启发性。生灭过程之所以重要，是因为它们有很重大的理论意义和应用价值。对生灭过程的研究不仅可以得出许多丰富而深刻的结果，而且往往是产生解决一般过程的问题的思想和方法的源泉。

第四章　双边生灭过程

§1. 引　　言

当状态空间E为一切整数而$Q=(q_{ij})$具有下列形状

$$\left.\begin{array}{l} q_{ij}=0,\ \text{如}\ |i-j|>1, \\ q_{ii-1}=a_i>0,\ q_{ii+1}=b_i>0,\ q_i=-q_{ii}=a_i+b_i. \end{array}\right\} \qquad (1)$$

时，称Q过程为双边生灭过程。本章中，Q 过程恒指双边生灭过程。双边生灭过程是保守的，因而必定满足向后方程组。

本章中我们圆满地解决了双边生灭过程的构造问题。换言之，我们构造了全部双边生灭过程。我们视双边生灭过程为扩散，使用的方法是分析方法。本章内容取自杨向群[6]。

§2. 自然尺度和标准测度

对于(1.1) 形的Q，称

$$
\left.\begin{array}{l}
z_i=-b_0\left(1+\dfrac{b_{-1}}{a_{-1}}+\dfrac{b_{-1}b_{-2}}{a_{-1}a_{-2}}+\cdots+\dfrac{b_{-1}b_{-2}\cdots b_{i+1}}{a_{-1}a_{-2}\cdots a_{i+1}}\right),\\
\quad \text{如} i<-1.\\
z_{-1}=-b_0,\ z_0=0,\ z_1=a_0,\\
z_i=a_0\left(1+\dfrac{a_1}{b_1}+\dfrac{a_1a_2}{b_1b_2}+\cdots+\dfrac{a_1a_2\cdots a_{i-1}}{b_1b_2\cdots b_{i-1}}\right),\\
\quad \text{如} i>1.
\end{array}\right\} \tag{1}
$$

为自然尺度，称

$$
r_1=\lim_{i\to-\infty} z_i,\quad r_2=\lim_{i\to+\infty} z_i \tag{2}
$$

为边界点。称

$$
\left.\begin{array}{l}
\mu_i=\dfrac{a_{-1}a_{-2}\cdots a_{i+1}}{b_0b_{-1}b_{-2}\cdots b_{i+1}b_i},\quad \text{如} i<-1,\\
\mu_{-1}=\dfrac{1}{b_0b_{-1}},\quad \mu_0=\dfrac{1}{a_0b_0},\quad \mu_1=\dfrac{1}{a_0a_1},\\
\mu_i=\dfrac{b_1b_2\cdots b_{i-1}}{a_0a_1a_2\cdots a_{i-1}a_i},\quad \text{如} i>1.
\end{array}\right\} \tag{3}
$$

为标准测度。

§3. 边界点的分类

通过自然尺度和标准测度可以将边界点分类。称边界点r_2为

正则　如果r_2有穷，$\sum\limits_{i>0}\mu_i$有穷；

流出　如果r_2非正则，但r_2有穷，$\sum\limits_{i>0}(r_2-z_i)\mu_i$有穷；

流入　如果r_2非正则，但$\sum\limits_{i>0}z_i\mu_i$有穷；

自然 其它情形。

对于r_1可类似地进行分类。

如果令

$$\left.\begin{aligned}
R_1 &= \sum_{i\leqslant 0}(z_i-r_1)\mu_i=\sum_{i\leqslant 0}(z_i-z_{i-1})\sum_{i\leqslant j\leqslant 0}\mu_j,\\
S_1 &= -\sum_{i\leqslant 0}z_i\mu_i,\\
R_2 &= \sum_{i\geqslant 0}(r_2-z_i)\mu_i=\sum_{i\geqslant 0}(z_{i+1}-z_i)\sum_{0\leqslant j\leqslant i}\mu_j,\\
S_2 &= \sum_{i\geqslant 0}z_i\mu_i.
\end{aligned}\right\}\qquad(1)$$

则当R_2有穷时，可得r_2有穷，当S_2有穷时可得$\sum\limits_{i\geqslant 0}\mu_i$有穷。

因此上述边界点的分类互相排斥。并且r_a为流入时，则r_a无穷。

定理1 边界点r_a为

正则 当且只当 $R_a<\infty$，$S_a<\infty$；

流出 当且只当 $R_a<\infty$，$S_a=\infty$；

流入 当且只当 $R_a=\infty$，$S_a<\infty$；

自然 当且只当 $R_a=\infty$，$S_a=\infty$。

证 对$a=2$予以证明。显然

$$R_2\leqslant r_2\sum_{i\geqslant 0}\mu_i,\ S_2\leqslant r_2\sum_{i\geqslant 0}\mu_i.$$

故如r_2正则，则$R_2<\infty$，$S_2<\infty$。反之，如果$R_2<\infty$，$S_2<\infty$由R_2的定义必定r_2有穷。又$\sum\limits_{i\geqslant 0}\mu_i=\dfrac{1}{r_2}(S_2+R_2)<\infty$。

如果r_2流出，按定义$R_2<\infty$，$\sum\limits_{i\geqslant 0}\mu_i=\infty$。由$S_2\geqslant z_1\sum\limits_{i\geqslant 1}\mu_i$而得$S_2=\infty$。反之，如果$R_2<\infty$，$S_2=\infty$，由第一段所证，$r_2$非正则，而由$R_2<\infty$显然得$r_2$有穷，因此$r_2$为流出。

如果r_2流入，按定义$S_2<\infty$，又由于r_2非正则及第一段所证，必定$R_2=\infty$。反之，如果$R_2=\infty$，$S_2<\infty$。由第一段所证r_2非正则，从而r_2流入。

由上面三段的证明，立即得r_2为自然的充要条件是$R_2=\infty$，$S_2=\infty$，证毕。

§4. 二阶差分算子

设u为E上的列矢量，定义u^+和$D_\mu u^+$如下：

$$\left.\begin{aligned} u_i^+&=\frac{u_{i+1}-u_i}{z_{i+1}-z_i},\quad i\in E,\\ (D_\mu u^+)_i&=\frac{u_i^+-u_{i-1}^+}{\mu_i},\quad i\in E.\end{aligned}\right\}\tag{1}$$

定理1 对任意列矢量u，

$$Qu=D_\mu u^+.\tag{2}$$

即 $$a_iu_{i-1}-(a_i+b_i)u_i+b_iu_{i+1}=(D_\mu u^+)_i.\tag{3}$$

证 因 $$\begin{aligned} a_i&=\frac{1}{(z_i-z_{i-1})\mu_i},\\ b_i&=\frac{1}{(z_{i+1}-z_i)\mu_i},\end{aligned}\tag{4}$$

故 $$\begin{aligned}(D_\mu u^+)_i&=\frac{\dfrac{u_{i+1}-u_i}{z_{i+1}-z_i}-\dfrac{u_i-u_{i-1}}{z_i-z_{i-1}}}{\mu_i}\\ &=b_i(u_{i+1}-u_i)-a_i(u_i-u_{i-1})\\ &=a_iu_{i-1}-(a_i+b_i)u_i+b_iu_{i+1},\end{aligned}$$

证毕。

设u为列矢量，$u\mu$表示分量为$u_j\mu_j$的行矢量。反之，如果v为行矢量，$v\mu^{-1}$表示分量为$v_i\mu_i^{-1}$的列矢量。

定理2 设v为行矢量，$u=v\mu^{-1}$，则

$$vQ=(Qu)\mu.\tag{5}$$

证 注意 $$\mu_{i-1}b_{i-1}\mu_i^{-1}=a_i,\quad a_{i+1}\mu_{i+1}\mu_i^{-1}=b_i,\tag{6}$$

则 $$\begin{aligned}(Qu)_i&=a_iv_{i-1}\mu_{i-1}^{-1}-(a_i+b_i)v_i\mu_i^{-1}+b_iv_{i+1}\mu_{i+1}^{-1}\\ &=v_{i-1}b_{i-1}\mu_i^{-1}-(a_i+b_i)v_i\mu_i^{-1}+v_{i+1}a_{i+1}\mu_i^{-1}\\ &=(v_{i-1}b_{i-1}-v_i(a_i+b_i)+v_{i+1}a_{i+1})\mu_i^{-1}\\ &=(vQ)_i\mu_i^{-1}.\end{aligned}$$

系　设u，f为列矢量，$v=u\mu$，$g=f\mu$，则u满足

$$Qu=f, \tag{7}$$

或　$\lambda u-Qu=f$，$\lambda>0$。　　(8)

当且仅当v满足

$$vQ=g; \tag{9}$$

或　$\lambda v-vQ=g$。　　(10)

引理3　方程组

$$\left.\begin{aligned}&u_i=f_i,\\&a_ku_{k-1}-(a_k+b_k)u_k+b_ku_{k+1}=-f_k,\quad i<k<n,\\&u_n=f_n.\end{aligned}\right\} \tag{11}$$

的解是

$$\begin{aligned}u_k=&f_i\frac{z_n-z_k}{z_n-z_i}+f_n\frac{z_k-z_i}{z_n-z_i}+\frac{z_n-z_k}{z_n-z_i}\sum_{j=i+1}^{k-1}(z_j-z_i)f_j\mu_j\\&+\frac{z_k-z_i}{z_n-z_i}\sum_{j=k}^{n-1}(z_n-z_j)f_j\mu_j.\end{aligned} \tag{12}$$

证　由定理1，方程(11) 成为

$$\left.\begin{aligned}&u_i=f_i,\\&u_k^+-u_{k-1}^+=-f_k\mu_k,\quad i<k<n,\\&u_n=f_n.\end{aligned}\right\} \tag{13}$$

由上式得

$$u_k^+=u_i^++\sum_{l=i+1}^{k}(u_l^+-u_{l-1}^+)=u_i^+-\sum_{l=i+1}^{k}f_l\mu_l,$$

由此得

$$\begin{aligned}u_k&=u_i+\sum_{l=i}^{k-1}(u_{l+1}-u_l)=u_i+\sum_{l=i}^{k-1}u_l^+(z_{l+1}-z_l)\\&=u_i+\sum_{l=i}^{k-1}u_i^+(z_{l+1}-z_l)-\sum_{l=i}^{k-1}\Big(\sum_{j=i+1}^{l}f_j\mu_j\Big)(z_{l+1}-z_l)\\&=f_i+u_i^+(z_k-z_i)-\sum_{j=i+1}^{k-1}\sum_{l=j}^{k-1}f_j\mu_j(z_{l+1}-z_l).\end{aligned}$$

故　$u_k=f_i+u_i^+(z_k-z_i)-\sum\limits_{j=i+1}^{k-1}(z_k-z_j)f_j\mu_j$。　　(14)

特别当$k=n$时得

$$f_n=f_i+u_i^+(z_n-z_i)-\sum_{j=i+1}^{n-1}(z_n-z_j)f_j\mu_j,$$

从而 $$u_i^+=\frac{f_n-f_i}{z_n-z_i}+\frac{1}{z_n-z_i}\sum_{j=i+1}^{n-1}(z_n-z_j)f_j\mu_j. \tag{15}$$

将(15) 代入(14) 并经整理得(12)，证毕。

§5. 方程$\lambda u-D_\mu u^+=0$的解

定理1 设u，v为方程

$$\lambda u-D_\mu u^+=0,\ \lambda>0. \tag{1}$$

的两个解，则

$$W(u,\ v)\equiv u^+v-uv^+=\text{常数}. \tag{2}$$

证 首先注意，对任意矢量s，t，

$$s_it_i-s_{i-1}t_{i-1}=s_i(t_i-t_{i-1})+t_{i-1}(s_i-s_{i-1})$$
$$=s_{i-1}(t_i-t_{i-1})+t_i(s_i-s_{i-1}),$$

故 $$[D_\mu(st)]_i=s_i(D_\mu t)_i+t_{i-1}(D_\mu s)_i$$
$$=s_{i-1}(D_\mu t)_i+t_i(D_\mu s)_i. \tag{3}$$

于是 $$\begin{aligned}D_\mu W(u,\ v)&=D_\mu(u^+v)-D_\mu(uv^+)\\&=v_i(D_\mu u^+)_i+u_{i-1}^+(D_\mu v)_i\\&\quad-u_i(D_\mu v^+)_i-v_{i-1}^+(D_\mu u)_i\\&=\lambda v_iu_i+u_{i-1}^+(D_\mu v)_i-\lambda u_iv_i-v_{i-1}^+(D_\mu u)_i\\&=\frac{u_i-u_{i-1}}{z_i-z_{i-1}}\cdot\frac{v_i-v_{i-1}}{u_i}\\&\quad-\frac{v_i-v_{i-1}}{z_i-z_{i-1}}\cdot\frac{u_i-u_{i-1}}{u_i}=0.\end{aligned}$$

方程(1) 可以改写为

$$u_i^+-u_{i-1}^+=\lambda u_i\mu_i,\ \lambda>0. \tag{4}$$

故如果u是方程(1) 的解，则当$i>0$时，

$$u_i^+=u_0^++\sum_{k=1}^{i}(u_k^+-u_{k-1}^+),$$

$$u_i^+ = u_0^+ + \lambda \sum_{k=1}^{i} u_k \mu_k, \quad i>0. \tag{5}$$

$$u_i = u_0 + \sum_{k=0}^{i-1} (u_{k+1} - u_k)$$

$$= u_0 + \sum_{k=0}^{i-1} u_k^+ (z_{k+1} - z_k), \quad i>0. \tag{6}$$

将(5)代入(6)式并整理得

$$u_i = u_0 + u_0^+ (z_i - z_0) + \lambda \sum_{k=1}^{i-1} u_k (z_i - z_k) \mu_k,$$

$$i>0. \tag{7}$$

类似地对$i<0$,

$$u_i^+ = u_0^+ - \lambda \sum_{i+1 \le k \le 0} u_k \mu_k, \quad i<0. \tag{8}$$

$$u_i = u_0 - u_0^+ (z_0 - z_i) + \lambda \sum_{i+1 \le k \le 0} u_k (z_k - z_i) \mu_k,$$

$$i<0. \tag{9}$$

反之，任意给定u_0，u_0^+，根据(5)、(6) 可确定u_1，u_1^+，u_2，u_2^+，…，类似可确定u_{-1}，u_{-1}^+，u_{-2}，u_{-2}^+，…。易见，这样确定的u是方程(1) 的解。

给定$u_0=1$，$u_0^+=0$时，方程(1) 的解记为v. 给定$u_0=0$，$u_0^+=1$时，方程(1) 的解记为s. 由 (7)、(9) 看出，当 $0<i\uparrow+\infty$时，v_i和s_i为正且严格增加，当$0>i\downarrow-\infty$时，v_i和$-s_i$为正且随i的绝对值增加而严格增加。而且由定理1,

$$W(s, v) = s_0^+ v_0 - s_0 v_0^+ = 1. \tag{10}$$

引理2 当$i>0$时，$\dfrac{v_i}{s_i} > \dfrac{v_i^+}{s_i^+}$.

证 $\dfrac{v_i}{s_i} - \dfrac{v_i^+}{s_i^+} = \dfrac{W(s,v)}{s_i s_i^+} = \dfrac{1}{s_i s_i^+} > 0$.

引理 3 当$0<i\uparrow+\infty$时，$\dfrac{v_i}{s_i}$严格减少。

证 $\left(\dfrac{v}{s}\right)_i^+ = \left(\dfrac{v_{i+1}}{s_{i+1}} - \dfrac{v_i}{s_i}\right) \cdot \dfrac{1}{z_{i+1} - z_i}$

$$= -\frac{W(s,v)}{s_i s_{i+1}} < 0.$$

引理4 当$0 < i \uparrow +\infty$时，$\frac{v_i^+}{s_i^+}$严格增加。

证
$$\left[D_\mu\left(\frac{v^+}{s^+}\right)\right]_i = \left(\frac{v_i^+}{s_i^+} - \frac{v_{i-1}^+}{s_{i-1}^+}\right)\mu_i^{-1}$$
$$= \frac{s_{i-1}^+ v_i^+ - s_i^+ v_{i-1}^+}{s_i^+ s_{i-1}^+ \mu_i}$$
$$= \frac{s_{i-1}^+ (D_\mu v^+)_i - v_{i-1}^+ (D_\mu s^+)_i}{s_i^+ s_{i-1}^+}$$
$$= \frac{\lambda(s_{i-1}^+ v_i - v_{i-1}^+ s_i)}{s_i^+ s_{i-1}^+}$$
$$= \frac{\lambda[(s_i^+ - \lambda s_i)v_i - (v_i^+ - \lambda v_i)s_i]}{s_i^+ s_{i-1}^+}$$
$$= \frac{\lambda W(s,\ v)}{s_i^+ s_{i-1}^+} = \frac{\lambda}{s_i^+ s_{i-1}^+} > 0.$$

引理5 设u是方程(1)的解，且当$0 < i \uparrow +\infty$时，u_i为正且严格增加。则$0 < i \uparrow +\infty$时，u_i^+也严格增加。$u(r_2) = \lim\limits_{i \to +\infty} u_i < \infty$当且只当$r_2$正则或流出，$u^+(r_2) = \lim\limits_{i \to +\infty} u_i^+ < \infty$当且只当$r_2$正则或流入。

证 由(5)可见$u_i^+ (i > 0)$为正且严格增加。

设 $u(r_2) < \infty$。由(7)得

$$u_i > \lambda u_0 \sum_{k=1}^{i-1} (z_i - z_k)\mu_k \tag{11}$$

当$i \to +\infty$时，可见$R_2 < \infty$，即r_2正则或流出。反之，设$R_2 < \infty$，由(5)，

$$u_{i+1} - u_i < u_0^+ (z_{i+1} - z_i) + \lambda u_i (z_{i+1} - z_i) \sum_{k=1}^{i} \mu_k,$$

$$\frac{u_{i+1}}{u_i} - 1 < \frac{u_0^+}{u_0}(z_{i+1} - z_i) + \lambda (z_{i+1} - z_i) \sum_{k=1}^{i} \mu_k.$$

上式右方是收敛级数的项，故$\sum\limits_{i>0} \log \frac{u_{i+1}}{u_i}$收敛，从而

$$\lim_{i \to +\infty} u_i < \infty.$$

设$u^+(r_2)<\infty$。由(7) 有$u_i>u_0^+z_i$。故由(5)，

$$u_i^+>u_0^-+u_0^+\lambda\sum_{k=1}^{i}z_k\mu_k,$$

故$S_2<\infty$，即r_2正则或流入。反之，设$S_2<\infty$，由(6)，

$$u_i<u_0+u_{i-1}^+(z_i-z_0)=u_0+u_{i-1}^+z_i,$$

故 $u_i^+-u_{i-1}^+=\lambda u_i\mu_i<\lambda u_0\mu_i+\lambda u_{i-1}^+z_i\mu_i,$

$$\frac{u_i^+}{u_{i-1}^+}-1<\frac{\lambda u_0}{u_0^+}\mu_i+\lambda z_i\mu_i.$$

因$S_2<\infty$蕴含$\sum\limits_{i>0}\mu_i<\infty$，故上式说明$\sum\limits_{i>0}\log\dfrac{u_i^+}{u_{i-1}^+}$收敛，从而$\lim\limits_{i\to+\infty}u_i^+<\infty$，证毕。

引理6 当$o<i\uparrow+\infty$时，

$$\frac{v_i}{s_i}-\frac{v_i^+}{s_i^+}=\frac{1}{s_is_i^+}\longrightarrow\begin{cases}o, & \text{当}r_2\text{非正则时，}\\ c>o, & \text{当}r_2\text{正则时。}\end{cases}$$

证 由引理2和引理5推出

由引理2至4，可以令

$$\overline{\theta}=\lim_{i\to+\infty}\frac{v_i}{s_i},\quad \underline{\theta}=\lim_{i\to+\infty}\frac{v_i^+}{s_i^+}. \tag{12}$$

而且$\underline{\theta}\leqslant\overline{\theta}$，当且只当$r_2$非正则时，$\underline{\theta}=\overline{\theta}$。

定理 7 u是方程(1) 满足条件$u_0=1$的正的严格下降解的充要条件是u具有下列形式

$$u=v-\theta s. \tag{13}$$

其中$\underline{\theta}\leqslant\theta\leqslant\overline{\theta}$。如果$r_2$正则，则上述解$u$有无穷多个，而且介于$\underline{u}=v-\overline{\theta}s$和$\overline{u}=v-\underline{\theta}s$之间。如果$r_2$非正则，则上述解$u$是唯一的。

证 由于v，s是方程 (1) 的两个线性独立解，故每个解u是v和s的线性组合，从而满足$u_0=1$的解u必具有形式 (13)。

设u为正的严格下降解，则$u=v-\theta s>0$，$u^+=v^+-\theta s^+>0$，故$\underline{\theta}\leqslant\theta\leqslant\overline{\theta}$。反之，设$\underline{\theta}\leqslant\theta\leqslant\overline{\theta}$，则当$i>0$时，$u_i>0$，$u_i^+<0$。即$u$在$i>0$上为正且严格下降。在$i<0$上，由于$v$和 $(-s)$ 为正且当$0\geqslant i\downarrow-\infty$时，随$i$的绝对值增大而严格增加。故$u$在$E$上为正

且严格下降，证毕。

引理 8 对定理7中的u, $\underline{u}$和$\overline{u}$，

$$u(r_2)=\begin{cases}0, & r_2\text{流出或自然，正则时}\ \underline{u}(r_2)=0.\\ \dfrac{1}{s^+(r_2)}, & r_2\text{流入时，正则时}\ \overline{u}(r_2)=\dfrac{1}{s^+(r_2)}.\end{cases}$$

$$u^+(r_2)=\begin{cases}0, & r_2\text{流入或自然，正则时}\ \overline{u}^+(r_2)=0.\\ -\dfrac{1}{s(r_2)}, & r_2\text{流出，正则时}\ \underline{u}^+(r_2)=-\dfrac{1}{s(r_2)}.\end{cases}$$

证 当r_2流出或正则时，由引理5，$v(r_2)<\infty$，$s(r_2)<\infty$。故$\underline{u}(r_2)=v(r_2)-\overline{\theta}s(r_2)=0$。当$r_2$正则时，

$$\overline{u}(r_2)=v(r_2)-\underline{\theta}s(r_2)=\frac{v(r_2)s^+(r_2)-v^+(r_2)s(r_2)}{s^+(r_2)}$$

$$=\frac{1}{s^+(r_2)}.$$

当r_2流入或自然时，由

$$u_i=\overline{u}_i=v_i-\underline{\theta}s_i\leqslant v_i-\frac{v_i}{s_i^+}s_i=\frac{1}{s_i^+},$$ 如果r_2自然，则因

$s^+(r_2)=\infty$，故$u(r_2)=\overline{u}(r_2)=0$。如果$r_2$流入，则

$$u(r_2)=\overline{u}(r_2)\leqslant\frac{1}{s^+(r_2)}.$$

对任意$\varepsilon>0$，当i充分大时

$$u(r_2)+\varepsilon>v_i-\underline{\theta}s_i.$$

固定一个i，当$j(>i)$充分大时，

$$u(r_2)+\varepsilon>v_i-\frac{v_j^+}{s_j^+}s_i.$$

但当j固定时，

$$\left(v-\frac{v_j^+}{s_j^+}s\right)_i^+=v_i^+-\frac{v_j^+}{s_j^+}s_i^+=\left(\frac{v_i^+}{s_i^+}-\frac{v_j^+}{s_j^+}\right)s_i^+<0,$$

故 $$u(r_2)+\varepsilon>v_j-\frac{v_j^+}{s_j^+}s_j=\frac{1}{s_j^+}\longrightarrow\frac{1}{s^+(r_2)}.$$

由于ε任意，$u(r_2)\geqslant\dfrac{1}{s^+(r_2)}$。所以$r_2$流入时，$u(r_2)=\dfrac{1}{s^+(r_2)}$。

对$u^+(r_2)$的证明类似，从略，证毕。

定理 9 方程（1）存在正的严格下降解$u_1(\lambda)$和正的严格上升解$u_2(\lambda)$，具有下列性质：

(i) $u_1^+(\lambda)<0$严格上升，$u_2^+(\lambda)>0$严格上升。

$$u_2^+(\lambda)u_1(\lambda)-u_2(\lambda)u_1^+(\lambda)=1,\ \lambda>0. \tag{14}$$

(ii) $u_a(r_a,\ \lambda)\equiv\lim\limits_{x_i\to r_a}u_{a\,i}(\lambda)$有穷当且只当$r_a$正则或流出；$u_a^+(r_a,\ \lambda)\equiv\lim\limits_{x_i\to r_a}u_{a\,i}^+(\lambda)$有穷，或等价地$\sum\limits_i u_{a\,i}(\lambda)\mu_i<\infty$，当且只当$r_a$正则或流入。

(iii) 如果r_a非流入，则$u_b(r_a,\ \lambda)=0$ $(b\neq a)$。如果r_a流入或自然，则$u_b^+(r_a,\ \lambda)=0(b\neq a)$。

证 根据定理7，方程（1）的正的严格下降解$u_1(\lambda)$存在，类似地，正的严格上升解$u_2(\lambda)$也存在。从（5）和（8）可见，$u_1^+(\lambda)<0$和$u_2^+(\lambda)>0$都严格上升。显然$W(u_2(\lambda),\ u_1(\lambda))>0$。适当规范后，可使(14)满足。

对$a=2$来证明(ii)、(iii)。由引理5推出(ii)。而

$$\lambda\sum_i u_{2\,i}(\lambda)\mu_i=u_2^+(r_2,\ \lambda)-u_2^+(r_1,\ \lambda). \tag{15}$$

因为显然$u_2^+(r_1,\ \lambda)$有穷，故$u_2^+(r_2,\ \lambda)$有穷等价于$\sum\limits_i u_{2\,i}(\lambda)\mu_i<\infty$。由引理8，可以选取$u_1(\lambda)$和$u_2(\lambda)$还满足(iii)，证毕。

定理10 $u_1(\lambda)\mu$和$u_2(\lambda)\mu$是方程

$$(V_\lambda)\quad \lambda v-vQ=0,\ \lambda>0. \tag{16}$$

的两个线性独立解。方程(V_λ)的任何解都是它们的线性组合。

证 由定理4.2的系得出。

§6. 最小解

今后将用$u_1(\lambda)$，$u_2(\lambda)$表示定理5.9中的解。令

$$\phi_{ij}(\lambda)=\begin{cases}u_{2\,i}(\lambda)u_{1\,j}(\lambda)\mu_j, & \text{如}i\leqslant j,\\ u_{1\,i}(\lambda)u_{2\,j}(\lambda)\mu_j, & \text{如}i>j.\end{cases} \tag{1}$$

则
$$\mu_i\phi_{ij}(\lambda)=\mu_j\phi_{ji}(\lambda). \tag{2}$$

设f为列矢量，g为行矢量，则

$$[\phi(\lambda)f]_i=\sum_j\phi_{ij}(\lambda)f_j$$

$$=u_{1i}(\lambda)\sum_{j\le i}u_{2j}(\lambda)f_j\mu_j+u_{2i}(\lambda)\sum_{j>i}u_{1j}(\lambda)f_j\mu_j. \tag{3}$$

$$[g\phi(\lambda)]_j=\sum_i g_i\phi_{ij}(\lambda)$$

$$=u_{1j}(\lambda)\mu_j\sum_{i\le j}g_iu_{2i}(\lambda)+u_{2j}(\lambda)\mu_j\sum_{i>j}g_iu_{1i}(\lambda). \tag{4}$$

如果　　$g=f\mu$，则

$$g\phi(\lambda)=[\phi(\lambda)f]\mu. \tag{5}$$

定理1

$$\lambda\sum_i\phi_{ij}(\lambda)=1-\frac{u_{1i}(\lambda)}{u_1(r_1,\lambda)}-\frac{u_{2j}(\lambda)}{u_2(r_2,\lambda)}. \tag{6}$$

（如果$u_a(r_a,\ \lambda)=\infty$，(6)中对应的项为0）

证　由（3）及（5.4），

$$\lambda\sum_i\phi_{ij}(\lambda)=u_{1i}(\lambda)\sum_{j\le i}\lambda u_{2j}(\lambda)\mu_j$$

$$+u_{2i}(\lambda)\sum_{j>i}\lambda u_{1j}(\lambda)\mu_j=u_{1i}(\lambda)\sum_{j\le i}\left[u_{2j}^+(\lambda)-u_{2j-1}^+(\lambda)\right]$$

$$+u_{2i}(\lambda)\sum_{j>i}\left[u_{1j}^+(\lambda)-u_{1j-1}^+(\lambda)\right]$$

$$=u_{1i}(\lambda)\left[u_{2i}^+(\lambda)-u_2^+(r_1,\ \lambda)\right]$$

$$+u_{2i}(\lambda)\left[u_1^+(r_2,\ \lambda)-u_{1i}^+(\lambda)\right]$$

$$=u_{1i}(\lambda)u_{2i}^+(\lambda)-u_{2i}(\lambda)u_{1i}^+(\lambda)$$

$$-u_{1i}(\lambda)u_2^+(r_1,\ \lambda)$$

$$+u_{2i}(\lambda)u_1^+(r_2,\ \lambda). \tag{7}$$

如果能证明

$$u_1^+(r_2,\ \lambda)=-\frac{1}{u_2(r_2,\lambda)},u_2^+(r_1,\lambda)=\frac{1}{u_1(r_1,\lambda)}. \tag{8}$$

则由（5.14），从（7）得（6）。

只证（8）第一式。实际上，当r_2流入或自然时，由定理5.9 (ii)、(iii)，$u_2(r_2,\ \lambda)=\infty$，$u_1^+(r_2,\lambda)=0$。故（8）第一式显然成立。当$r_2$正则或流出时，由于（5.14），只需证明

$$\lim_{z_i\to r_2} u_{1i}(\lambda)u_{2i}^+(\lambda)=0. \tag{9}$$

对于r_2正则，由于$u_2^+(r_2,\ \lambda)<\infty$，$u_1(r_2,\ \lambda)=0$，上式显然成立。对于$r_2$流出，则由于$u_1(r_2,\ \lambda)=0$，并且$u_2^+(\lambda)$增加，$-u_1^+(\lambda)$减小，故

$$\begin{aligned}0&\leqslant u_{1i}(\lambda)u_{2i}^+(\lambda)=u_{2i}^+(\lambda)[u_{1i}(\lambda)-u_1(r_2,\ \lambda)]\\&=u_{2i}^+(\lambda)\sum_{j\geqslant i}\left[-u_{1j}^+(\lambda)(z_{j+1}-z_j)\right]\\&\leqslant -u_{1i}^+(\lambda)\sum_{j\geqslant i}u_{2j}^+(\lambda)(z_{j+1}-z_j)\\&=-u_{1i}^+(\lambda)[u_2(r_2,\ \lambda)-u_{2i}(\lambda)]\\&\to -u_1^+(r_2,\ \lambda)[u_2(r_2,\ \lambda)-u_2(r_2,\ \lambda)]=0,\ (z_i\to r_2),\end{aligned}$$

证毕。

定理 2 设$f\in \boldsymbol{m}$，$g\in \boldsymbol{l}$，则$\phi(\lambda)f\in \boldsymbol{m}$，$g\phi(\lambda)\in \boldsymbol{l}$，且

$$\lambda\phi(\lambda)f-Q[\phi(\lambda)f]=f,\ \lambda>0. \tag{10}$$

$$\lambda g\phi(\lambda)-[g\phi(\lambda)]Q=g,\ \lambda>0. \tag{11}$$

证 由定理1可见$\phi(\lambda)f\in \boldsymbol{m}$，$g\phi(\lambda)\in \boldsymbol{l}$。

由于（5）及定理4.2系，只需证（10）即可。实际上，由（3）得

$$[\phi(\lambda)f]_i^+=u_{1i}^+(\lambda)\sum_{j<i}u_{2j}(\lambda)f_j\mu_j+u_{2i}^+(\lambda)\sum_{j>i}u_{1j}(\lambda)f_j\mu_j. \tag{12}$$

$$D_\mu[\phi(\lambda)f]_i^+ = D_\mu u_{1i}^+(\lambda)\sum_{j\le i}u_{2j}(\lambda)f_j\mu_j +$$

$$+D_\mu u_{2i}^+(\lambda)\sum_{j>i}u_{1j}(\lambda)f_j\mu_j$$

$$+[u_{1\,i-1}^+(\lambda)u_{2i}(\lambda)-u_{2\,i-1}^+(\lambda)u_{1i}(\lambda)]f_i$$

$$=\lambda u_{1i}(\lambda)\sum_{j\le i}u_{2j}(\lambda)f_j\mu_j+\lambda u_{2i}(\lambda)\sum_{j>i}u_{1j}(\lambda)f_j\mu_j$$

$$-[u_{2i}^+(\lambda)u_{1i}(\lambda)-u_{2i}(\lambda)u_{1i}^+(\lambda)]f_i$$

$$=\lambda[\phi(\lambda)f]_i-f_i.$$

引理 3 设$f\in\mathfrak{m}$，r_a正则或流出。则

$$[\phi(\lambda)f](r_a)=\lim_{z_i\to r_a}[\phi(\lambda)f]_i=0. \tag{13}$$

证 因r_a正则或流出时，$u_a(r_a,\ \lambda)<\infty$，且当$z_i\to r_a$时

$$\frac{u_{ai}(\lambda)}{u_a(r_a,\ \lambda)}\longrightarrow 1,\quad u_{bi}(\lambda)\to 0,\ (b\ne a). \tag{14}$$

故由定理1，$[\phi(\lambda)1](r_a)=0$。由此得（13）。

定理 4 $\phi(\lambda)$是最小Q过程。$\phi(\lambda)$不中断的充要条件是r_1和r_2均为流入或自然。

证 $\phi(\lambda)\geqslant 0$不待证。由（6）可见范条件成立，而且（6）式等号成立的充要条件是r_1和r_2均为流入或自然。由（10）、（11）可见，对$\phi(\lambda)$的B条件和F条件成立。

设$f\in\mathfrak{m}$，令$F(\lambda)=\phi(\lambda)f$。由定理2可见$F(\lambda)-F(\mu)+(\lambda-\mu)\phi(\lambda)F(\mu)\in\mathfrak{m}$且是方程（5.1）的解，因此是$u_1(\lambda)$和$u_2(\lambda)$的线性组合，即

$$F(\lambda)-F(\mu)+(\lambda-\mu)\phi(\lambda)F(\mu)=c_1u_1(\lambda)+c_2u_2(\lambda). \tag{15}$$

其中c_1，c_2为常数。

如果r_2正则或流出，由引理3，在（15）中令$z_i\to r_2$得

$$0=c_1u_1(r_2,\lambda)+c_2u_2(r_2,\lambda)=c_2u_2(r_2,\lambda).$$

故$c_2=0$。如果r_2流入或自然，则由于$u_2(r_2,\lambda)=\infty$，$u_1(r_2,\ \lambda)$

$<\infty$，而（15）左方有界，因此也有$c_2=0$。同样可证$c_1=0$。于是（15）成为

$$F(\lambda)-F(\mu)+(\lambda-\mu)\phi(\lambda)F(\mu)=0.$$

令$f_i=\delta_{ij}$，上式成为对$\phi(\lambda)$的预解方程。

这样，$\phi(\lambda)$是满足向后和向前方程组的Q过程。往证$\phi(\lambda)$的最小性。

设$\psi(\lambda)$是Q过程。由于Q保守，$\psi(\lambda)$满足B条件，因此当j固定时，$\psi_{ij}(\lambda)-\phi_{ij}(\lambda)$是方程（5.1）的解，因而

$$\psi_{ij}(\lambda)-\phi_{ij}(\lambda)=c_1u_{1i}(\lambda)+c_2u_{2i}(\lambda). \tag{16}$$

其中c_1，c_2是与i无关的常数。

如果r_2正则或流出，由$\psi(\lambda)\geqslant 0$及引理3，在(16)中令$z_i\to r_2$得$c_1u_1(r_2,\lambda)+c_2u_2(r_2,\lambda)=c_2u_2(r_2,\lambda)\geqslant 0$，故$c_2\geqslant 0$。如果$r_2$自然或流入，则（16）左方有界，而$u_1(r_2,\lambda)<\infty$，$u_2(r_2,\lambda)=\infty$，故$c_2=0$。因此恒有$c_2\geqslant 0$。同样可证$c_1\geqslant 0$。从而$\psi(\lambda)\geqslant\phi(\lambda)$，证毕。

§7．若干引理

以后我们简记

$$X_i^1(\lambda)=\frac{u_{1i}(\lambda)}{u_1(r_1,\lambda)},\quad X_i^2(\lambda)=\frac{u_{2i}(\lambda)}{u_2(r_2,\lambda)}. \tag{1}$$

$$X_i^1=\frac{r_2-z_i}{r_2-r_1},\qquad X_i^2=\frac{z_i-r_1}{r_2-r_1}. \tag{2}$$

当r_a正则或流出时，$X^a(\lambda)\neq 0$；当r_a流入或自然时，$X^a(\lambda)=0$。如果r_1有穷，r_2无穷，约定$X^1=1$。如果r_1无穷，r_2无穷，约定$X^1=0$。类似可对X^2进行约定。X^a是方程(1．11、18)的解。(6.6)成为

$$\lambda\phi(\lambda)1=1-X^1(\lambda)-X^2(\lambda). \tag{3}$$

引理 1　$X^a(\lambda)\in\mathscr{M}_1^+(1)$　$(a=1,2)$是列协调族，且

$$\lambda\phi(\lambda)X^a=X^a-X^a(\lambda),\quad a=1,2. \tag{4}$$

证 $X^a(\lambda)\in\mathscr{A}_1^+(1)$明显。由(4)及$\phi(\lambda)$的预解方程知$X^a(\lambda)$是列协调族。往证(4)。

如r_1，r_2均无穷，(4)当然成立。如r_a有穷$r_b(b\neq a)$无穷，则$X^b=X^b(\lambda)=0$，$X^a=1$，而(3)成为

$$\lambda\phi(\lambda)1=1-X^a(\lambda). \tag{5}$$

故(4)成立。

如果r_1,r_2有穷，只需证(4)对$a=2$成立，$a=1$可类似证明。由(6.3)

$$\lambda\sum_i\phi_{ij}(\lambda)(r_2-z_j)=u_{1i}(\lambda)\sum_{j\leq i}\lambda u_{2j}(\lambda)\mu_j\sum_{k\geq j}(z_{k+1}-z_k)$$
$$+u_{2i}(\lambda)\sum_{j>i}\lambda u_{1j}(\lambda)\,\mu_j\sum_{k\geq j}(z_{k+1}-z_k). \tag{6}$$

$$\text{第一项}=u_{1i}(\lambda)\Big[\sum_{k<i}(z_{k+1}-z_k)\sum_{j\leq k}\lambda u_{2j}(\lambda)\mu_j$$
$$+\sum_{k\geq i}(z_{k+1}-z_k)\sum_{j\leq i}\lambda u_{2j}(\lambda)\mu_j\Big]$$
$$=u_{1i}(\lambda)\Big\{\sum_{k<i}(z_{k+1}-z_k)\Big[u_{2k}^+(\lambda)-u_2^+(r_1,\lambda)\Big]$$
$$+\sum_{k\geq i}(z_{k+1}-z_k)[u_{2i}^+(\lambda)-u_2^+(r_1,\lambda)]\Big\}$$
$$=u_{1i}(\lambda)\{u_{2i}(\lambda)-u_2(r_1,\lambda)+u_{2i}^+(\lambda)(r_2-z_i)$$
$$-\sum_k(z_{k+1}-z_k)u_2^+(r_1,\lambda)\Big\},$$

$$\text{第二项}=u_{2i}(\lambda)\sum_{k>i}(z_{k+1}-z_k)\sum_{i<j\leq k}\lambda u_{1j}(\lambda)\mu_j$$
$$=u_{2i}(\lambda)\sum_{k>i}(z_{k+1}-z_k)[u_{1k}^+(\lambda)-u_{1i}^+(\lambda)]$$
$$=u_{2i}(\lambda)\sum_{k\geq i}(z_{k+1}-z_k)[u_{1k}^+(\lambda)-u_{1i}^+(\lambda)]$$
$$=u_{2i}(\lambda)[u_1(r_1,\lambda)-u_{1i}(\lambda)-u_{1i}^+(\lambda)(r_2-z_i)]$$

$$=u_{2i}(\lambda)[-u_{1i}(\lambda)-u_{1i}^{+}(\lambda)(r_2-z_i)].$$

这样，代入（6）后并注意（5.14)得

$$\lambda\sum_j \phi_{ij}(\lambda)(r_2-z_j)=(r_2-z_i)-u_{1i}(\lambda)u_2(r_1,\lambda)$$

$$-u_{2i}(\lambda)(r_2-r_1)u_2^{+}(r_1,\lambda). \tag{7}$$

因r_1有限必非流入，故$u_2(r_1,\lambda)=0$，再注意(6.8)，在(7)两边除r_2-r_1得(4)对$a=2$成立，证毕。

引理 2 设r_1流入，r_2正则或流出。令

$$\eta_{1j}=(r_2-z_j)\mu_j, \qquad \eta_{1j}(\lambda)=-\frac{u_{1j}(\lambda)\mu_j}{u_1^{+}(r_1,\lambda)}. \tag{8}$$

则$\eta_1(\lambda)\in\mathscr{L}_1^{+}$是行协调族，而且

$$\lambda\eta_1\phi(\lambda)=\eta_1-\eta_1(\lambda). \tag{9}$$

证 由定理4.2系，$\eta_1(\lambda)\in\mathscr{L}_1^{+}$。如果能证明

$$u_2(r_1,\lambda)=-\frac{1}{u_1^{+}(r_1,\lambda)}. \tag{10}$$

则由（7），注意（6.5)而得(9)。

为证(10)，由(5.14)，只需证

$$\lim_{z_i\to r_1}[u_{1i}(\lambda)u_{2i}^{+}(\lambda)]=0. \tag{11}$$

因$u_2^{+}(r_1,\lambda)=0$，故

$$0\leqslant u_{1i}(\lambda)u_{2i}^{+}(\lambda)=u_{1i}(\lambda)[u_{2i}^{+}(\lambda)-u_2^{+}(r_1,\lambda)]$$

$$=u_{1i}(\lambda)\sum_{j<i}\lambda u_{2j}(\lambda)\mu_j$$

$$\leqslant u_{2i}(\lambda)\sum_{j<i}\lambda u_{1j}(\lambda)\mu_i$$

$$=u_{2i}(\lambda)[u_{1i}^{+}(\lambda)-u_1^{+}(r_1,\lambda)]\to 0,\ (z_i\to r_1).$$

此即（11）。由(9)及$\phi(\lambda)$的预解方程知$\eta_1(\lambda)$是列协调族，证毕。

引理 3 设r_2正则或流出。行协调族$\bar{\eta}(\lambda)\in\mathscr{L}_1^{+}(\lambda>0)$的充要条件是$\bar{\eta}(\lambda)$有下列$Riesz$表现：

$$\bar{\eta}(\lambda)=p_1\Phi_1(\lambda)+p_2X^2(\lambda)\mu. \tag{12}$$

其中常数$p_a\geqslant 0(a=1,2)$。当r_a流出或自然时$p_a=0$。而

$$\Phi_1(\lambda)=\begin{cases}\eta_1(\lambda), & \text{如}r_1\text{流入。}\\ X^1(\lambda)\mu, & \text{如}r_1\text{正则。}\end{cases}\tag{13}$$

证 由定理5.10，$\bar{\eta}(\lambda)\in\mathscr{L}_\lambda^+$有形式$\bar{\eta}(\lambda)=c_{1\lambda}u_1(\lambda)\mu+c_{2\lambda}u_2(\lambda)\mu$。因$\bar{\eta}(\lambda)\in\mathbf{I}$，故由定理5.9(ii)，当$r_a$为流出或自然时，$c_{a\lambda}=0$。因此$\bar{\eta}(\lambda)=p_{1\lambda}\Phi_1(\lambda)+p_{2\lambda}X^2(\lambda)\mu$，且$r_a$流出或自然时$p_{a\lambda}=0$。因为当$r_a$正则时，$X^a(\lambda)$是列协调族，故$X^a(\lambda)\mu$是行协调族。既然$\bar{\eta}(\lambda)$，$\Phi_1(\lambda)$，$X^2(\lambda)\mu$都是行协调族，故$p_{a\lambda}=p_a$与$\lambda$无关。因此$\bar{\eta}(\lambda)$有表现(12)。因为$\bar{\eta}(\lambda)\geqslant 0$，由定理5.9(iii)，$p_a\geqslant 0(a=1,2)$。反之，由(12)给出的$\bar{\eta}(\lambda)\in\mathscr{L}_\lambda^+$并且是行协调族，证毕。

引理 4 设r_1，r_2正则或流出。如r_a正则，则当$\lambda\uparrow\infty$时

$$U_\lambda^{ab}\equiv\lambda[X^a(\lambda)\mu,\ X^b]\uparrow U^{ab}=\begin{cases}+\infty, & \text{如}b=a,\\ \dfrac{1}{r_2-r_1}, & \text{如}b\neq a\end{cases}\tag{14}$$

证 象(1.11.46)一样可得

$$U_\lambda^{ab}-U_\nu^{ab}=(\lambda-\nu)[X^a(\nu)\mu,\ X^b(\lambda)].\tag{15}$$

得证单调性。其次，不妨设r_2正则。由(4)，(6.12)，

$$\begin{aligned}[\lambda\phi(\lambda)X^b]_i^+&=\lambda u_{1i}^+(\lambda)\sum_{j\leqslant i}u_{2j}(\lambda)X_j^b\mu_j\\ &\quad+\lambda u_{2i}^+(\lambda)\sum_{j>i}u_{1j}(\lambda)X_j^b\mu_j\\ &=\frac{(-1)^b}{r_2-r_1}-[X_i^b(\lambda)]^+.\end{aligned}\tag{16}$$

令$z_i\to r_2$得

$$-U_\lambda^{2b}=\frac{(-1)^b}{r_2-r_1}-X^{b+}(r_2,\ \lambda).\tag{17}$$

其中$X^{b+}(r_2,\lambda)=\lim\limits_{z_i\to r_2}[X_i^b(\lambda)]^+$。为证(14)，只需证

$$\lim_{\lambda\to\infty} X^{2+}(r_2,\ \lambda)=+\infty, \tag{18}$$

$$\lim_{\lambda\to\infty} X^{1+}(r_2,\ \lambda)=0. \tag{19}$$

因为$X^2(\lambda)$，$[X^2(\lambda)]^+$都是增函数，故

$$\frac{X^2(r_2,\ \lambda)-X_i^2(\lambda)}{r_2-z_i}<X^{2+}(r_2,\ \lambda).$$

由于(1.11.29)有$X^a(\lambda)\downarrow 0(\lambda\uparrow\infty,\ a=1,\ 2)$。由上式得

$$\frac{1}{r_2-z_i}\leqslant\lim_{\lambda\to\infty}X^{2+}(r_2,\lambda).$$

令$z_i\to r_2$而得(18)。又

$$0\leqslant -X^{1+}(r_2,\ \lambda)<-[X_0^1(\lambda)]^+=\frac{X_0^1(\lambda)-X_1^1(\lambda)}{z_1-z_0}.$$

再由$X^1(\lambda)\downarrow 0$而得(19)，证毕。

引理5 r_a流出或正则时，$X^a(\lambda)$的标准映象为X^a。

证 对$a=2$证明。因$X^2(\lambda)\leqslant X^2$，故$X^2(\lambda)$的标准映象$\overline{X^2}\leqslant X^2$，因而

$$\lambda\phi(\lambda)\overline{X^2}=\overline{X^2}-X^2(\lambda), \tag{20}$$

而且$u\equiv X^2-\overline{X^2}$是方程(1.11.18)的解。由(4)(20)有

$$\lambda\phi(\lambda)u=u. \tag{21}$$

设r_1无穷。因r_2-z_i和z_i是$Qu=0$的两个线性独立解，故

$$u=c_1(r_2-z_i)+c_2z_i=c_1r_2+(c_2-c_1)z_i$$

因左方有界，故必定$c_1=c_2$，从而$u=c_1r_2$，

$$\lambda\phi(\lambda)u=c_1r_2\lambda\phi(\lambda)1=c_1r_2(1-X^2(\lambda))=u-c_1r_2X^2(\lambda).$$

比较(21)得$c_1=0$。于是$u=0$，$X^2=\overline{X^2}$，

设r_1有穷。由于X^1，X^2是(1.11.18)的两个线性独立解，故$u=c_1X^1+c_2X^2$。由(4)(20)得

$$\lambda\phi(\lambda)u=u-c_1X^1(\lambda)-c^2X^2(\lambda).$$

比较(21)，$c_1X^1(\lambda)+c_2X^2(\lambda)=0$。倘若$r_1$流出或正则，则$c_1=c_2=0$，因而$u=0$；倘若$r_1$为自然（因$r_1$有穷，$r_1$不可能为流入）

则$c_2=0$，因而$0\leqslant c_1X_i^1=u_i\leqslant X_i^2$。令 $i\to-\infty$得$c_1=0$。从而$u=0$，证毕。

§8. r_1,r_2一个流入或自然，另一个流出或正则

从本节开始，我们将着手构造一切Q过程。由定理6.4，当r_1和r_2均为流入或自然时，最小解$\phi(\lambda)$不中断，因而Q过程唯一。本节我们假定一个边界点正则或流出（例如r_2），另一个边界点为流入或自然(例如r_1)。此时 $u_1(r_1,\lambda)=\infty$，$u_2(r_2,\lambda)<\infty$。因此方程(5.1)只有一个线性独立的非零非负有界解$\overline{X}(\lambda)=X^2(\lambda)$，即$\mathscr{M}_\lambda^+$的维数 $m^+=1$。此时构造问题已在定理2.2.1中解决。当然，在现在的情形下，取比较特殊的形式。

如果我们用$\bar{c}$表示定理2.2.1中的$c-[\alpha,X^0]-\bar{\sigma}^0$，则

$$
\begin{aligned}
& c+[\alpha,\overline{X}-\overline{X}(\lambda)]+\lambda[\bar{\eta}(\lambda),\overline{X}]\\
&=\bar{c}+[\alpha,X^0+\overline{X}-\overline{X}(\lambda)]+\lambda[\bar{\eta}(\lambda),X^0+\overline{X}]\\
&=\bar{c}+[\alpha,1-\overline{X}(\lambda)]+\lambda[\bar{\eta}(\lambda),1]. \qquad (1)
\end{aligned}
$$

这样做了之后，再将$\bar{c}$记为c，则定理2.2.1现在取下面的形式。

定理1 设r_1流入或自然，r_2流出或正则。$\psi(\lambda)$是Q过程的充要条件是，或者 $\psi(\lambda)=\phi(\lambda)$，或者$\psi(\lambda)$可如下得到：取行矢量$\alpha\geqslant0$使$\alpha\phi(\lambda)\in\mathscr{l}$，取常数 $p_a\geqslant0(a=1,2)$且当r_a流出或自然时$p_a=0$。按(7.12)取$\bar{\eta}(\lambda)$，满足

$$\eta(\lambda)=\alpha\phi(\lambda)+\bar{\eta}(\lambda)\neq0. \qquad (2)$$

取常数$c\geqslant0$。最后令

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+X_i^2(\lambda)\frac{\sum\limits_k\alpha_k\phi_{kj}(\lambda)+\bar{\eta}_j(\lambda)}{c+[\alpha,1-X^2(\lambda)]+\lambda[\eta(\lambda),1]}. \qquad (3)$$

过程$\psi(\lambda)$不中断的充要条件$c=0$。过程$\psi(\lambda)$满足向前方程组

的充要条件是$a=0$。

§9. r_1，r_2正则或流出：线性相关的情形

本节设r_1，r_2均正则或流出。此时非零$X^a(\lambda)\in\mathscr{M}_\lambda^+(1)(a=1,2)$且是列协调族。设$\psi(\lambda)$为$Q$过程，由于$\psi(\lambda)$和$\phi(\lambda)$都满足$B$条件，故当$j$固定时，$\psi_{ij}(\lambda)-\phi_{ij}(\lambda)$是程(5.1)的解，故

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+X_i^1(\lambda)F_j^1(\lambda)+X_i^2(\lambda)F_j^2(\lambda). \qquad (1)$$

令$z_i\to r_a$得$F^a(\lambda)\geqslant 0$。

我们将确定$F^a(\lambda)(a=1,2)$使(1)中的$\psi(\lambda)$是Q过程。由于(1)中的$\psi(\lambda)$满足B条件，故我们只需考虑范条件和预解方程

由于(7.3)，注意$X^a(r_a,\ \lambda)=1$，$X^a(r_b,\ \lambda)=0(b\neq a)$，易见$\psi(\lambda)$的范条件等价于

$$F^a(\lambda)\geqslant 0,\ \lambda[F^a(\lambda),\ 1]\leqslant 1,\ a=1,2. \qquad (2)$$

由于$\phi(\lambda)$满足预解方程，将(1)中的$\psi(\lambda)$代入预解方程后，注意$X^a(\lambda)(a=1,\ 2)$是列协调族及它们的线性独立性，我们得$\psi(\lambda)$的预解方程等价于

$$F^a(\lambda)A(\lambda,\ \nu)=F^a(\nu)+(\nu-\lambda)\sum_{b=1}^{2}[F^a(\lambda),\ X^b(\nu)]F^b(\nu),$$

$$a=1,2,\lambda,\nu>0. \qquad (3)$$

先假设对某个$\nu>0$，$F^1(\nu)$，$F^2(\nu)$线性相关，即

$$F^a(\nu)=m_{a\nu}\eta(\nu),\ m_{a\nu}\geqslant 0,\ \eta(\nu)\geqslant 0,\ a=1,\ 2. \qquad (4)$$

由(1.10.8)，将$A(\nu,\ \lambda)$右乘(3)两边得

$$F^a(\lambda)=\left\{m_{a\nu}+(\nu-\lambda)\sum_{b=1}^{2}[F^a(\lambda),\ X^b(\nu)]m_{b\nu}\right\}$$

$$\cdot\eta(\nu)A(\nu,\ \lambda).$$

从而$F^a(\lambda)(a=1,\ 2)$对任意$\lambda>0$线性相关。于是(4)对一切$\nu>0$成立。

由于(1.10.8)，在(2)和(4)之下，(3)等价于

$$F^a(\lambda)=m_{a\lambda}\eta(\lambda),\ \text{数量}m_{a\lambda}\geqslant 0,\ \text{行矢量}\eta(\lambda)\geqslant\mathbf{0};\tag{5}$$

$$\eta(\lambda)(\lambda>0)\text{是行协调族};\tag{6}$$

$$m_{a\lambda}=m_{a\nu}+(\nu-\lambda)\sum_{b=1}^{2}m_{a\nu}[\eta(\lambda),\ X^b(\nu)]m_{b\nu},$$

$$a=1,2.\tag{7}$$

由(7)，如果对某a及$\lambda>0$有 $m_{a\lambda}=0$，则有 $m_{a\lambda}=0$对一切$\lambda>0$。因此或者$m_{1\lambda}=m_{2\lambda}=0$，此时$\psi(\lambda)=\psi(\lambda)$；否则$\eta(\lambda)\neq\mathbf{0}$，$m_{1\lambda}+m_{2\lambda}>0$。由(6)及引理7.3，

$$\eta(\lambda)=\alpha\phi(\lambda)+p_1X^1(\lambda)\mu+p_2X^2(\lambda)\mu\neq\mathbf{0}.\tag{8}$$

其中常数$p_a\geqslant 0$，r_a流出时$p_a=0$，行矢量$\alpha\geqslant\mathbf{0}$使$\alpha\phi(\lambda)\in\boldsymbol{l}$。其次，由（7）得$m_{1\lambda}m_{2\lambda}=m_{1\nu}m_{2\nu}$。因而存在常数$d_a\geqslant 0$，$d_1+d_2>0$使

$$d_1m_{2\lambda}=d_2m_{1\lambda}.\tag{9}$$

不妨设$d_2>0$，故$m_{2\lambda}>0$。而(7)成为

$$d_2m_{2\lambda}=d_2m_{2\nu}+(\nu-\lambda)m_{2\lambda}\sum_{b=1}^{2}d_b[\eta(\lambda),\ X^b(\nu)]m_{2\nu}.$$

除以$m_{2\lambda}m_{2\nu}$得

$$d_2m_{2\nu}^{-1}=d_2m_{2\lambda}^{-1}+(\nu-\lambda)\sum_{b=1}^{2}d_b[\eta(\lambda),\ X^b(\nu)].\tag{10}$$

由于(1.11.46)有

$$(\nu-\lambda)[\eta(\lambda),\ X^b(\nu)]=\nu[\eta(\nu),\ X^b]-\lambda[\eta(\lambda),\ X^b].\tag{11}$$

因此 $$d_2m_{2\lambda}^{-1}-\sum_{b=1}^{2}d_b\lambda[\eta(\lambda),\ X^b]=c\ (\text{常数}).\tag{12}$$

由(9)得

$$m_{a\lambda}=d_a\left(c+\sum_{b=1}^{2}d_b\lambda[\eta(\lambda),\ X^b]\right)^{-1},\ a=1,2.\tag{13}$$

将(5)代入(2)并注意$X^1+X^2=\mathbf{1}$，范条件成为

$$\left.\begin{aligned}(d_1-d_2)\lambda[\eta(\lambda),X^2]\leqslant c,\\(d_2-d_1)\lambda[\eta(\lambda),X^1]\leqslant c.\end{aligned}\right\}\tag{14}$$

等号成立当且只当$d_1=d_2$，$c=0$。由(8)及(7.4)，

$$\lambda[\eta(\lambda),\ X^a]=\lambda[\alpha\phi(\lambda),\ X^a]+\sum_{b=1}^{2}p_b\lambda[X^b(\lambda)\mu,\ X^a]$$
$$=[\alpha,\ X^a-X^a(\lambda)]+p_1U_\lambda^{1a}+p_2U_\lambda^{2a}.$$

由于$X^a(\lambda)\downarrow 0(\lambda\uparrow\infty)$及(7.14)，

$$\lambda[\eta(\lambda),\ X^a]\uparrow W_a\equiv[\alpha,\ X^a]+p_1U^{1a}+p_2U^{2a},\quad \lambda\uparrow\infty. \tag{15}$$

由(7.14)，为使W_a有穷当且只当

$$[\alpha,\ X^a]<\infty,\quad p_a=0. \tag{16}$$

并且此时有

$$W_a=[\alpha,\ X^a]+\frac{p_b}{r_2-r_1},\quad (b\neq a). \tag{17}$$

这样，(14)等价于

$$\left.\begin{aligned}&c\geqslant 0,\ \text{如}d_1=d_2.\\&c\geqslant(d_1-d_2)W_2,\ \text{如}d_1>d_2.\\&c\geqslant(d_2-d_1)W_1,\ \text{如}d_1<d_2.\end{aligned}\right\} \tag{18}$$

定理1 设r_1，r_2流出或正则，又设$F^a(\lambda)$ $(a=1,\ 2)$对某个(从而一切)$\lambda>0$线性相关。

为使(1)中的$\psi(\lambda)$是Q过程的充要条件是，或者$\psi(\lambda)=\phi(\lambda)$，或者$\psi(\lambda)$可如下得到：取常数$d_a\geqslant 0$，$d_1+d_2>0$，$P_a\geqslant 0$（$r_a$流出时$P_a=0$）。取行矢量$\alpha\geqslant\mathbf{0}$使$\alpha\phi(\lambda)\in\boldsymbol{l}$，而且(8)成立。如果$d_a<d_b$ $(b\neq a)$，还要求(16)成立。取常数c满足(18)(其中W_a如(17))。最后令

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+$$
$$\frac{\{d_1X_i^1(\lambda)+d_2X_i^2(\lambda)\}\left\{\sum\alpha_h\phi_{hj}(\lambda)+p_1X_j^1(\lambda)\mu_j+p_2X_j^2(\lambda)\mu_j\right\}}{c+\sum\limits_{b=1}^{2}d_b\{[\alpha,\ X^b-X^b(\lambda)]+\lambda[p_1X^1(\lambda)\mu+p_2X^2(\lambda)\mu,\ X^b]\}}. \tag{19}$$

过程$\psi(\lambda)$不中断的充要条件是$d_1=d_2$，$c=0$。过程满足向前方程组的充要条件是$\alpha=\mathbf{0}$。

§10. r_1、r_2正则或流出：线性独立的情形

上面假定$F^a(\lambda)(a=1,2)$线性相关。本节假定$F^a(\lambda)(a=1,2)$线性独立。

采用矩阵符号是方便的。记

$$[y]=\binom{y^1}{y^2},\quad [y]'=(y^1,\ y^2).$$

其中y^1，y^2是数量或矢量。如果y^a，$v^a(a=1,\ 2)$是矢量，记$\{[y,v]\}$为以$[y^a,\ v^b]$为元素的二阶方阵，I也表示二阶幺矩阵。

这样，(9.1)、(9.2)、(9.3)成为

$$\psi(\lambda)=\phi(\lambda)+[X(\lambda)]'[F(\lambda)],\tag{1}$$

$$[F(\lambda)]\geqslant[\mathbf{0}],\quad \{\lambda[F(\lambda),\ X]\}[1]\leqslant[1],\tag{2}$$

$$[F(\lambda)A(\lambda,\ \nu)]$$
$$=[F(\nu)]+(\nu-\lambda)\{[F(\lambda),\ X(\nu)]\}[F(\nu)].\tag{3}$$

引理1 设$\psi(\lambda)$为Q过程，形如(1)。则存在行矢量$\alpha^a\geqslant\mathbf{0}(a=1,\ 2)$使$\alpha^a\phi(\lambda)\in\boldsymbol{l}$，存在二阶方阵$\mathscr{R}_\lambda=(\gamma_\lambda^{ab})\geqslant\mathbf{0}$，$\mathscr{M}_\lambda=(M_\lambda^{ab})\geqslant\mathbf{0}$($r_a$流出时，$M_\lambda^{1a}=M_\lambda^{2a}=0$)使

$$[F(\lambda)]=\mathscr{R}_\lambda[\alpha\phi(\lambda)]+\mathscr{M}_\lambda[X(\lambda)\mu].\tag{4}$$

证 既然$\psi(\lambda)$为Q过程，故(2)(3)成立。

由(2)、(3)可见，对任意λ，$\nu>0$，$[F(\lambda)A(\lambda,\nu)]\geqslant[\mathbf{0}]$。令暂时固定$a$及$\lambda>0$，令$\eta(\nu)=F^a(\lambda)A(\lambda,\ \nu)$。则$\eta(\nu)$是行协调族，因而依引理1.11.3及引理7.3，存在与ν无关(但与a及λ有关)的矢量$\beta_\lambda^a\geqslant\mathbf{0}$使$\beta_\lambda^a\phi(\lambda)\in\boldsymbol{l}$，并且

$$\nu F^a(\lambda)A(\lambda,\ \nu)-F^a(\lambda)A(\lambda,\ \nu)Q=\beta_\lambda^a,\tag{5}$$

$$F^a(\lambda)A(\lambda,\ \nu)=\beta_\lambda^a\phi(\nu)+M_\lambda^{a1}X^1(\lambda)\mu+M_\lambda^{a2}X^2(\lambda)\mu.\tag{6}$$

其中$\mathscr{M}_\lambda=(M_\lambda^{ab})\geqslant0$与$\nu$无关，并且$r_a$流出时，$M_\lambda^{1a}=M_\lambda^{2a}=0$。特别地，当$\nu=\lambda$时有

$$\lambda F^a(\lambda)-F^a(\lambda)Q=\beta_\lambda^a,\tag{7}$$

$$[F(\lambda)]=[\beta_\lambda^a\phi(\lambda)]+\mathscr{M}_\lambda[X(\lambda)\mu]. \tag{8}$$

因此，如能证明：存在与λ无关的行矢量$\alpha^a\geqslant 0(a=1,\ 2)$，使

$$\beta_\lambda^a=r_\lambda^{a1}\alpha^1+r_\lambda^{a2}\alpha^2, \tag{9}$$

其中 $\mathscr{R}_\lambda=(r_\lambda^{ab})\geqslant 0$。则将(9)代入(8)便得(4)，并由 $\beta_\lambda^a\phi(\lambda)\in\mathbf{I}$ 可推得$\alpha^a\phi(\lambda)\in\mathbf{I}$(如果对某$b$有$r_\lambda^{1b}\equiv r_\lambda^{2b}\equiv 0$,则可取$\alpha^b=0$而不影响$\beta_\lambda^a$的值)，引理便证明了。

由于(5)、(7)，将$(\nu I-Q)$左乘(3)两边后得

$$[\beta_\lambda]=[\beta_\nu]+(\nu-\lambda)\{[F(\lambda),\ X(\nu)]\}[\beta_\nu]. \tag{10}$$

由此可见$\beta_\lambda^a\downarrow(\lambda\uparrow)$，并且如果对某$\lambda>0$及某$a$有$\beta_\lambda^a=0$，则对$\nu>\lambda$有$\beta_\nu^1=\beta_\nu^2=0$，从而对一切$\lambda>0,\beta_\lambda^1=\beta_\lambda^2=0$。此时(9)是平凡的，只需取$\alpha^1=\alpha^2=0,\mathscr{R}_\lambda=0$即可。设对任意$\lambda>0$有$\beta_\lambda^a\neq 0(a=1,2)$。设

$$\mathscr{T}_{\lambda\nu}\equiv I+(\nu-\lambda)\{[F(\lambda),\ X(\nu)]\},$$

它的元素为$t_{\lambda\nu}^{ab}$，则(10)可以写为

$$\beta_\lambda^a=t_{\lambda\nu}^{a1}\beta_\nu^1+t_{\lambda\nu}^{a2}\beta_\nu^2,\quad a=1,2. \tag{11}$$

并且当$\nu>\lambda$时

$$0\leqslant t_{\lambda\nu}^{ab}\beta_\nu^a\leqslant\beta_\lambda^b,\quad a,b=1,2. \tag{12}$$

倘若对任意$\lambda>0$，当$\nu\to\infty$时，

$$t_{\lambda\nu}^{a1}\beta_\nu^1\to 0,\quad a=1,\ 2. \tag{13}$$

则由(11)，对任意$\lambda>0$，当$\nu\to\infty$时

$$t_{\lambda\nu}^{a2}\beta_\nu^2\to\beta_\lambda^a\neq 0,\quad a=1,2. \tag{14}$$

任取固定的$\lambda_0>0$，令 $\alpha^2=\beta_{\lambda_0}^1$。由(11)，

$$\beta_\lambda^a=t_{\lambda\nu}^{a1}\beta_\nu^1+\frac{t_{\lambda\nu}^{a2}}{t_{\lambda_0\nu}^{12}}t_{\lambda_0\nu}^{12}\beta_\nu^2.$$

上式左方与ν无关，右方第一项由(13)当$\nu\to\infty$时趋于0。由于(14)，右方第二项分式必收敛于某有穷非负数r_λ^{a2}，从而取极限时我们得

$$\beta_\lambda^a=r_\lambda^{a2}\alpha^2.$$

此时当然有形状(9)。

倘若(13)不成立，则存在$\lambda_1>0$，某a_1(不失一般性不妨设a_1

$=1$)及子列$\nu_n\to\infty$使

$$t_{\lambda,\nu_n}^{1\,1}\beta_{\nu_n}^1\to\alpha^1\neq 0,\quad 0\leqslant\alpha^1\leqslant\beta_\lambda^1. \tag{15}$$

又如果对任意$\lambda>0$，当$\nu_n\to\infty$时有

$$t_{\lambda\,\nu_n}^{a\,2}\beta_{\nu_n}^2\to 0,\quad a=1,\ 2,$$

则象上一段证明一样，同样地可以证明

$$\beta_\lambda^a=r_\lambda^{a\,1}\alpha^1,\quad 0\leqslant\alpha^1\neq 0,$$

因而有形状(9)。否则，存在$\lambda_2>0$及某a_2(不失一般性设$a_2=2$)及ν_n的子列$\nu_n(1)\to\infty$使

$$t_{\lambda_2\nu_n(1)}^{2\,2}\beta_{\nu_n(1)}^2\to\alpha^2\neq 0,\quad 0\leqslant\alpha^2\leqslant\beta_{\lambda_2}^2. \tag{16}$$

现在对任意$\lambda>0$，(11)可以写为

$$\beta_\lambda^a=\frac{t_{\lambda\nu}^{a\,1}}{t_{\lambda_1\nu}^{1\,1}}t_{\lambda_1\nu}^{1\,1}\beta_\nu^1+\frac{t_{\lambda\nu}^{a\,2}}{t_{\lambda_2\nu}^{2\,2}}t_{\lambda_2\nu}^{2\,2}\beta_\nu^2.$$

我们还可以选$\nu_n(1)$的子列$\nu_n(2)\to\infty$(子列可能随λ而不同)，使当$\nu=\nu_n(2)\to\infty$时，上式中二分数收敛于非负的$r_\lambda^{a\,1}$，$r_\lambda^{a\,2}$。于是由(15)(16)，在上式中令$\nu=\nu_n(2)\to\infty$时得

$$\beta_\lambda^a=r_\lambda^{a\,1}\alpha^1+r_\lambda^{a\,2}\alpha^2.$$

由于α^1，α^2不为$\mathbf{0}$，故$\mathscr{R}_\lambda=(r_\lambda^{a\,b})$有穷。此时也有形状(9)，证毕。

由于引理1，以后我们只需考虑形如(1)、(4)的Q过程$\psi(\lambda)$即可。

引进记号：

$$h_\lambda^{ab}=\lambda[\alpha^a\phi(\lambda),\ X^b]=[\alpha^a,\ X^b-X^b(\lambda)],$$

$$\uparrow h^{ab}=[\alpha^a,\ X^b],\quad \lambda\uparrow\infty. \tag{17}$$

$$\mathscr{H}_\lambda=(h_\lambda^{ab})\uparrow\mathscr{H}=(h^{ab}),\quad \lambda\uparrow\infty. \tag{18}$$

上列关系式是由于$X^a(\lambda)\downarrow 0(\lambda\uparrow\infty)$及(7.4)。由于(1.11.40)，

$$(\nu-\lambda)[\alpha^a\phi(\lambda),\ X^b(\nu)]=h_\nu^{ab}-h_\lambda^{ab}. \tag{19}$$

先考虑一特殊情形：$\mathscr{R}_\lambda=0$，$\mathscr{H}<\infty$。

即 $$[F(\lambda)]=\mathscr{R}_\lambda[\alpha\varphi(\lambda)], \tag{20}$$

$$[\alpha^a,\ 1]<\infty,\quad a=1,2. \tag{21}$$

定理2 设$F^1(\lambda)$，$F^2(\lambda)$线性独立，并且具有形状(20)、(21)。则由(1)确定的$\psi(\lambda)$是Q过程的充要条件是它可如下得到：

取非负行矢量α^1，α^2线性独立且满足

$$[\alpha^a,\ \mathbf{1}]\leqslant 1,\ a=1,2. \tag{22}$$

然后令

$$\psi(\lambda)=\phi(\lambda)+[X(\lambda)]'(I-\mathscr{T}_\lambda)^{-1}[\alpha\phi(\lambda)], \tag{23}$$

其中$\mathscr{T}_\lambda=\{[\alpha,\ X(\lambda)]\}$。

过程$\psi(\lambda)$不满足向前方程组。矢量α^1和α^2（它们满足(22)）由过程唯一决定。过程$\psi(\lambda)$不中断的充要条件是

$$[\alpha^a,\ \mathbf{1}]=1,\ a=1,2. \tag{24}$$

证 （一）设Q过程$\psi(\lambda)$有形状(20)(21)。将(20)代入(2)得

$$\mathscr{R}_\lambda\mathscr{H}_\lambda[1]\leqslant[1]. \tag{25}$$

又将(20)代入(3)，注意(1.10.9)及(19)得

$$\mathscr{R}_\lambda[\alpha\phi(\nu)]=\mathscr{R}_\nu[\alpha\phi(\nu)]+\mathscr{R}_\lambda(\mathscr{H}_\nu-\mathscr{H}_\lambda)\mathscr{R}_\nu[\alpha\phi(\nu)]. \tag{26}$$

将$(\nu I-Q)$右乘上式后得

$$\mathscr{R}_\lambda[\alpha]=\mathscr{R}_\nu[\alpha]+\mathscr{R}_\lambda(\mathscr{H}_\nu-\mathscr{H}_\lambda)\mathscr{R}_\nu[\alpha].$$

既然$F^a(\lambda)(a=1,2)$线性独立，由(20)，α^1和α^2也线性独立，故由上式得

$$\mathscr{R}_\lambda=[I-\mathscr{R}_\lambda(\mathscr{H}_\lambda-\mathscr{H}_\nu)]\mathscr{R}_\nu. \tag{27}$$

选取子列$\lambda\to\infty$使

$$\mathscr{R}_\lambda\to\mathscr{R}\geqslant\mathbf{0}. \tag{28}$$

由(25)(27)和(22)得

$$\mathscr{R}\mathscr{H}[1]\leqslant[1], \tag{29}$$

$$\mathscr{R}=[I-\mathscr{R}(\mathscr{H}-\mathscr{H}_\nu)]\mathscr{R}_\nu. \tag{30}$$

令 $[\overline{\alpha}]=\mathscr{R}[\alpha]\geqslant[\mathbf{0}]$，则(29)、(30)成为

$$[\overline{\alpha}^a,\ \mathbf{1}]\leqslant 1,\ a=1,2. \tag{31}$$

$$\mathscr{R}=(I-\overline{\mathscr{T}}_\lambda)\mathscr{R}_\lambda,\ \overline{\mathscr{T}}_\lambda=\{[\overline{\alpha},\ X(\lambda)]\}. \tag{32}$$

但由(31)，$[\overline{\alpha}^a,\ X^1(\lambda)+X^2(\lambda)]<1(a=1,2)$，因此逆矩阵$(I-\overline{\mathscr{T}}_\lambda)^{-1}$存在且非负。由(32)得

$$\mathscr{R}_\lambda=(I-\overline{\mathscr{T}}_\lambda)^{-1}\mathscr{R}.$$

代入(20)得

$$[F(\lambda)]=\overline{\mathscr{R}}_\lambda[\overline{\alpha}\phi(\lambda)],\quad \overline{\mathscr{R}}_\lambda=(I-\overline{\mathscr{T}}_\lambda)^{-1}. \tag{33}$$

剩下需证$\overline{\alpha}^1$和$\overline{\alpha}^2$线性独立。这从上式可见，因为如果$\overline{\alpha}^1$和$\overline{\alpha}^2$线性相关，则$F^1(\lambda)$，$F^2(\lambda)$也线性相关。

（二）设α^1，α^2非负，线性独立并满足（22)。在(一)中已证明，逆矩阵$\mathscr{R}_\lambda=(I-\mathscr{T}_\lambda)^{-1}$存在且非负。

由$[F(\lambda)]=(I-\mathscr{T}_\lambda)^{-1}[\alpha\phi(\lambda)]$定义的$F^a(\lambda)(a=1,2)$线性独立。实际上，设

$$0=[c]'[F(\lambda)]=[c]'(I-\mathscr{T}_\lambda)^{-1}[\alpha\phi(\lambda)],$$

则右乘$(\lambda I-Q)$后得

$$0=[c]'(I-\mathscr{T}_\lambda)^{-1}[\alpha].$$

由$\alpha^a(a=1,2)$的独立性，$[c]'(I-\mathscr{T}_\lambda)^{-1}=[0]'$，$[c]'=[0]'$。

由(22)，$\mathscr{H}[1]\leqslant[1]$，即$(I-\mathscr{H})[1]\geqslant[0]$。因$(I-\mathscr{T}_\lambda)[1]\geqslant\mathscr{H}_\lambda[1]$，既然$(I-\mathscr{T}_\lambda)^{-1}$非负，故$[1]\geqslant(I-\mathscr{T}_\lambda)^{-1}\mathscr{H}_\lambda[1]$。因此(25)满足，从而(2)成立。直接验证便知(27)满足，因此(26)成立，从而(3)成立。

（三）因为α^1,α^2线性独立，而$u_j=X_i^1(\lambda)F_j^1(\lambda)+X_i^2(\lambda)F_j^2(\lambda)$满足

$$(\lambda u-uQ)_j=[X_i(\lambda)]'[\alpha_j]\neq 0,$$

故Q过程$\psi(\lambda)$不满足向前方程组。

为使(2)即(25)成立等式，充要条件是$(I-\mathscr{T}_\lambda)^{-1}\mathscr{H}_\lambda[1]=[1]$，即

$$(\mathscr{H}-\mathscr{T}_\lambda)[1]=\mathscr{H}_\lambda[1]=(I-\mathscr{T}_\lambda)[1],$$

即$\mathscr{H}[1]=[1]$，得(24)。

设α^a及$\overline{\alpha}^a(a=1,2)$均满足(22)，并对应同一过程，即

$$(I-\mathscr{T}_\lambda)^{-1}[\alpha\phi(\lambda)]=(I-\overline{\mathscr{T}}_\lambda)^{-1}[\overline{\alpha}\phi(\lambda)],$$

右乘$\lambda I-Q$后得

$$(I-\mathscr{T}_\lambda)^{-1}[\alpha]=(I-\overline{\mathscr{T}}_\lambda)^{-1}[\overline{\alpha}].$$

由于$X^a(\lambda)\downarrow 0(\lambda\uparrow\infty)$及(22)，在上式中令$\lambda\uparrow\infty$时得$[a]=[\bar{a}]$，证毕。

现在考虑一般情形。

定理3 设$F^a(\lambda)(a=1,2)$线性独立。为使形如(1)和(4)的$\psi(\lambda)$是Q过程的充要条件是它可以如下得到：取非负行矢量$\bar{\alpha}^a(a=1,2)$使$\bar{\alpha}^a\phi(\lambda)\in\mathbf{I}$，取非负矩阵$\mathscr{S}=\begin{pmatrix}0, & \bar{s}^{12}\\ \bar{s}^{21}, & 0\end{pmatrix}$，$\mathscr{M}=\begin{pmatrix}\overline{M}^{11} & 0\\ 0 & \overline{M}^{22}\end{pmatrix}$具有下列性质：

(i) 如r_a流出，则$\overline{M}^{aa}=0$；

(ii) 或者$\overline{M}^{aa}>0(a=1,2)$；或者$\overline{M}^{aa}=0$，$\overline{M}^{bb}>0(b\neq a)$，$\bar{\alpha}^a\neq 0$；或者$\overline{M}^{aa}=0(a=1,2)$，$\bar{\alpha}^1$，$\bar{\alpha}^2$线性独立。

(iii) $\bar{h}^{ab}<\infty(a\neq b)$。

(iv) $\bar{s}^{12}\leqslant 1$，$\bar{s}^{21}\leqslant 1$，

$$\bar{s}^{ab}\geqslant\bar{h}^{ab}+\frac{\overline{M}^{aa}}{r_2-r_1},\quad(a\neq b)。$$

令

$$\left.\begin{aligned}\mathscr{R}_\lambda&=(I-\overline{\mathscr{S}}+\overline{\mathscr{H}}_\lambda+\overline{\mathscr{M}}\mathscr{U}_\lambda)^{-1},\\ \mathscr{M}_\lambda&=(I-\overline{\mathscr{S}}+\overline{\mathscr{H}}_\lambda+\overline{\mathscr{M}}\mathscr{U}_\lambda)^{-1}\overline{\mathscr{M}},\end{aligned}\right\}\tag{33}$$

其中$\overline{\mathscr{H}}_\lambda\uparrow\overline{\mathscr{H}}(\lambda\uparrow\infty)$对$\bar{\alpha}^a(a=1,2)$按(17)、(18)确定，而由(7.14)

$$0<\mathscr{U}_\lambda=(U_\lambda^{ab})\uparrow\mathscr{U}=\begin{pmatrix}+\infty & \dfrac{1}{r_2-r_1}\\ \dfrac{1}{r_2-r_1} & +\infty\end{pmatrix},\lambda\uparrow\infty。\tag{34}$$

$\psi(\lambda)$按(1)、(4)(33)确定。

过程$\psi(\lambda)$不中断的充要条件是$\bar{s}^{12}=\bar{s}^{21}=1$。过程满足向前方程的充要条件是$[\bar{\alpha}]=[0]$。过程$\psi(\lambda)$具有形状(1)、(20)、(21)

的充要条件是$\overline{\overline{M}}^{11}=\overline{\overline{M}}^{22}=0$，$\bar{h}^{aa}<\infty(a=1,2)$。

证 分几步证明。

（一）设Q过程$\psi(\lambda)$形如(1)、(4)，而且使$F^a(\lambda)(a=1,2)$线性独立。

将(4)代入(2)得

$$\mathscr{S}_\lambda[1]\leqslant[1],\quad \mathscr{S}_\lambda\equiv\mathscr{R}_\lambda\mathscr{H}_\lambda+\mathscr{M}_\lambda\mathscr{U}_\lambda. \tag{35}$$

将(4)代入(3)，由于(1.10.9)、$X^a(\lambda)$的列协调性、(7.15)和(19)，

$$\begin{aligned}\mathscr{R}_\lambda[\alpha\phi(\nu)]+\mathscr{M}_\lambda[X(\nu)\mu]=&\mathscr{R}_\nu[\alpha\phi(\nu)]+\mathscr{M}_\nu[X(\nu)\mu]\\&+(\mathscr{R}_\lambda\mathscr{H}_\nu+\mathscr{M}_\lambda\mathscr{U}_\nu-\mathscr{S}_\lambda)\{\mathscr{R}_\nu[\alpha\phi(\nu)]\\&+\mathscr{M}_\nu[X(\nu)\mu]\}.\end{aligned} \tag{36}$$

将$(\nu I-Q)$右乘上式后得

$$\mathscr{R}_\lambda[\alpha]=\mathscr{R}_\nu[\alpha]+(\mathscr{R}_\lambda\mathscr{H}_\nu+\mathscr{M}_\lambda\mathscr{U}_\nu-\mathscr{S}_\lambda)\mathscr{R}_\nu[\alpha],$$

$$\mathscr{R}_\lambda[\alpha]=(I-\mathscr{S}_\lambda+\mathscr{R}_\lambda\mathscr{H}_\nu+\mathscr{M}_\lambda\mathscr{U}_\nu)\mathscr{R}_\nu[\alpha]. \tag{37}$$

代入(36)得

$$\mathscr{M}_\lambda=(I-\mathscr{S}_\lambda+\mathscr{R}_\lambda\mathscr{H}_\nu+\mathscr{M}_\lambda\mathscr{U}_\nu)\mathscr{M}_\nu. \tag{38}$$

令 $\delta_\lambda^a=1-s_\lambda^{aa}$，则$\delta_\lambda^a>0$。实际上，由(35)，$\delta_\lambda^a\geqslant0$。倘若$\delta_\lambda^a=0$，则$s_\lambda^{aa}=1$。由(35)，$s_\lambda^{ab}=\sum\limits_{t=1}^{2}(r_\lambda^{at}h_\lambda^{tb}+M_\lambda^{at}U_\lambda^{tb})=0$ $(b\neq a)$。故 $r_\lambda^{at}\alpha^t=M_\lambda^{at}=0$ $(t=1,2)$。从而

$$s_\lambda^{aa}=\sum_{t=1}^{2}(r_\lambda^{at}h_\lambda^{ta}+M_\lambda^{at}U_\lambda^{ta})=0,\text{矛盾。}$$

在(35)、(37)、(38)中除第a行以δ_λ^a得

$$\overline{\mathscr{S}}_\lambda[1]\leqslant[1],\quad \overline{\mathscr{S}}_\lambda\equiv\begin{pmatrix}0&\bar{s}_\lambda^{12}\\\bar{s}_\lambda^{21}&0\end{pmatrix}. \tag{39}$$

$$\overline{\mathscr{R}}_\lambda[\alpha]=(I-\overline{\mathscr{S}}_\lambda+\overline{\mathscr{R}}_\lambda\mathscr{H}_\nu+\overline{\mathscr{M}}_\lambda\mathscr{U}_\nu)\mathscr{R}_\nu[\alpha]. \tag{40}$$

$$\overline{\mathscr{M}}_\lambda=(I-\overline{\mathscr{S}}_\lambda+\overline{\mathscr{R}}_\lambda\mathscr{H}_\nu+\overline{\mathscr{M}}_\lambda\mathscr{U}_\nu)\mathscr{M}_\nu. \tag{41}$$

其中 $\bar{s}_\lambda^{aa}=0$ $(a=1,2)$， (42)

$$\bar{s}_\lambda^{ab} = \sum_{t=1}^{2} \bar{r}_\lambda^{at} h_\lambda^{tb} + \sum_{t=1}^{2} \overline{M}_\lambda^{at} U_\lambda^{tb}, \quad (a \neq b). \tag{43}$$

$$\bar{r}_\lambda^{ab} = \frac{r_\lambda^{ab}}{\delta_\lambda^a}, \quad \overline{M}_\lambda^{ab} = \frac{M_\lambda^{ab}}{\delta_\lambda^a}. \tag{44}$$

取子列$\lambda \to \infty$使

$$\overline{\mathscr{S}}_\lambda \to \overline{\mathscr{S}}, \quad \overline{\mathscr{R}}_\lambda \to \overline{\mathscr{R}}, \quad \overline{\mathscr{M}}_\lambda \to \overline{\mathscr{M}}. \tag{45}$$

则$\overline{\mathscr{S}} = (\bar{s}^{ab})$，$\overline{\mathscr{R}} = (\bar{r}^{ab})$，$\overline{\mathscr{M}} = (\overline{M}^{ab})$非负，并由(39)—(44)，

$$\bar{s}^{aa} = 0 \quad (a = 1,2), \quad \bar{s}^{12} \leqslant 1, \quad \bar{s}^{21} \leqslant 1. \tag{46}$$

$$\bar{s}^{ab} \geqslant \sum_{t=1}^{2} \bar{r}^{at} h^{tb} + \sum_{t=1}^{2} \overline{M}^{at} U^{tb}, \quad (a \neq b). \tag{47}$$

$$\overline{\mathscr{R}}[\alpha] = (I - \overline{\mathscr{S}} + \overline{\mathscr{R}}\mathscr{H}_v + \overline{\mathscr{M}}\mathscr{U}_v)\mathscr{R}_v[\alpha]. \tag{48}$$

$$\overline{\mathscr{M}} = (I - \overline{\mathscr{S}} + \overline{\mathscr{R}}\mathscr{H}_v + \overline{\mathscr{M}}\mathscr{U}_v)\mathscr{M}_v. \tag{49}$$

(约定$0 \cdot \infty = 0$)

由(47)得

$$\overline{M}^{12} = \overline{M}^{21} = 0, \tag{50}$$

$$\text{如果} h^{tb} = \infty, \text{则} \bar{r}^{at} = 0 \quad (a \neq b). \tag{51}$$

由引理1，本定理的(i)成立。今令$[\bar{\alpha}] = \overline{\mathscr{R}}[\alpha]$，则$\bar{\alpha}^a \geqslant 0$且$\bar{\alpha}^a \phi(\lambda) \in \mathbf{I}(a = 1,2)$。(50)(46)(47)成为(iv)。从而有(iii)。(48)(49)成为

$$[\bar{\alpha}] = (I - \overline{\mathscr{S}} + \overline{\mathscr{H}}_v + \overline{\mathscr{M}}\mathscr{U}_v)\mathscr{R}_v[\alpha]. \tag{52}$$

$$\overline{\mathscr{M}} = (I - \overline{\mathscr{S}} + \overline{\mathscr{H}}_v + \overline{\mathscr{M}}\mathscr{U}_v)\mathscr{M}_v. \tag{53}$$

$$\text{往证} \bar{\alpha}^a \neq \mathbf{0}, \text{如果} \overline{M}^{aa} = 0. \tag{54}$$

因为不然的话，不妨设$\overline{M}^{11} = 0$，$\bar{\alpha}^1 = \mathbf{0}$。则由(52)、(53)得

$$r_\lambda^{11}\alpha^1 + r_\lambda^{12}\alpha^2 = \bar{s}^{12}(r_\lambda^{21}\alpha^1 + r_\lambda^{22}\alpha^2),$$

$$M_\lambda^{1a} = \bar{s}^{12} M_\lambda^{2a} (a = 1, 2).$$

于是$F^1(\lambda) = \bar{s}^{12} F^2(\lambda)$。这与$F^1(\lambda), F^2(\lambda)$线性独立矛盾。

往证逆矩阵

$$\mathscr{Z}_v^{-1} \equiv (I - \overline{\mathscr{S}} + \overline{\mathscr{H}}_v + \overline{\mathscr{M}}\mathscr{U}_v)^{-1} \tag{55}$$

存在且非负。从而由(52)(53)，$\mathscr{R}_v[\alpha] = \mathscr{Z}_v^{-1}[\bar{\alpha}]$，$\mathscr{M}_v = \mathscr{Z}_v^{-1}\overline{\mathscr{M}}$，

因而

$$[F(\lambda)]=\mathscr{Z}_{\lambda}^{-1}\overline{\mathscr{R}}[\alpha\phi(\lambda)]+\mathscr{Z}_{\lambda}^{-1}\overline{\mathscr{M}}[X(\lambda)\mu],$$

$$[F(\lambda)]=\mathscr{Z}_{\lambda}^{-1}[\bar{\alpha}\phi(\lambda)]+\mathscr{Z}_{\lambda}^{-1}\overline{\mathscr{M}}[X(\lambda)\mu]. \tag{56}$$

即$\psi(\lambda)$按(1)、(4)、(3)确定。

由(iv)及(54)、(34)得

$$1>\bar{s}^{ab}-(\bar{h}^{ab}-\overline{M}^{aa}U_{\nu}^{ab})\geqslant\bar{s}^{ab}-(\bar{h}^{ab}-\overline{M}^{aa}U^{ab})\geqslant 0,\quad (a$$

$\neq b$)。因此$I-\overline{\mathscr{S}}+\overline{\mathscr{H}}_{\nu}+\overline{\mathscr{M}}\mathscr{U}_{\nu}$具有形状

$$\begin{pmatrix}1+l^{11} & -l^{12}\\ -l^{21} & 1+l^{22}\end{pmatrix},\quad \begin{matrix}1+l^{11}>l_{12}\geqslant 0,\\ 1+l^{22}>l_{21}\geqslant 0.\end{matrix}$$

故其行列式$\Delta>0$，且逆矩阵$\mathscr{Z}_{\nu}^{-1}$为

$$\frac{1}{\Delta}\begin{pmatrix}1+l^{22} & l^{21}\\ l^{12} & 1+l^{11}\end{pmatrix}\geqslant 0.$$

（二） 往证$F^{a}(\lambda)(a=1,2)$线性独立的充要条件是（ii）。

显然$F^{a}(\lambda)(a=1,2)$线性独立等价于$[\bar{\alpha}\phi(\lambda)]+\overline{\mathscr{M}}[X(\lambda)\mu]$线性独立。如果

$$[c]'([\bar{\alpha}\phi(\lambda)]+\overline{\mathscr{M}}[X(\lambda)\mu])=0. \tag{57}$$

则右乘$(\lambda I-Q)$后得

$$[c]'[\bar{\alpha}]=\mathbf{0},\ [c]'\overline{\mathscr{M}}=[0]'. \tag{58}$$

反之，(58)成立，(57)当然成立。为了由(58)推出$[c]=[0]$，必须而且只需(ii)成立。

（三） 设$[\bar{\alpha}]$，$\overline{\mathscr{S}}$，$\overline{\mathscr{M}}$满足定理中的条件。

因为(ii)蕴含(54)，故在（一）中已指出逆矩阵(55)存在且非负，因而可按(1)(4)(其中的a^{a}换成$\bar{a}^{a}$，$a=1,2.$)(33)确定$\psi(\lambda)$。往证(2)、(3)成立，即(35)、(36)中将$[\alpha]$换成$[\bar{\alpha}]$后成立。因为

$$(\mathscr{R}_{\lambda}\overline{\mathscr{H}}_{\lambda}+\mathscr{M}_{\lambda}\mathscr{U}_{\lambda})[1]=\mathscr{Z}_{\lambda}^{-1}(\overline{\mathscr{H}}_{\lambda}+\overline{\mathscr{M}}\mathscr{U}_{\lambda})[1]$$

$$=\mathscr{Z}_{\lambda}^{-1}\{\mathscr{Z}_{\lambda}-(I-\overline{\mathscr{S}})\}[1]=[1]-\mathscr{Z}_{\lambda}^{-1}(I-\overline{\mathscr{S}})[1].$$

由于(iv)及$\mathscr{Z}_{\lambda}^{-1}\geqslant 0$，故将$[\bar{\alpha}]$代替$[\alpha]$后的(35)式成立。

其次

$$I-\mathscr{R}_{\lambda}\overline{\mathscr{H}}_{\lambda}+\mathscr{M}_{\lambda}\mathscr{U}_{\lambda}+\mathscr{R}_{\lambda}\overline{\mathscr{H}}_{\nu}+\mathscr{M}_{\lambda}\mathscr{U}_{\nu}$$

$$=\mathscr{Z}_{\lambda}^{-1}(\mathscr{Z}_{\lambda}-\overline{\mathscr{H}}_{\lambda}-\overline{\mathscr{M}}\mathscr{U}_{\lambda}+\overline{\mathscr{H}}_{\nu}+\overline{\mathscr{M}}\mathscr{U}_{\nu})$$

$$=\mathscr{Z}_{\lambda}^{-1}(I-\overline{\mathscr{S}}+\overline{\mathscr{H}}_{\nu}+\overline{\mathscr{M}}\mathscr{U}_{\nu})=\mathscr{Z}_{\lambda}^{-1}\mathscr{Z}_{\nu}. \tag{59}$$

因此，用$\bar{\alpha}^a$，$\overline{\mathscr{H}}_\nu$代替$\alpha^a$，$\mathscr{H}_\nu$后的(37)(38)成立，从而将$[\bar{\alpha}]$代替$[\alpha]$后的(36)成立。

（四）往证（56）具有形状（20）、（21）的充要条件是$\overline{M}^{11}=\overline{M}^{22}=0$, $\bar{h}^{aa}<\infty(a=1,2)$.

设(56)具有形状(20)、(21)。由定理2，即

$$\mathscr{Z}_\lambda^{-1}([\bar{\alpha}\phi(\lambda)]+\overline{\mathscr{M}}[X(\lambda)\mu])$$
$$=(I-\mathscr{T}_\lambda)^{-1}[\alpha\phi(\lambda)]. \tag{60}$$

右乘$\lambda I-Q$ 得

$$\mathscr{Z}_\lambda^{-1}[\bar{\alpha}]=(I-\mathscr{T}_\lambda)^{-1}[\alpha]. \tag{61}$$

故$\mathscr{Z}_\lambda^{-1}\overline{\mathscr{M}}=0$，$\overline{\mathscr{M}}=0$，从而$\overline{M}^{11}=\overline{M}^{22}=0$。由(61)得$[\bar{\alpha}]=\mathscr{Z}_\lambda\cdot(I-\mathscr{T}_\lambda)^{-1}[\alpha]$.因$[\alpha^a,1]\leqslant 1$，故$\bar{h}^{aa}<\infty$ $(a=1,2)$。反之，如果$\overline{M}^{11}=\overline{M}^{22}=0$及$\bar{h}^{aa}<\infty(a=1,2)$，则(56)中的$[F(\lambda)]$当然有形状(20)、(21)。

（五）关于过程不中断或满足向前方程组的充要条件明显，证毕。

§11. 关于$\alpha\phi$ (λ) $\in \mathscr{l}$ 的条件

设$\alpha\geqslant 0$。由于(7.3)，$\alpha\phi(\lambda)\in \mathscr{l}$等价于

$$\sum_i \alpha_i[1-X_i^1(\lambda)-X_i^2(\lambda)]<\infty,\ \lambda>0. \tag{1}$$

本节将给出直接由Q判断$\alpha\phi(\lambda)\in \mathscr{l}$的充要条件。显然，只需对(1)中的$\sum\limits_{i\geqslant 0}$及$\sum\limits_{i<0}$考虑即可。我们只考虑$\sum\limits_{i\geqslant 0}$，对于$\sum\limits_{i<0}$是完全类似的。

引理1 对定理5.9中的$u_1(\lambda)$和$u_2(\lambda)$，有

$$u_{1i}(\lambda)=u_{2i}(\lambda)\sum_{j\geqslant i}\frac{z_{j+1}-z_j}{u_{2j}(\lambda)u_{2j+1}(\lambda)}. \tag{2}$$

证。设(2)右方确定的量为$v_i(\lambda)$。首先证$v(\lambda)$是方程(5.1)的下降解，且对用$v(\lambda)$代替$u_1(\lambda)$后的(5.14)成立。

由于$u_2(\lambda)$和$u_2^+(\lambda)$增加，

$$0<v_i(\lambda)=u_{2i}(\lambda)\sum_{j\geqslant i}\frac{z_{j+1}-z_j}{u_{2j}(\lambda)u_{2j+1}(\lambda)}$$

$$=u_{2i}(\lambda)\sum_{j\geqslant i}\frac{1}{u_{2j}^+(\lambda)}\left[\frac{1}{u_{2j}(\lambda)}-\frac{1}{u_{2j+1}(\lambda)}\right]$$

$$<\frac{u_{2i}(\lambda)}{u_{2i}^+(\lambda)}\sum_{j\geqslant i}\left[\frac{1}{u_{2j}(\lambda)}-\frac{1}{u_{2j+1}(\lambda)}\right]$$

$$=\frac{u_{2i}(\lambda)}{u_{2i}^+(\lambda)}\left[\frac{1}{u_{2i}(\lambda)}-\frac{1}{u_2(r_2,\lambda)}\right]$$

$$\leqslant\frac{1}{u_{2i}^+(\lambda)}<\infty. \tag{3}$$

因此级数(2)收敛．其次，

$$v_i^+(\lambda)=u_{2i}^+(\lambda)\sum_{j\geqslant i}\frac{z_{j+1}-z_j}{u_{2j}(\lambda)u_{2j+1}(\lambda)}-\frac{1}{u_{2i}(\lambda)}. \tag{4}$$

由(3) 可见

$$v_i^+(\lambda)<\frac{1}{u_{2i}(\lambda)}-\frac{1}{u_{2i}(\lambda)}=0.$$

因而$v(\lambda)$下降。由(2),(4)得用$v(\lambda)$代替$u_1(\lambda)$后的(5.14) 成立。由(4) 得

$$D_\mu v_i^+(\lambda)=D_\mu u_{2i}^+(\lambda)\sum_{j\geqslant i}\frac{z_{j+1}-z_j}{u_{2j}(\lambda)u_{2j+1}(\lambda)}$$

$$=\lambda u_{2i}(\lambda)\sum_{j\geqslant i}\frac{z_{j+1}-z_j}{u_{2j}(\lambda)u_{2j+1}(\lambda)}$$

$$=\lambda v_i(\lambda),$$

即$v(\lambda)$是方程(5.1)的正的严格下降解。

当r_2非正则时，依定理5.7，满足(5.14)的方程(5.1)的下降解$u_1(\lambda)$唯一。故$v(\lambda)=u_1(\lambda)$。

当r_2正则时，依定理5.7，

$$v_i(\lambda)=K[v_i-\theta s_i]，K为常数。$$

由(2)，$0=v(r_2,\lambda)=K[v(r_2)-\theta s(r_2)]$，故$\theta=\dfrac{v(r_2)}{s(r_2)}=\underline{\theta}$。于是

$\nu(\lambda)=Ku_1(\lambda)$。从而$\nu^+(\lambda)=Ku_1^+(\lambda)$。由(4)及(6.8)第一式得

$$-\frac{1}{u_2(r_2,\lambda)}=\nu^+(r_2,\lambda)=Ku_1^+(r_2,\lambda)=-\frac{K}{u_2(r_2,\lambda)}.$$

故$K=1$，从而$\nu(\lambda)=u_1(\lambda)$。证完。

引理2 设r_2正则或流出，则

$$\lim_{z_n\to r_2}\frac{u_{1n}(\lambda)}{r_2-z_n}=\frac{1}{u_2(r_2,\lambda)}. \tag{5}$$

证。由(2)，

$$u_{1n}(\lambda)<\frac{u_{2n}(\lambda)}{[u_{2n}(\lambda)]^2}\sum_{j\geq n}(z_{j+1}-z_j)=\frac{r_2-z_n}{u_{2n}(\lambda)}, \tag{6}$$

$$u_{1n}(\lambda)>\frac{u_{2n}(\lambda)}{[u_2(r_2,\lambda)]^2}\sum_{j\geq n}(z_{j+1}-z_j)$$

$$=\frac{u_{2n}(\lambda)(r_2-z_n)}{[u_2(r_2,\lambda)]^2}. \tag{7}$$

即 $$\frac{u_{2n}(\lambda)}{[u_2(r_2,\lambda)]^2}<\frac{u_{1n}(\lambda)}{r_2-z_n}<\frac{1}{u_{2n}(\lambda)}.$$

由此得(5)，证毕。

定理3 设r_2正则，则

$$\sum_{i\geq 0}\alpha_i[1-X_i^1(\lambda)-X_i^2(\lambda)]<\infty \tag{8}$$

当且只当

$$\sum_{i\geq 0}\alpha_i(r_2-z_i)<\infty. \tag{9}$$

或等价地

$$\sum_{i\geq 0}\alpha_iN_i<\infty. \tag{10}$$

其中

$$N_i=\sum_{j\geq i}(z_{j+1}-z_j)\sum_{k=0}^{j}\mu_k=\left(\sum_{k=0}^{i}\mu_k\right)(r_2-z_i)$$
$$+\sum_{k\geq i+1}(r_2-z_k)\mu_k. \tag{11}$$

证 因

$$1-X_i^1(\lambda)-X_i^2(\lambda)=\lambda u_{1i}(\lambda)\sum_{j\leqslant i}u_{2j}(\lambda)\mu_j$$

$$+\lambda u_{2i}(\lambda)\sum_{j>i}u_{1j}(\lambda)\mu_j \tag{12}$$

而r_2正则时$\sum\limits_{j>0}\mu_j<\infty$，由(5)，当$i\to+\infty$时，

$$0<\frac{u_{2i}(\lambda)}{r_2-z_i}\sum_{j>i}u_{1j}(\lambda)\mu_j<\frac{u_{2i}(\lambda)u_{1i}(\lambda)}{r_2-z_i}\sum_{j>i}\mu_j\to 1\cdot 0$$

$$=0.$$

因此

$$\lim_{i\to+\infty}\frac{1-X_i^1(\lambda)-X_i^2(\lambda)}{r_2-z_i}=\lim_{i\to+\infty}\frac{\lambda u_{1i}(\lambda)}{r_2-z_i}\sum_{j\leqslant i}u_{2j}(\lambda)\mu_j$$

$$=\lambda[X^2(\lambda)\mu,1].$$

因r_2正则时，$0<\lambda[X^2(\lambda)\mu,1]<\infty$。由此便知(8) 与(9) 等价。其次，(10) 显然蕴含(9)。由 (11)

$$N_i<(r_2-z_i)\sum_{j>0}\mu_j.$$

故r_2正则时(9) 也蕴含(10)，证毕。

定理4 设r_2流出，则 (8) 与(10) 等价，并且推出(9)成立。

证 由(6)、(7)及(11)，当$j\geqslant i\geqslant 0$时

$$u_{1j}(\lambda)<\frac{r_2-z_j}{u_{2j}(\lambda)}\leqslant\frac{r_2-z_j}{u_{2i}(\lambda)}\leqslant\frac{r_2-z_i}{u_{20}(\lambda)}, \tag{13}$$

$$u_{1j}(\lambda)>\frac{u_{2j}(\lambda)}{[u_2(r_2,\lambda)]^2}(r_2-z_j)$$

$$\geqslant\frac{u_{2i}(\lambda)}{[u_2(r_2,\lambda)]^2}(r_2-z_j), \tag{14}$$

$$\frac{u_{1j}(\lambda)}{N_j}<\frac{r_2-z_j}{u_{2j}(\lambda)N_j}\leqslant\frac{\mu_0(r_2-z_j)}{\mu_0 u_{20}(\lambda)N_j}$$

$$<\frac{1}{u_{20}(\lambda)\mu_0}. \tag{15}$$

又当$i\geqslant0$时，

$$u_{1i}(\lambda)\sum_{j=0}^{i}u_{2j}(\lambda)\mu_j$$

$$\leqslant\frac{u_{1i}(\lambda)u_{2i}(\lambda)}{r_2-z_i}\Big(\sum_{j=0}^{i}\mu_j\Big)(r_2-z_i)。\tag{16}$$

$$u_{1i}(\lambda)\sum_{j=0}^{i}u_{2j}(\lambda)\mu_j$$

$$\geqslant\frac{u_{1i}(\lambda)u_{20}(\lambda)}{r_2-z_i}\Big(\sum_{j=0}^{i}\mu_j\Big)(r_2-z_i)。\tag{17}$$

于是由(12)、(15)、(16)、(13)，

$$\frac{1-X_i^1(\lambda)-X_i^2(\lambda)}{N_i}\leqslant\frac{\lambda}{u_{20}(\lambda)\mu_0}\sum_{j<0}u_{2j}(\lambda)\mu_j$$

$$+\lambda\frac{u_{1i}(\lambda)u_{2i}(\lambda)}{r_2-z_i}\cdot\frac{\Big(\sum_{j=0}^{i}\mu_j\Big)(r_2-z_i)}{N_i}$$

$$+\lambda\frac{u_{2i}(\lambda)}{N_i}\sum_{j>i}\frac{(r_2-z_j)\mu_j}{u_{20}(\lambda)}\leqslant\frac{\lambda}{u_{20}(\lambda)\mu_0}\sum_{j<0}u_{2j}(\lambda)\mu_j$$

$$+\lambda\frac{u_{1i}(\lambda)u_{2i}(\lambda)}{r_2-z_i}+\lambda\frac{u_{2i}(\lambda)}{u_{20}(\lambda)}。$$

由(5)，

$$\varlimsup_{i\to+\infty}\frac{1-X_i^1(\lambda)-X_i^2(\lambda)}{N_i}\leqslant\frac{\lambda}{u_{20}(\lambda)\mu_0}\sum_{j<0}u_{2j}(\lambda)\mu_j$$

$$+\lambda+\lambda\frac{u_2(r_2,\lambda)}{u_{20}(\lambda)}<\infty。\tag{18}$$

同样地，由(12)(17)(14)，

$$\frac{1-X_i^1(\lambda)-X_i^2(\lambda)}{N_i}>\lambda\frac{u_{1i}(\lambda)}{N_i}\sum_{j=0}^{i}u_{2j}(\lambda)\mu_j$$

$$+\lambda\frac{u_{2i}(\lambda)}{N_i}\sum_{j>i}u_{1j}(\lambda)\mu_j$$

$$>\lambda\frac{u_{i1}(\lambda)u_{20}(\lambda)}{r_2-z_i}\cdot\frac{\Big(\sum_{j=0}^{i}\mu_j\Big)(r_2-z_i)}{N_i}$$

$$+\frac{\lambda u_{2i}(\lambda)}{N_i}\sum_{j>i}\frac{u_{2j}(\lambda)}{[u_2(r_2,\lambda)]^2}(r_2-z_j)\mu_j$$

$$>\frac{\lambda u_{2i}(\lambda)u_{20}(\lambda)}{[u_2(r_2,\lambda)]^2}\cdot\frac{\left(\sum\limits_{j=0}^{i}\mu_j\right)(r_2-z_i)}{N_i}$$

$$+\frac{\lambda u_{2i}(\lambda)u_{20}(\lambda)}{[u_2(r_2,\lambda)]^2}\cdot\frac{\sum\limits_{j>i}(r_2-z_j)\mu_j}{N_j}=\lambda X_i^2(\lambda)X_0^2(\lambda),$$

$$\lim_{i\to+\infty}\frac{1-X_i^1(\lambda)-X_i^2(\lambda)}{N_i}\geqslant\lambda X_0^2(\lambda)>0. \tag{19}$$

由(18)(19)便得证定理，证毕。

定理5 设r_2流入或自然，则(8)成立当且只当

$$\sum_{i\geqslant0}a_i<\infty. \tag{20}$$

证 充分性是明显的。必要性从

$$\lim_{i\to+\infty}(1-X_i^1(\lambda))=1-X^1(r_2,\lambda)>0$$

得出。

第五章　生灭过程

§1. 引　言

设$E=\{0,1,2,3,\cdots\}$，Q具有形状

$$Q=\begin{pmatrix} -(a_0+b_0) & b_0 & 0 & 0 & \cdots \\ a_1 & -(a_1+b_1) & b_1 & 0 & \cdots \\ 0 & a_2 & -(a_2+b_2) & b_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \tag{1}$$

其中$a_0\geqslant 0,b_0>0,a_i>0,b_i>0(i>0)$。我们称形如(1) 的 Q过程为生灭过程。本章中Q恒有形状(1)，Q过程恒指生灭过程。

我们指出，对(1)中的Q，关系式

$$q_i=\sum_{j\neq i}q_{ij} \tag{2}$$

当$a_0=0$时满足，当$a_0>0$时，上式对$i=0$不成立．即$a_0=0$时，Q保守，当$a_0>0$时，Q单非保守。当$a_0\geqslant 0$时，解空间 $\mathscr{M}_\lambda$ 的维数$m^+\leqslant 1$。因此当$a_0=0$时或 $a_0>0$，$m^+=0$时，生灭过程的构造已在定理2.2.1和定理2.3.3中解决．当 $a_0>0$，$m^+=1$ 时情况稍复杂，因为此时可能存在满足向后或向前方程组的Q 过程，也可能存在既不满足向前也不满足向后方程组的Q过程。

Feller[5]中视生灭过程为扩散，并在$a_0\geqslant 0$的情形下，找出了同时满足向后和向前方程组的全部生灭过程．作者[2] 中构造了全部生灭过程。

本章叙述生灭过程构造论，与前一章双边生灭过程构造论有许多相似之处，因此有的地方我们将从简叙述或从简证明。

§2．边界点的分类和二阶差分算子

对于(1.1)的Q，称

$$\begin{cases} z_0=\dfrac{1}{a_0}，\text{如果}a_0>0；\ z_0=0，\text{如果}a_0=0。\\ z_1=z_0+\dfrac{1}{b_0}, \\ \cdots \\ z_n=z_0+\dfrac{1}{b_0}+\cdots+\dfrac{a_1a_2\cdots a_{n-1}}{b_0b_1b_2\cdots b_{n-1}},(n=2,3,\cdots)。\end{cases} \tag{1}$$

为自然尺度；称

$$z=\lim_{n\to+\infty}z_n \tag{2}$$

为边界点，称

$$\mu_0=1，\mu_n=\frac{b_0b_1\cdots b_{n-1}}{a_1\cdots a_{n-1}a_n}，(n>1)。 \tag{3}$$

为标准测度。

通过自然尺度和标准测度可以将边界点z分类。说边界点z为

正则　如果$z<\infty$，$\sum\limits_i \mu_i<\infty$。

流出　如果z非正则，$\sum\limits_i (z-z_i)\mu_i<\infty$。

流入　如果z非正则，$\sum\limits_i z_i\mu_i<\infty$。

自然　其他情形。

还引进下列特征数：

$$\left.\begin{aligned} m_0&=\frac{1}{b_0}=(z_1-z_0)\mu_0, \\ m_i&=\frac{1}{b_i}+\sum_{k=0}^{i-1}\frac{a_ia_{i-1}\cdots a_{i-k}}{b_ib_{i-1}\cdots b_{i-k}b_{i-k-1}} \\ &=(z_{i+1}-z_i)\times\sum_{k=0}^{i}\mu_k,\ i>0。\end{aligned}\right\} \tag{4}$$

$$\left.\begin{aligned}&e_0=z_0\sum_{k=0}^{\infty}\mu_k.\\&e_i=\frac{1}{a_i}+\sum_{k=0}^{\infty}\frac{b_ib_{i+1}\cdots b_{i+k}}{a_ia_{i+1}\cdots a_{i+k}a_{i+k+1}}=(z_i-z_{i-1})\sum_{k=i}^{\infty}\mu_k,i>0.\end{aligned}\right\}\tag{5}$$

$$N_i=\sum_{j=i}^{\infty}m_j=(z-z_i)\sum_{j=0}^{i}\mu_j+\sum_{j=i+1}^{\infty}(z-z_j)\mu_j.\tag{6}$$

$$\left.\begin{aligned}&R=\sum_{j=0}^{\infty}m_j=\sum_{j=0}^{\infty}(z-z_j)\mu_j,\\&S=\sum_{j=0}^{\infty}e_j=\sum_{j=0}^{\infty}z_j\mu_j.\end{aligned}\right\}\tag{7}$$

定理1 边界点z为

正则 当且仅当 $R<\infty,S<\infty$;

流出 当且仅当 $R<\infty,S=\infty$;

流入 当且仅当 $R=\infty,S<\infty$;

自然 当且仅当 $R=\infty,S=\infty$.

证 同定理4.3.1

可以按(4.4.1)引进二阶差分算子，但在 $i=0$ 上要作适当修改。

设u为E上的列矢量，定义u^+如下:

$$u_i^+=\frac{u_{i+1}-u_i}{z_{i+1}-z_i},\quad(i\geqslant0)\tag{8}$$

为了方便，以后约定

$$u_{-1}^+=a_0u_0,u_{-1}=0.\tag{9}$$

设u为$\{-1\}\cup E$上的列矢量，定义$D_\mu u$如下:

$$D_\mu u_i=\frac{u_i-u_{i-1}}{\mu_i},\quad(i\geqslant0)\tag{10}$$

定理2 设u为E上的列矢量，则

$$Qu=D_\mu u^+\tag{11}$$

即 $a_iu_{i-1}-(a_i+b_i)u_i+b_iu_{i+1}=D_\mu u_i^+,\ i\geqslant0.$ (12)

证 同定理4.4.1，但需注意约定(9)。

定理3 定理4.4.2及其系，引理4.4.3仍然正确，只要将 E 理解为非负整数集，Q 为(1.1)的矩阵，甚至连记号都不用改变。

引理4 方程

$$\left.\begin{aligned}&-(a_0+b_0)u_0+b_0u_1=-f_0,\\&a_iu_{i-1}-(a_i+b_i)u_i+b_iu_{i+1}=-f_i,0<i<n,\\&u_n=f_n.\end{aligned}\right\}\tag{13}$$

的解是

$$\begin{aligned}u_i=&\frac{z_n-z_i}{a_0(z_n-z_0)+1}f_0+\frac{a_0(z_i-z_0)+1}{a_0(z_n-z_0)+1}f_n\\&+\frac{z_n-z_i}{a_0(z_n-z_0)+1}\sum_{j=1}^{i-1}[a_0(z_j-z_0)+1]f_j\mu_j\\&+\frac{a_0(z_i-z_0)+1}{a_0(z_n-z_0)+1}\sum_{j=i}^{n-1}(z_n-z_j)f_j\mu_j.\end{aligned}\tag{14}$$

证 由(4.4.12)，

$$\begin{aligned}u_i=&\frac{z_n-z_i}{z_n-z_0}u_0+\frac{z_i-z_0}{z_n-z_0}f_n+\frac{z_n-z_i}{z_n-z_0}\sum_{j=1}^{i-1}(z_j-z_0)f_j\mu_j\\&+\frac{z_i-z_0}{z_n-z_0}\sum_{j=i}^{n-1}(z_n-z_j)f_j\mu_j,\ 0<i<n.\end{aligned}\tag{15}$$

特别

$$\begin{aligned}u_1=&\frac{z_n-z_1}{z_n-z_0}u_0+\frac{z_1-z_0}{z_n-z_0}f_n\\&+\frac{z_1-z_0}{z_n-z_0}\sum_{j=1}^{n-1}(z_n-z_j)f_j\mu_j.\end{aligned}\tag{16}$$

但由(13)

$$\begin{aligned}u_0&=\frac{b_0}{a_0+b_0}u_1+\frac{f_0}{a_0+b_0}\\&=\frac{u_1}{a_0(z_1-z_0)+1}+\frac{(z_1-z_0)f_0}{a_0(z_1-z_0)+1}.\end{aligned}\tag{17}$$

从(16)、(17)解出

$$u_0=\frac{z_n-z_0}{a_0(z_n-z_0)+1}f_0+\frac{1}{a_0(z_n-z_0)+1}f_n$$

$$+\frac{1}{a_0(z_n-z_0)+1}\sum_{j=1}^{n-1}(z_n-z_j)f_j\mu_j,$$

代入(15)中得(14)，证毕。

引理5 方程组

$$D_\mu u^+=f, \tag{18}$$

即
$$\left.\begin{aligned}&-(a_0+b_0)u_0+b_0u_1=f_0,\\&a_iu_{i-1}-(a_i+b_i)u_i+b_iu_{i+1}=f_i(i>0).\end{aligned}\right\} \tag{19}$$

的解是

$$u_i=[a_0\ (z_i-z_0)\ +1]u_0+\sum_{i=0}^{i-1}(z_i-z_j)f_ju_j. \tag{20}$$

证 (19)即

$$\left.\begin{aligned}&u_0^+=a_0u_0+f_0\mu_0\\&u_i^+-u_{i-1}^+=f_i\mu_i,\quad i>0.\end{aligned}\right\} \tag{21}$$

故
$$u_i^+=u_0^++\sum_{j=1}^{i}(u_j^+-u_{j-1}^+)=a_0u_0+f_0\mu_0+\sum_{j=1}^{i}f_j\mu_j$$

$$=a_0u_0+\sum_{j=0}^{i}f_j\mu_j,$$

$$u_i=u_0+\sum_{j=0}^{i-1}(u_{j+1}-u_j)=u_0+\sum_{j=0}^{i-1}u_j^+(z_{j+1}-z_j)$$

$$=u_0+\sum_{j=0}^{i-1}\left(a_0u_0+\sum_{K=0}^{j}f_k\mu_k\right)(z_{j+1}-z_j)$$

$$=u_0+a_0(z_i-z_0)u_0+\sum_{j=0}^{i-1}(z_{j+1}-z_j)\sum_{K=0}^{j}f_k\mu_k$$

$$=[a_0(z_i-z_0)+1]u_0+\sum_{j=0}^{i-1}(z_i-z_j)f_j\mu_j.$$

系 方程

$$Qu=0 \tag{22}$$

的解为

$$u_i=[a_0(z_i-z_0)+1]u_0,\quad i\geqslant 0. \tag{23}$$

§3. 方程 $\lambda u-D_\mu u^+=0$的解

定理1 对每个$\lambda>0$，方程

$$\lambda u-D_\mu u^+=0,\quad u_0=1 \tag{1}$$

的解$u(\lambda)$存在而且唯一，具有下列性质：

(i) $u(\lambda)$和$u^+(\lambda)$均严格增加；

(ii) $u(z,\lambda)\equiv\lim\limits_{i\to\infty}u_i(\lambda)<\infty$当且仅当$z$为流出或正则；

(iii) $u(\lambda)\mu\in\ell$即$u^+(z,\lambda)\equiv\lim\limits_{i\to\infty}u_i^+(\lambda)<\infty$当且仅当$z$为正则或流入。

证 由(2.20)，$u_i(\lambda)$必定满足下式

$$u_i(\lambda)=1+a_0(z_i-z_0)+\sum_{j=0}^{i-1}(z_i-z_j)u_j(\lambda)\mu_j. \tag{2}$$

由此可确定$u_1(\lambda)$，$u_2(\lambda)$，…，因而$u(\lambda)$存在且唯一。由(2)可见$u(\lambda)$严格增加。

其次，
$$u_i^+(\lambda)=a_0+\sum_{j=0}^{i}D_\mu u_j^+(\lambda)\mu_j$$
$$=a_0+\lambda\sum_{j=0}^{i}u_j(\lambda)\mu_j. \tag{3}$$

故$u^+(\lambda)$严格增加。

(ii)、(iii) 的证明同引理4.5.5，证毕。

定理2 设$u(\lambda)$为(1)的解，令

$$\nu_i(\lambda)=u_i(\lambda)\sum_{j=i}^{\infty}\frac{z_{j+1}-z_j}{u_j(\lambda)u_{j+1}(\lambda)}. \tag{4}$$

则$\nu(\lambda)$严格下降，$\nu^+(\lambda)$严格上升，且

$$\lambda\nu_i(\lambda)-D_\mu\nu_i^+(\lambda)=\begin{cases}0, & 如i>0.\\ 1, & 如i=0.\end{cases} \tag{5}$$

$$u_i^+(\lambda)\nu_i(\lambda)-\nu_i^+(\lambda)u_i(\lambda)=1,\quad i\geqslant 0. \tag{6}$$

$$\nu^+(z,\ \lambda)\equiv\lim_{i\to\infty}\nu_i^+(\lambda)=-\frac{1}{u(z,\ \lambda)}. \tag{7}$$

($u(z,\ \lambda)=\infty$时，右方为0)。

证 象引理4.11.1一样，类似(4.11.3)可得(4) 中级数的收敛性。类似(4.11.4)有

$$\nu_i^+(\lambda)=u_i^+(\lambda)\sum_{j\geqslant i}\frac{z_{j+1}-z_j}{u_j(\lambda)u_{j+1}(\lambda)}-\frac{1}{u_i(\lambda)}. \tag{8}$$

$$\nu_i^+(\lambda)<\frac{1}{u_i(\lambda)}-\frac{1}{u_i(\lambda)}=0. \tag{9}$$

故$\nu(\lambda)$严格下降。由(8)得(6)。再由(8)，

$$\begin{aligned}-\frac{1}{u_i(\lambda)}\leqslant\nu_i^+(\lambda)&\leqslant\sum_{j=i}^{\infty}u_j^+(\lambda)\ \frac{z_{j+1}-z_j}{u_j(\lambda)u_{j+1}(\lambda)}-\frac{1}{u_i(\lambda)}\\&=\sum_{j=i}^{\infty}\left\{\frac{1}{u_j(\lambda)}-\frac{1}{u_{j+1}(\lambda)}\right\}-\frac{1}{u_i(\lambda)}\\&=-\frac{1}{u(z,\ \lambda)}.\end{aligned} \tag{10}$$

由此得(7)。从(8)及(2.10)得：当$i>0$时，

$$D_\mu\nu_i^+(\lambda)=D_\mu u_i^+(\lambda)\ \sum_{j\geqslant i}\frac{z_{j+1}-z_j}{u_j(\lambda)u_{j+1}(\lambda)}=\lambda\nu_i(\lambda).$$

得证(5)第一行。为证第二行，注意按$\nu(\lambda)$的定义(4)，

$$\nu_1(\lambda)=\nu_0(\lambda)u_1(\lambda)-\frac{z_1-z_0}{u_0(\lambda)}=\nu_0(\lambda)u_1(\lambda)-b_0^{-1}. \tag{11}$$

由于$u(\lambda)$是(1)的解，故

$$b_0u_1(\lambda)=(\lambda+a_0+b_0)u_0(\lambda)=\lambda+a_0+b_0. \tag{12}$$

代入(11)得(5)第二行，证毕。

§4. 最小解的构造

对(3.1)的解$u(\lambda)$及按(3.4)确定的$\nu(\lambda)$，令

$$\phi_{ij}(\lambda)=\begin{cases}u_i(\lambda)v_j(\lambda)\mu_j, & \text{如} j\geqslant i,\\ v_i(\lambda)u_j(\lambda)\mu_j, & \text{如} j\leqslant i,\end{cases} \tag{1}$$

类似(4.6.1)——(4.6.5)有

$$\mu_i\phi_{ij}(\lambda)=\mu_j\phi_{ji}(\lambda). \tag{2}$$

设f为列矢量，g为行矢量，则

$$[\phi(\lambda)f]_i=\sum_j\phi_{ij}(\lambda)f_j$$

$$=v_i(\lambda)\sum_{j=0}^{i}u_j(\lambda)f_j\mu_j+u_i(\lambda)\sum_{j=i+1}v_j(\lambda)f_j\mu_j \tag{3}$$

$$[g\phi(\lambda)]_j=\sum_i g_i\phi_{ij}(\lambda)$$

$$=v_j(\lambda)\mu_j\sum_{i=0}^{j}g_iu_i(\lambda)+u_j(\lambda)\mu_j\sum_{i=j+1}^{\infty}g_iv_i(\lambda). \tag{4}$$

如果$g=f\mu$，则

$$g\phi(\lambda)=(\phi(\lambda)f)\mu. \tag{5}$$

定理1

$$\lambda\sum_j\phi_{ij}(\lambda)=1-a_0v_i(\lambda)-\frac{u_i(\lambda)}{u(z,\lambda)}. \tag{6}$$

（约定$\frac{0}{\infty}=0$）。

证 由(3)，回忆约定(2.9)，

$$\lambda\sum_j\phi_{ij}(\lambda)=v_i(\lambda)\sum_{j=0}^{i}(u_j^+(\lambda)-u_{j-1}^+(\lambda))+$$

$$u_i(\lambda)\sum_{j=i+1}^{\infty}(v_j^+(\lambda)-v_{j-1}^+\lambda))$$

$$=v_i(\lambda)u_i^+(\lambda)-u_i(\lambda)v_i^+(\lambda)-a_0v_i(\lambda)$$

$$+u_i(\lambda)v^+(z,\lambda).$$

由于(3.6) 及(3.7)，上式即(6)。

定理2 设$f\in\boldsymbol{m}$，$g\in\boldsymbol{l}$，则$\phi(\lambda)f\in\boldsymbol{m}$，$g\phi(\lambda)\in\boldsymbol{l}$，且

$$\lambda[\phi(\lambda)f]-Q[\phi(\lambda)f]=f,\ \lambda>0. \tag{7}$$

$$\lambda[g\phi(\lambda)]-[g\phi(\lambda)]Q=g,\ \lambda>0. \tag{8}$$

证 只证(7)。对$i>0$，证明同定理4.6.2，但对$i=0$则需要小心。由(3)得

$$[\phi(\lambda)f]_i^+=\nu_i^+(\lambda)\sum_{j=0}^{i}u_j(\lambda)f_j\mu_j$$

$$+u_i^+(\lambda)\sum_{j=i+1}^{\infty}v_j(\lambda)f_j\mu_j \tag{9}$$

$$[\phi(\lambda)f]_0^+=\nu_0^+(\lambda)f_0+u_0^+(\lambda)\sum_{j=1}^{\infty}\nu_j(\lambda)f_j\mu_j \tag{10}$$

回忆约定(2.9)，

$$[\phi(\lambda)f]_{-1}^{\pm}=a_0[\phi(\lambda)f]_0$$

$$=a_0\nu_0(\lambda)f_0+a_0u_0(\lambda)\sum_{j=1}^{\infty}\nu_j(\lambda)f_j\mu_j$$

$$=\nu_{-1}^{\pm}(\lambda)f_0+u_{-1}^{\pm}(\lambda)\sum_{j=1}^{\infty}\nu_j(\lambda)f_j\mu_j. \tag{11}$$

由(10)、(11)及(3.5)，

$$D_\mu[\phi(\lambda)f]_0^+=D_\mu\nu_0^+(\lambda)f_0+D_\mu u_0^+(\lambda)\sum_{j=1}^{\infty}\nu_j(\lambda)f_j\mu_j$$

$$=(\lambda\nu_0(\lambda)-1)f_0+\lambda u_0(\lambda)\sum_{j=1}^{\infty}\nu_j(\lambda)f_j\mu_j$$

$$=[\phi(\lambda)f]_0-f_0.$$

引理3 设$f\in \boldsymbol{m}$，z为正则或流出。则

$$[\phi(\lambda)f](z)=\lim_{i\to\infty}[\phi(\lambda)f]_i=0. \tag{12}$$

证 因为z正则或流出时，$z<\infty$，$u(z,\lambda)<\infty$。故由$\nu(\lambda)$的定义(3.4)得$\lim\limits_{i\to\infty}\nu_i(\lambda)=\nu(z,\lambda)=0$。由(6)得$[\lambda\phi(\lambda)1](z)=0$。由此得(12)。

定理4 $\phi(\lambda)$是最小Q过程。$\phi(\lambda)$不中断的充要条件是$a_0=0$

且z为流入或自然。

当z自然时，$\phi(\lambda)$是唯一满足向前或向后方程组的Q过程。

当z流入时，$\phi(\lambda)$是唯一满足向后方程组的Q过程。

当z流出时，$\phi(\lambda)$是唯一满足向前方程组的Q过程。

证 $\phi(\lambda)$的范条件由定理1得出。$\phi(\lambda)$的B条件和F条件由定理2得出。为证$\phi(\lambda)$的预解方程，令$f\in \boldsymbol{m}$，$F(\lambda)=\phi(\lambda)f$。则$F(\lambda)-F(\nu)+(\lambda-\nu)\phi(\lambda)F(\nu)\in \boldsymbol{m}$且是方程(3.1)的解，故

$$F(\lambda)-F(\nu)+(\lambda-\nu)\phi(\lambda)F(\nu)=cu(\lambda). \tag{13}$$

其中c为常数。如果z为正则或流出，由引理3，在上式中令$i\to\infty$得$cu(z,\lambda)=0$，$c=0$。如果z为流入或自然，则由于(13)左方有界，$u(\lambda)$无界，故$c=0$。所以恒有$c=0$。取$f_i=\delta_{ij}$便得$\phi(\lambda)$的预解方程。这样，$\phi(\lambda)$是满足向后和向前方程组的Q过程。

设$\psi(\lambda)$为任意Q过程，则由于向后不等式（1.8.10）和定理1.7.3，当j固定时，$u_i\equiv\psi_{ij}(\lambda)-\phi_{ij}(\lambda)$满足

$$\lambda u_i-\sum_{K} q_{ik}u_k=\begin{cases}c_1\geqslant 0, & \text{当} i=0 \text{时},\\ 0, & \text{当} i>0 \text{时}.\end{cases} \tag{14}$$

故$u_i-c_1v_i(\lambda)$是方程（3.1）的解，因而$u_i-c_1v_i(\lambda)=c_2u_i(\lambda)$，即

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+c_1v_i(\lambda)+c_2u_i(\lambda). \tag{15}$$

其中c_1，c_2为与i无关的常数，$c_1\geqslant 0$。

如果z流出或正则，由引理3及其证明中指出的$v(z,\lambda)=0$。在（15）中令$i\to\infty$得$c_2u(z,\lambda)\geqslant 0$，故$c_2\geqslant 0$。如$z$流入或自然，由于（15）左方有界，右方中仅$u(\lambda)$无界，故$c_2=0$。这样恒有$c_2\geqslant 0$，即$\psi(\lambda)\geqslant\phi(\lambda)$。得证$\phi(\lambda)$的最小性。

为使$\lambda\phi(\lambda)\mathbf{1}=\mathbf{1}$，由定理1知当且当只$a_0=0$且$z$流入或自然。

如果$\psi(\lambda)$满足向后方程组，则$u_1=\psi_{ij}(\lambda)-\phi_{ij}(\lambda)$是方程(3.1)的解，故

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+cu_i(\lambda). \tag{16}$$

其中$c\geqslant 0$与i无关。如果z流入或自然，象证明$\phi(\lambda)$的最小性一样，可证$c=0$，因此$\phi(\lambda)$是唯一的满足向后方程组的Q过程。

如果$\psi(\lambda)$满足向前方程组，则$v_j=\psi_{ij}(\lambda)-\phi_{ij}(\lambda)$是方程组

$$\lambda v-vQ=0 \tag{17}$$

的解，而$u(\lambda)\mu$是(17)的唯一线性独立解，故

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+cu_j(\lambda)\mu_j \tag{18}$$

其中c为与j无关的常数．如果z流出或自然，则由于$\psi(\lambda)$及$\phi(\lambda)$的范条件，$c=0$．故$\phi(\lambda)$是唯一满足向前方程组的Q过程，证毕．

注 如果$a_0=0$，则(14)中的$c_1=0$．因此从定理证明可见，任何Q过程$\psi(\lambda)$具有形状

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+a_0\nu_i(\lambda)F_j^1(\lambda)+\frac{u_i(\lambda)}{u(z,\lambda)}F_j^2(\lambda). \tag{19}$$

其中$F^a(\lambda)\geqslant 0$．$\psi(\lambda)$满足向后方程组当且只当$F^1(\lambda)=0$．$\psi(\lambda)$满足向前方程组当且只当$\psi(\lambda)$有形状(18)．

§5．一些引理

以后我们简记

$$X_i^1(\lambda)=a_0\nu_i(\lambda),\qquad X_i^2(\lambda)=\frac{u_i(\lambda)}{u(z,\lambda)}. \tag{1}$$

$$X_i^1=\frac{a_0(z-z_i)}{a_0(z-z_0)+1},\qquad X_i^2=\frac{a_0(z_i-z_0)+1}{a_0(z-z_0)+1}. \tag{2}$$

显然$X^1+X^2=1$，且(4.6)成为

$$\lambda\phi(\lambda)1=1-X^1(\lambda)-X^2(\lambda). \tag{3}$$

引理1 $X^a(\lambda)(a=1,2)$是列协调族，且

$$X^1(\lambda)\downarrow 0,\qquad \lambda X_i^1(\lambda)\to\begin{cases}0, & \text{如}i>0,\\ a_0, & \text{如}i=0.\end{cases}\quad(\lambda\uparrow\infty) \tag{4}$$

$$X_2(\lambda)\downarrow 0,\ \lambda X_2(\lambda)\to 0,\quad(\lambda\uparrow\infty). \tag{5}$$

$$\lambda\phi(\lambda)X^a = X^a - X^a(\lambda), \quad a=1, \ 2. \tag{6}$$

证 只需证(6) 即可。当$a_0=0$时，(6) 对$a=1$显然，对$a=2$由(3)得出。下面设$a_0>0$。

如果$z=\infty$，则$X^2=X^2(\lambda)=0$，$X^1=1$，同样由(3)得(6)。

如果$z<\infty$，则

$$
\begin{aligned}
&\lambda\sum_j \phi_{ij}(\lambda)(z-z_j)\\
&=\nu_i(\lambda)\sum_{j=0}^{i}\lambda u_j(\lambda)\mu_j\sum_{k=j}^{\infty}(z_{k+1}-z_k)\\
&\quad+u_i(\lambda)\sum_{j=i+1}^{\infty}\lambda\nu_j(\lambda)\mu_j\sum_{k=j}^{\infty}(Z_{k+1}-Z_k)\\
&=\nu_i(\lambda)\left\{\sum_{K=0}^{i}(z_{k+1}-z_k)\sum_{j=0}^{K}\lambda u_j(\lambda)\mu_j+\sum_{k=i+1}^{\infty}(z_{k+1}-z_k)\right.\\
&\quad\left.\times\sum_{j=0}^{i}\lambda u_j(\lambda)\mu_j\right\}+u_i(\lambda)\sum_{k=i+1}^{\infty}(z_{k+1}-z_k)\\
&\quad\times\sum_{j=i+1}^{K}\lambda\nu_j(\lambda)\mu_j\\
&=\nu_i(\lambda)\left\{\sum_{k=0}^{i}(z_{k+1}-z_k)[u_k^{+}(\lambda)-u_{-1}^{+}(\lambda)]\right.\\
&\quad\left.+\sum_{k=i+1}^{\infty}(z_{k+1}-z_k)[u_i^{+}(\lambda)-u_{-1}^{+}(\lambda)]\right\}\\
&\quad+u_i(\lambda)\sum_{k=i+1}^{\infty}(z_{k+1}-z_k)[\nu_k^{+}(\lambda)-\nu_i^{+}(\lambda)]\\
&=\nu_i(\lambda)\left\{\sum_{k=0}^{i}[u_{k+1}(\lambda)-u_k(\lambda)]+u_i^{+}(\lambda)(z-z_{i+1})\right.\\
&\quad\left.-u_{-1}^{+}(\lambda)(z-z_0)\right\}\\
&\quad+u_i(\lambda)\left\{\sum_{k=i+1}^{\infty}[\nu_{k+1}(\lambda)-\nu_k(\lambda)]-\nu_i^{+}(\lambda)(z-z_{i+1})\right\}
\end{aligned}
$$

$$
\begin{aligned}
&=\nu_i(\lambda)\{u_i(\lambda)-u_0(\lambda)+u_i^+(\lambda)(z-z_i)\\
&\quad -a_0u_0(\lambda)(z-z_0)\}\\
&\quad +u_i(\lambda)\{\nu(z,\ \lambda)-\nu_i(\lambda)-\nu_i^+(\lambda)(z-z_i)\}\\
&=[u_i^+(\lambda)\nu_i(\lambda)-u_i(\lambda)\nu_i^+(\lambda)](z-z_i)-[a_0(z-z_0)\\
&\quad +1]\nu_i(\lambda)\\
&=(z-z_i)-[a_0(z-z_0)+1]\nu_i(\lambda) \qquad (7)
\end{aligned}
$$

两边乘$\dfrac{a_0}{a_0(z-z_0)+1}$得当$a=1$的(6) 式。由此及(3)得(6)对$a=2$成立，证毕。

引理2 (i) 设X^0和$\overline{X}$的定义1.11.3中的最大消极解和最大流出解。则当z正则或流出时，$X^0=0$，$\overline{X}=X^2$；当z流入或自然时$X^0=X^2$，$\overline{X}=0$。

(ii) $X^1(\lambda)$的标准映象为X^1。如果z正则或流出，则$X^2(\lambda)$的标准映象为X^2。

证 由引理2.5系，方程 (1.11.18) 的唯一线性独立解是X^2，而X^0和$\overline{X}$是(1.11.18) 的解，故$X^0=cX^2$，$\overline{X}=\bar{c}X^2$ (c，$\bar{c}$为常数)。

设z流出或正则，则$X^2\neq 0$。由(6)，

$$\lambda\phi(\lambda)X^0=c[X^2-X^2(\lambda)]=X^0-cX^2(\lambda),$$

$$\lambda\phi(\lambda)\overline{X}=\bar{c}[X^2-X^2(\lambda)]=\overline{X}-\bar{c}X^2(\lambda).$$

比较 (1.11.17)(1.11.19) 得$c=0$，$\bar{c}=1$。

设z流入或自然。倘若$a_0z=\infty$，此时$X^2=0$，因而$X^0=X^2$，$\overline{X}=0$。倘若$a_0z<\infty$，则$X^2\neq 0$。且由于$X^2(\lambda)=0$，故解空间$\mathscr{M}_1^+(1)$ 为零空间，因而$\overline{X}(\lambda)=0$的标准映象为$\overline{X}=0$。而(6) 成为

$$\lambda\phi(\lambda)X^2=X^2.$$

由引理1.11.2中X^0的最大性，$X^2\leqslant X^0=cX^2$，故$c\geqslant 1$，但显然$X_i^0=cX_i^2\leqslant 1$，令$i\to\infty$得$c\leqslant 1$，所以$c=1$，$X^0=X^2$。

由(6)，$X^2(\lambda)$的标准映象$\overline{X}^2\leqslant X^2$，象引理4.7.5一样，$u=X^2-\overline{X}^2$是方程(1.11.18) 满足 (4.7.21) 的解，由X^0的最大性，$X^2-\overline{X}^2\leqslant X^0$。当$z$正则或流出时，$X^0=0$，$X^2=\overline{X}^2$。即$X^2(\lambda)$的

标准映象为X^2，从而$X^1(\lambda)$的标准映象为X^1。

如果z流入或自然，依（1.11.13），$X^1(\lambda)$的标准映象$\bar{X}^1=1-X^0-\bar{X}=X^1$，证毕。

引理3 令

$$\left.\begin{aligned}\eta_i&=\begin{cases}X_i^2\mu_i, & \text{如}z\text{正则。}\\ [a_0(z_i-z_0)+1]\mu_i, & \text{如}z\text{流入。}\end{cases}\\ \eta_i(\lambda)&=\begin{cases}X_i^2(\lambda)\mu_i, & \text{如}z\text{正则；}\\ \dfrac{a_0u_i(\lambda)\mu_i}{u^+(z,\ \lambda)}, & \text{如}z\text{流入。}\end{cases}\end{aligned}\right\}\tag{9}$$

则$\eta(\lambda)\in\mathscr{L}_1^+(\lambda>0)$是行协调族，且

$$\lambda\eta\phi(\lambda)=\eta-\eta(\lambda).\tag{10}$$

证 只需证(10)。由(6)及(4.5)得z正则时的(10)。对z流入，$u(z,\ \lambda)=\infty$，$u^+(z,\ \lambda)<\infty$。由(3.7)，

$$\nu^+(z,\ \lambda)=0.\tag{11}$$

还有
$$\nu(z,\lambda)=\frac{1}{u^+(z,\ \lambda)}\tag{12}$$

实际上，注意$u(\lambda)$增加，$\nu(\lambda)$减少，故

$$0\leqslant-u_i(\lambda)\nu_i^+(\lambda)=u_i(\lambda)[\nu^+(z,\ \lambda)-\nu_i^+(\lambda)]$$

$$=u_i(\lambda)\sum_{k=i+1}^{\infty}\lambda\nu_k(\lambda)\mu_k\leqslant\nu_0(\lambda)\sum_{k=i+1}^{\infty}\lambda u_k(\lambda)\mu_k$$

$$=\nu_0(\lambda)[u^+(z,\ \lambda)-u_i^+(\lambda)]\to0\quad(i\to\infty).$$

由此及(3.6)得(12)。

其次，注意(11)(12)，

$$\lambda\sum_k\phi_{ik}(\lambda)(z_k-z_0)$$

$$=\nu_i(\lambda)\sum_{k=0}^{i}\lambda u_k(\lambda)\mu_k\sum_{j=0}^{K-1}(z_{j+1}-z_j)$$

$$+u_i(\lambda)\sum_{k=i+1}^{\infty}\lambda\nu_k(\lambda)\mu_k\sum_{j=0}^{K-1}(z_{j+1}-z_j)$$

$$= v_i(\lambda)\sum_{j=0}^{i-1}(z_{j+1}-z_j)\sum_{k=i+1}^{\infty}\lambda u_k(\lambda)\mu_k$$

$$+u_i(\lambda)\left\{\sum_{j=0}^{i}(z_{j+1}-z_j)\sum_{k=i+1}^{\infty}\lambda v_k(\lambda)\mu_k\right.$$

$$\left.+\sum_{j=i+1}^{\infty}(z_{j+1}-z_j)\sum_{k=j+1}^{\infty}\lambda v_k(\lambda)\mu_k\right\}$$

$$= v_i(\lambda)\sum_{j=0}^{i-1}(z_{j+1}-z_j)[u_i^+(\lambda)-u_j^+(\lambda)]$$

$$+u_i(\lambda)\left\{\sum_{j=0}^{i}(z_{j+1}-z_j)[v^+(z,\lambda)-v_i^+(\lambda)]\right.$$

$$\left.+\sum_{j=i+1}^{\infty}(z_{j+1}-z_j)[v^+(z,\ \lambda)-v_j^+(\lambda)]\right\}$$

$$= v_i(\lambda)\{u_i^+(\lambda)(z_i-z_0)-[u_i(\lambda)-u_0(\lambda)]\}$$

$$+u_i(\lambda)\{-v_i^+(\lambda)(z_{j+1}-z_0)-[v(z,\ \lambda)-v_{i+1}(\lambda)]\}$$

$$= v_i(\lambda)\{u_i^+(\lambda)(z_i-z_0)-u_i(\lambda)+1\}$$

$$+u_i(\lambda)\{-v_i^+(\lambda)(z_i-z_0)-\frac{1}{u^+(z,\ \lambda)}+v_i(\lambda)\}$$

$$= [u_i^+(\lambda)v_i(\lambda)-u_i(\lambda)v_i^+(\lambda)](z_i-z_0)+v_i(\lambda)$$

$$-\frac{u_i(\lambda)}{u^+(z,\ \lambda)}=(z_i-z_0)-v_i(\lambda)-\frac{u_i(\lambda)}{u^+(z,\ \lambda)}.$$

所以由（4.5）及（6）得（10），证毕。

引理4 $\eta(\lambda)(\lambda>0)$是行协调族的充要条件是有下列Riesz表现：

$$\eta(\lambda)=a\phi(\lambda)+d\bar{\eta}(\lambda). \tag{13}$$

其中$a\geqslant 0$使$a\phi(\lambda)\in\mathscr{I}$，常数$d\geqslant 0$，$z$流出或自然时$d=0$，而

$$\bar{\eta}(\lambda)=\begin{cases}X^2(\lambda)\mu, & \text{如}z\text{正则},\\ \dfrac{a_0 u(\lambda)\mu}{u^+(z,\lambda)}, & \text{如}z\text{流入}.\end{cases} \tag{14}$$

证 因为当且只当z正则或流入时，$\mathscr{L}_i^+$含非零行协调族$\eta(\lambda)$，

故本引理是引理1.11.3的特款。

引理5 设z正则。则

$$U_\lambda^a=\lambda[X^2(\lambda)\mu,\ X^a]\uparrow U^a,\ \lambda\uparrow\infty. \tag{15}$$

其中 $U^1=\dfrac{a_0}{a_0(z-z_0)+1}$, $U^2=+\infty$. (16)

证 (1.11.40)现在成为

$$\lambda[X^2(\lambda)\mu,\ X^a]-\nu[X^2(\nu)\mu,\ X^a]=(\lambda-\nu)[X^2(\lambda)\mu,$$

$$X^a(\nu)]=(\lambda-\nu)[X^2(\nu)\mu,\ X^a(\lambda)],\ \lambda,\ \nu>0. \tag{17}$$

由此得U_λ^a的单调性。

其次，由(6)及(4.9),

$$[\lambda\phi(\lambda)X^a]_i^+=\lambda\nu_i^+(\lambda)\sum_{j=0}^{i}u_j(\lambda)X_j^a\mu_j$$

$$+\lambda u_i^+(\lambda)\sum_{j=i+1}^{\infty}\nu_j(\lambda)X_j^a\mu_j$$

$$=[X^a]_i^+-[X^a(\lambda)]_i^+.$$

注意(3.7)，在上式中令$i\to\infty$得

$$-U_\lambda^a=[X^a]^+(z)-[X^a(\lambda)]^+(z).$$

由(2)得

$$[X^1]^+(z)=-\frac{a_0}{a_0(z-z_0)+1},$$

$$[X^2](z)=\frac{a_0}{a_0(z-z_0)+1}.$$

故为证引理，只需证

$$\lim_{\lambda\to\infty}[X^2(\lambda)]^+(z)=\infty,\quad \lim_{\lambda\to\infty}[X^1(\lambda)]^+(z)=0. \tag{18}$$

而这可象证(4.7.18)(4.7.19)一样进行，证毕。

§6. 满足向后方程组的Q过程的构造

由定理4.4，z流入或自然时，满足向后方程组的Q过程唯一，

因此我们假设z流出或正则，此时$z<\infty$，$u(z,\lambda)<\infty$。

由定理4.4的注，Q过程$\psi(\lambda)$满足向后方程组当且只当$\psi(\lambda)$具有形状：

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+X_i^2(\lambda)F_j(\lambda). \tag{1}$$

其中$F(\lambda)\geqslant \mathbf{0}$。因此满足向后方程组的$Q$过程的构造问题已含于定理2.2.1。当然，在生灭过程的情形下将取更具体而简单的形式。

此时，定理2.2.1中的$H=\{0\}$，$\overline{\eta}(\lambda)$为$dX^2(\lambda)\mu$，而常数$d\geqslant 0$，z流出时$d=0$。其中的$\overline{V^1}=dU^1=\dfrac{da_0}{a_0(z-z_0)+1}$。注意根据引理5.2，$X^0=\mathbf{0}$，$\overline{X}=X^2$。因此定理2.2.1现在取下面形式。

定理1 设z正则或流出．$\psi(\lambda)$是满足向后方程组的Q过程的充要条件是，或者$\psi(\lambda)=\phi(\lambda)$，或者$\psi(\lambda)$可如下得到：取行矢量$\alpha\geqslant 0$使$\alpha\phi(\lambda)\in \boldsymbol{l}$，取常数$d\geqslant 0$，当$z$流出时$d=0$。并且

$$\eta(\lambda)=\alpha\phi(\lambda)+dX^2(\lambda)\mu\neq\mathbf{0}. \tag{2}$$

取常数c满足

$$c\geqslant[\alpha,X^1]+\frac{da_0}{a_0(z-z_0)+1}, \tag{3}$$

其中$X^a(a=1,2)$由(5.2)确定。最后令

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+X_i^2(\lambda)\frac{\sum_k\alpha_k\phi_{kj}(\lambda)+dX_j^2(\lambda)\mu_j}{c+[\alpha,X^2-X^2(\lambda)]+d\lambda[X^2(\lambda)\mu,X^2]}. \tag{4}$$

过程$\psi(\lambda)$不中断的充要条件是$a_0=0$，$c=0$。过程$\psi(\lambda)$满足向前方程组的充要条件是$\alpha=\mathbf{0}$。

§7. 满足向前方程组的Q过程的构造

由于定理4.4，我们只需考虑z为正则或流入。由于Q过程$\psi(\lambda)$满足向前方程组当且只当$\psi(\lambda)$有形状(4.18)，即

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+F_i(\lambda)\overline{\eta}_j(\lambda). \tag{1}$$

其中 $\overline{\eta}(\lambda)$由(5.14)确定。因此，满足向前方程组的Q过程的构造问题已在定理2.4.1中解决。

先设z正则。此时 $\overline{\eta}(\lambda)=X^2(\lambda)\mu$。按照定理2.4.1中的记号，或者 $\overline{\xi}(\lambda)=\mathbf{0}$，或 者 $\overline{\xi}(\lambda)=X^2(\lambda)$。

如果 $\overline{\xi}(\lambda)=\mathbf{0}$，则$\overline{W}_\lambda=\lambda[\overline{\eta}(\lambda),\ \overline{X}]=U_\lambda^2\uparrow U^2=+\infty$，这与(2.4.3)冲突。因此必定 $\overline{\xi}(\lambda)=X^2(\lambda)$，因而$\delta>0$。按照同样的考虑，(2.4.3)中的$k$必定为$\delta^{-1}$。根据(2.4.4)，$\beta^0\leqslant\delta$。又 $\overline{V}^0=U^1=\dfrac{a_0}{a_0(z-z_0)+1}$，$X^0=\mathbf{0}$，因此如果用常数$\beta$代替$\beta^0$，则定理2.4.1取下面形式。

定理1 设z正则。$\psi(\lambda)$是满足向前方程组的Q过程的充要条件是，或者$\psi(\lambda)=\phi(\lambda)$，或者$\psi(\lambda)$可如下得到：取非负常数 β及正常数δ，$\beta\leqslant\delta$，常数c满足

$$\frac{(\delta-\beta)a_0}{a_0(z-z_0)+1}\leqslant c. \tag{2}$$

令
$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+\frac{[\beta X_i^1(\lambda)+\delta X_i^2(\lambda)]X_j^2(\lambda)\mu_j}{c+\lambda[X^2(\lambda)\mu,\beta X^1+\delta X^2]}. \tag{3}$$

过程$\psi(\lambda)$不中断的充要条件是$a_0>0$，$\beta=\delta$，$c=0$ 或者 $a_0=0$，$c=0$。过程$\psi(\lambda)$满足向后方程组的充要条件是$a_0=0$，或者$a_0>0$，$\beta=0$。

对于z流入时满足向前方程组的 Q 过程的构造，当然可以由定理2.4.1导出。但我们将按下面方式进行。

假定z流入或自然，此时$u(z,\lambda)=\infty$。如果 $a_0=0$，则最小解不中断，Q过程唯一。因此我们进一步设 $a_0>0$。于是 (5.3)成为

$$\lambda\phi(\lambda)\mathbf{1}=\mathbf{1}-X^1(\lambda).$$

根据(4.19)，任何Q过程$\psi(\lambda)$具有形状

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+X_i^1(\lambda)F_j(\lambda). \tag{4}$$

因此，此时Q过程的构造是定理2.3.3的特殊情形。

由于$\nu(\lambda)$的下降性，

$$\sup_i X_i^{\frac{1}{i}}(\lambda) = a_0 \nu_0(\lambda) < 1. \tag{5}$$

因此，如果z自然因而零流入，根据定理2.3.3，Q过程唯一。

如果z流入，依定理2.3.3，每个非最小Q过程由定理2.3.1得到。按照定理2.3.1的记号，此时$\bar{\eta}(\lambda) = d\frac{u(\lambda)\mu}{u^+(z,\lambda)}$，常数$d \geqslant 0$。而

$$Y_\lambda = \lambda[\bar{\eta}(\lambda),\ 1] = \frac{d}{u^+(z,\lambda)} \sum_i \lambda u_i(\lambda)\mu_i$$

$$= \frac{d}{u^+(z,\lambda)}[u^+(z,\lambda) - u_{-1}^+(\lambda)] = d - \frac{da_0}{u^+(z,\lambda)}.$$ 故 $Y \leqslant d < \infty$。由（2.3.5），如果$\alpha \neq 0$，则必定$[\alpha,\ 1] = \infty$。由于(5)及引理2.3.2，这样的α不存在。因此必定$\alpha = 0$。从而$d > 0$。这样，定理2.3.1中的Q过程满足向前方程组。如果用$\bar{c}$表示定理2.3.1中的$\frac{c - \bar{\sigma}^0}{d}$，则

$$c + d\lambda[\bar{\eta}(\lambda),\ 1 - X^0] = d(\bar{c} + \lambda[\bar{\eta}(\lambda), 1]).$$

如果仍用c表示$\bar{c}$，则定理2.3.3取下面形式。

定理2 设z自然，$a_0 \geqslant 0$，则Q过程唯一。设z流入。如$a_0 = 0$，则Q过程唯一；如$a_0 > 0$，则每个Q过程$\psi(\lambda)$满足向前方程组，而且或者$\psi(\lambda) = \phi(\lambda)$，或者

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \frac{X_i^{\frac{1}{i}}(\lambda)\bar{\eta}_j(\lambda)}{c + \lambda[\bar{\eta}(\lambda),\ 1]}. \tag{6}$$

其中常数 $c \geqslant 0$，$\bar{\eta}(\lambda) = \frac{u(\lambda)\mu}{u^+(z,\lambda)}$。过程 $\psi(\lambda)$ 不中断当且只当 $c = 0$。

§8. 不满足向后、向前方程组的Q过程的构造

当 z 流入或自然时，上节已研究过了。我们假定 z 正则或流

出。由于$a_0=0$时任何Q过程都满足向后方程组，因此我们进一步设$a_0>0$。

根据(4.19)，每个Q过程$\psi(\lambda)$有形状

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+X_i^1(\lambda)F_j^1(\lambda)+X_i^2(\lambda)F_j^2(\lambda). \tag{1}$$

其中$F^a(\lambda)\geqslant 0$。由定理1.12.1，$\lambda[F^1(\lambda),1]\leqslant 1$。

我们将确定$F^a(\lambda)(a=1,\ 2)$ 使 (1)中确定的$\psi(\lambda)$ 满足范条件、预解方程和Q条件。

如果对$\psi(\lambda)$的范条件成立，由(5.3)，即

$$X^1(\lambda)\lambda[F^1(\lambda),\ 1]+X^2(\lambda)\lambda[F^2(\lambda),\ 1]\leqslant X^1(\lambda)+X^2(\lambda).$$

令$i\to\infty$，注意$X^1(z,\lambda)=0$，$X^2(z,\lambda)=1$得$\lambda[F^2(\lambda),\ 1]\leqslant 1$。因此，范条件等价于

$$F^a(\lambda)\geqslant 0,\ \lambda[F^a(\lambda),\ 1]\leqslant 1,\ a=1,\ 2. \tag{2}$$

象双边生灭过程一样，$\psi(\lambda)$的预解方程仍等价于

$$F^a(\lambda)A(\lambda,\ \nu)=F^a(\nu)+(\nu-\lambda)\sum_{b=1}^{a}[F^a(\lambda),\ X^b(\nu)]\cdot F^b(\nu), \tag{3}$$

$$(a=1,\ 2,\ \lambda,\ \nu>0)$$

(一) 现在设对某个$\nu>0$，$F^1(\nu)$，$F^2(\nu)$线性相关。于是第四章§9的讨论几乎仍然有效，只需作适当的修改。我们叙述成下面的引理。

引理1 设z为正则或流出，$F^a(\lambda)$ $(a=1,\ 2)$对某个(从而一切)$\lambda>0$线性相关。

为使(1)中的$\psi(\lambda)$ 满足范条件和预解方程的充要条件是，或者$\psi(\lambda)=\phi(\lambda)$，或者$\psi(\lambda)$可如下得到：取常数$d_a\geqslant 0$ $(a=1,2)$，$d_1+d_2>0$，$p\geqslant 0$ (z流出时$p=0$)。取行矢量$\alpha\geqslant 0$ 使 $\alpha\phi(\lambda)\in l$，且

$$\eta(\lambda)=\alpha\phi(\lambda)+pX^2(\lambda)\mu\neq 0, \tag{4}$$

如果$d_1>d_2$，还要求

$$[\alpha,\ X^2]<\infty,\ p=0. \tag{5}$$

取常数c满足

$$\left.\begin{array}{l} c\geqslant 0,\ \text{如}d_1=d_2, \\ c\geqslant(d_1-d_2)W_2,\ \text{如}d_1>d_2, \\ c\geqslant(d_2-d_1)W_1,\ \text{如}d_1<d_2. \end{array}\right\} \tag{6}$$

其中
$$\left.\begin{array}{l} W_1=[\alpha,\ X^1]+\dfrac{pa_0}{a_0(z-z_0)+1}, \\ W_2=[\alpha,\ X^2]+PU^2. \end{array}\right\} \tag{7}$$

最后令

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+\frac{[d_1X_i^1(\lambda)+d_2X_i^2(\lambda)]\eta_j(\lambda)}{c+\lambda[\eta(\lambda),d_1X^1+d_2X^2]}. \tag{8}$$

注 比较定理4.9.1，本引理并未要求

$$W_1=\lim_{\lambda\to\infty}\lambda[\eta(\lambda),\ X^1]<\infty.$$

因为依引理1.11.4，W_1本来就是有限的。

为了使引理1中的$\psi(\lambda)$是Q过程，尚需验证Q条件，即

$$\lim_{\lambda\to\infty}\lambda X_i^a(\lambda)\lambda F_j^a(\lambda)=0,\ a=1,\ 2. \tag{9}$$

由于(8)及(1.11.26)，

$$\lim_{\lambda\to\infty}\lambda F_j^a(\lambda)=\frac{d_a\alpha_j}{c+d_1W_1+d_2W_2}. \tag{10}$$

故由(5.4)(5.5)，(9)对除$a=1$，$i=0$以外成立。对$a=1$，$i=0$，(9)成为

$$\frac{a_0d_1\alpha}{c+d_1W_1+d_2W_2}=0. \tag{11}$$

现在设$d_1>d_2\geqslant 0$。由(5)(6)得$W_2=[\alpha,\ X^2]<\infty$，$p=0$。由(4)，$\alpha\neq 0$。上面的注已指出$W_1<\infty$。因此(11)不可能成立。所以必定$d_2\geqslant d_1\geqslant 0$。

这样，(6)成为

$c\geqslant 0$，如$d_1=d_2$，

$$c \geqslant (d_2 - d_1)\left([\alpha, X^1] + \frac{p a_*}{a_0(z - z_0) + 1}\right), \text{ 如 } d_1 < d_2. \quad \Big\}$$

(12)

(11)成立的充要条件为 $d_1\alpha = 0$，或者 $W_2 = \infty$。由于 $[\alpha, X^1] \leqslant W_1 < \infty$，充要条件成为

$$d_1\alpha = 0 \text{ 或 } p > 0 \text{ 或 } p = 0,\ [\alpha, 1] = \infty. \tag{13}$$

我们得到下面的

定理2 设z正则或流出，$a_0 > 0$。$F^a(\lambda)(a = 1,2)$ 线性相关。为使(1)确定的 $\psi(\lambda)$是Q过程的充要条件是，或者$\psi(\lambda) = \phi(\lambda)$，或者$\psi(\lambda)$可如下得到：取$\alpha \geqslant 0$使$\alpha\phi(\lambda) \in \boldsymbol{l}$，取常数 $d_2 \geqslant d_1 \geqslant 0$，$d_2 > 0$，$p \geqslant 0$ （z流出时$p = 0$）及c满足(4)(12)(13)。令

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \frac{[d_1 X_i^1(\lambda) + d_2 X_i^2(\lambda)]\left[\sum_k \alpha_k \phi_{kj}(\lambda) + p X_j^2(\lambda)\mu_j\right]}{c + \sum_{b=1}^{2} d_b\{[\alpha, X^b - X^b(\lambda)] + p\lambda[X^2(\lambda)\mu, X^b]\}}. \tag{14}$$

过程不中断当且只当$d_1 = d_2$，$c = 0$，过程不满足向前、向后方程组的充要条件是$d_1 > 0$，$\alpha \neq 0$。

最后一句话证明如下：左乘(1)两边 $(\lambda I - Q)$ 得B条件等价于

$$0 = (\lambda I - Q)[X_i^1(\lambda)F_j^1(\lambda) + X_i^2(\lambda)F_j^2(\lambda)] = \begin{cases} 0 & \text{如 } i > 0, \\ a_0 F_j^1(\lambda), & \text{如 } i = 0. \end{cases}$$

右乘(1)两边$(\lambda I - Q)$得F条件等价于

$$\begin{aligned} 0 &= [X_i^1(\lambda)F_j^1(\lambda) + X_i^2(\lambda)F_j^2(\lambda)](\lambda I - Q) \\ &= \frac{[d_1 X_i^1(\lambda) + d_2 X_i^2(\lambda)]\alpha_j}{c + \lambda[\eta(\lambda), d_1 X^1 + d_2 X^2]}. \end{aligned}$$

而 $X^1(\lambda)$，$X^2(\lambda)$ 线性独立，且 $d_1 + d_2 > 0$。因而 $d_1 X^1(\lambda) + d_2 X^2(\lambda) \neq 0$。

(二) 在(一)中假定了 $F^a(\lambda)(a = 1,2)$ 线性相关。现在假定

$F^a(\lambda)(a=1,2)$对某个(从而一切)$\lambda>0$线性独立。

此时可以仿照第四章§10的记号和讨论.特别地,引理4.10.1作显然的修改后仍然有效。

引理3 设$\psi(\lambda)$为Q过程，形如(1)。则存在行矢量$\alpha^a\geqslant 0(a=1,2)$使$\alpha^a\phi(\lambda)\in\boldsymbol{l}$，存在二阶方阵$\mathscr{R}_\lambda=(r_\lambda^{ab})\geqslant 0$及数量$P_\lambda^a\geqslant 0$ $(a=1,2)$(Z流出时$[P_\lambda]=[0]$)使

$$[F(\lambda)]=\mathscr{R}_\lambda[\alpha\phi(\lambda)]+[P_\lambda]X^2(\lambda)\mu. \tag{15}$$

引进记号

$$h_\lambda^{ab}=\lambda[\alpha^a\phi(\lambda),\ X^b]=[\alpha^a,X^b-X^b(\lambda)]\uparrow h^{ab}$$
$$=[\alpha^a,X^b],\ \lambda\uparrow\infty. \tag{16}$$

$$\mathscr{H}_\lambda=(h_\lambda^{ab})\uparrow\mathscr{H}=(h^{ab}),\ \lambda\uparrow\infty. \tag{17}$$

考虑一特殊情形：$[p_\lambda]=[0]$，$\mathscr{H}<\infty$，即

$$[F(\lambda)]=\mathscr{R}_\lambda[\alpha\phi(\lambda)], \tag{18}$$

$$[\alpha^a,1]<\infty,\quad a=1,2. \tag{19}$$

定理4 设z正则或流出，$a_0>0$。任意形如(1)的Q过程$\psi(\lambda)$不可能形如(18)、(19)而使$F^a(\lambda)(a=1,2)$线性独立。

证 设$\psi(\lambda)$是Q过程，形如(18)、(19)，且$F^a(\lambda)$ $(a=1,2)$线性独立。

象定理4.10.2一样，根据$\psi(\lambda)$的范条件和预解方程可得$[F(\lambda)]$有形状(4.10.33)，即

$$[F(\lambda)]=(I-\overline{\mathscr{T}}_\lambda)^{-1}[\vec{\bar{\alpha}}\phi(\lambda)], \tag{20}$$

其中$\bar{\alpha}^a\geqslant 0$使$[\bar{\alpha}^a,\ 1]\leqslant 1$，$\overline{\mathscr{T}}_\lambda=\{[\bar{\alpha},\ X(\lambda)]\}$。

$\psi(\lambda)$还应满足Q条件，即

$$0=\lim_{\lambda\to\infty}\lambda[X(\lambda)]'\lambda[F(\lambda)]$$
$$=\lim_{\lambda\to\infty}\lambda[X(\lambda)]'(I-\overline{\mathscr{T}}_\lambda)^{-1}[\lambda\bar{\alpha}\phi(\lambda)]. \tag{21}$$

因为由(5.4)(5.5)及控制收敛定理，

$$\lim_{\lambda\to\infty}(I-\overline{\mathscr{T}}^\lambda)^{-1}=\lim_{\lambda\to\infty}\sum_{n=0}^{\infty}\overline{\mathscr{T}}_\lambda^n=I.$$

故(21)对$i>0$是成立的。对$i=0$，(21)成为

$$\mathbf{0}=\begin{pmatrix}a_0\\0\end{pmatrix}[\bar{a}]=a_0\bar{a}^1,$$

于是$\bar{a}^1=\mathbf{0}$。由(20)得$F^a(\lambda)(a=1,2)$线性相关，证毕。

注 从证明可见，按(20)(1)确定的$\psi(\lambda)$虽然不是Q过程，但却是$\bar{Q}$过程，其中$\bar{q}_{ij}=q_{ij}(i>0,\ j\in E)$，$\bar{q}_{0j}=q_{0j}+a_0\bar{a}^1_j$。

现在考虑(15)中的一般情形。

引理5 设z流出或正则，$a_0>0$。设$F^a(\lambda)$ $(a=1,2)$线性独立。为使(1)中的$\psi(\lambda)$满足范条件、预解方程的充要条件是$\psi(\lambda)$可如下得到：取非负行矢量$\bar{a}^a(a=1,2)$使$\bar{a}^a\phi(\lambda)\in\boldsymbol{l}$，取非负矩阵$\overline{\mathscr{S}}=\begin{pmatrix}0, & \bar{s}^{12}\\ \bar{s}^{21}, & 0\end{pmatrix}$，常数$\bar{p}^2\geqslant 0$($z$流出时$\bar{p}^2=0$)，具有下列性质：

(i) 或者$\bar{p}^2>0$，或者$\bar{p}^2=0$且$\bar{a}^1$与$\bar{a}^2$线性独立。

(ii) $\bar{h}^{ab}<\infty(a\neq b)$。

(iii) $\bar{s}^{12}\leqslant 1$，$\bar{s}^{21}\leqslant 1$，

$$\bar{s}^{ab}\geqslant\bar{h}^{ab}+\frac{\bar{p}^2a_0}{a_0(z-z_0)+1},\quad(a\neq b).$$

最后令：$[\bar{p}]=\begin{pmatrix}0\\ \bar{p}^2\end{pmatrix}$,

$$\left.\begin{aligned}&\mathscr{R}_\lambda=(I-\overline{\mathscr{S}}+\overline{\mathscr{H}}_\lambda+[\bar{p}][U_\lambda]')^{-1},\\&[p_\lambda]=(I-\overline{\mathscr{S}}+\overline{\mathscr{H}}_\lambda+[\bar{p}][U_\lambda]')^{-1}[\bar{p}].\end{aligned}\right\}\tag{22}$$

而$\psi(\lambda)$按(1)、(15)、(22)确定。这里$\overline{\mathscr{H}}_\lambda=(\bar{h}^{ab}_\lambda)\uparrow\overline{\mathscr{H}}=(\bar{h}^{ab})$ $(\lambda\uparrow\infty)$对$\bar{a}^a$按(16)、(17)确定。按(5.15)、(5.16)，

$$[U_\lambda]'=(U^1_\lambda,U^2_\lambda)\uparrow[U]'=\left(\frac{a_0}{a_0(z-z_0)+1},\ +\infty\right).\tag{23}$$

$\psi(\lambda)$具有形状(1)、(18)、(19)的充要条件是$\bar{p}^2=0$，$\bar{h}^{22}<\infty$。

证 重复定理4.10.3的证明，只需作显然的修改即可。还需

注意，依引理1.11.4，如果$\bar{a}^1 \geqslant 0$使$\bar{a}^1\phi(\lambda) \in \mathscr{I}$，则$\bar{h}^{11} = \lim\limits_{\lambda\to\infty} \bar{h}_\lambda^{11} < \infty$。证完。

考虑引理5中的$\psi(\lambda)$是Q过程的条件。由定理4，必然

$$\bar{p}^2 > 0 \text{或者} \bar{h}^{22} = \infty. \tag{24}$$

为使$\psi(\lambda)$是Q过程，尚需验证Q条件，即

$$\lim_{\lambda\to\infty} [\lambda X(\lambda)]' \mathscr{Z}_\lambda^{-1} \lambda([\bar{a}\phi(\lambda)] + [\bar{p}]X^2(\lambda)\mu) = 0. \tag{25}$$

其中
$$\mathscr{Z}_\lambda = (I - \overline{\mathscr{S}} + \overline{\mathscr{H}}_\lambda + [\bar{p}][U_\lambda]')$$
$$= \begin{pmatrix} 1 + \bar{h}_\lambda^{11} & -\bar{s}^{12} + \bar{h}_\lambda^{12} \\ -\bar{s}^{21} + \bar{h}_\lambda^{21} + \bar{p}^2 U_\lambda^1 & 1 + \bar{h}_\lambda^{22} + \bar{p}^2 U_\lambda^2 \end{pmatrix}.$$

由于(24)及引理5(iii)，$\lim\limits_{\lambda\to\infty} det \mathscr{Z}_\lambda = +\infty$。注意$\bar{h}^{11} < \infty$，$\bar{h}^{ab} < \infty$ $(a \neq b)$，故

$$\lim_{\lambda\to\infty} \mathscr{Z}_\lambda^{-1} = \lim_{\lambda\to\infty} \frac{1}{det \mathscr{Z}_\lambda} \begin{pmatrix} 1 + \bar{h}_\lambda^{22} + \bar{p}^2 U_\lambda^2 & \bar{s}^{21} - \bar{h}^{21} - \bar{p}^2 U_\lambda^1 \\ \bar{s}^{12} - \bar{h}_\lambda^{12} & 1 + \bar{h}_\lambda^{11} \end{pmatrix}$$
$$= \begin{pmatrix} (1 + \bar{h}^{11})^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

由于(5.4)、(5.5)，(25)对$i > 0$成立。对$i = 0$，(25) 左方极限为

$$\begin{pmatrix} a_0 \\ 0 \end{pmatrix}' \begin{pmatrix} (1 + \bar{h}^{11})^{-1} & 0 \\ 0 & 0 \end{pmatrix} [\bar{a}] = \frac{a_0 \bar{a}^1}{1 + \bar{h}^{11}}.$$

故为使(25)成立，当且只当$\bar{a}^1 = 0$。

这样，依(22)(15)，

$$[F(\lambda)] = \mathscr{Z}_\lambda^{-1} \begin{pmatrix} 0 \\ \bar{a}^2\phi(\lambda) + \bar{p}^2 X^2(\lambda)\mu \end{pmatrix}.$$

故$F^a(\lambda)(a = 1, 2)$线性相关。这与$F^a(\lambda)(a = 1, 2)$ 线性独立相冲突，于是Q条件(25)不可能成立。

结合定理4，我们实际上已得到下面的定理。

定理6 设z正则或流出，$a_0 > 0$。则不存在Q过程 $\psi(\lambda)$ 形如(1)而使$F^1(\lambda)$，$F^2(\lambda)$线性独立。

注 从定理 6 前面的论证看出，引理 5 中满足条件（24）的 $\psi(\lambda)$ 虽然不是Q过程，但却是$\bar{Q}$过程，其中$\bar{q}_{ij}=q_{ij}(i>0, j\in E)$，$\bar{q}_{0j}=q_{0j}+\dfrac{\alpha_3\bar{\alpha}_j^1}{1+\bar{h}^{11}}$。

§9. 关于$\alpha\phi(\lambda)\in \mathbb{I}$ 的条件

设$\alpha\geqslant 0$。由(5.3)，$\alpha\phi(\lambda)\in \mathbb{I}$等价于

$$\sum_{i=0}^{\infty}\alpha_i\{1-X_i^1(\lambda)-X_i^2(\lambda)\}<\infty。\tag{1}$$

本节给出直接由Q来判断$\alpha\phi(\lambda)\in \mathbb{I}$的条件。

引理1 设$a_0\geqslant 0$，z正则或流出。则

$$\lim_{i\to\infty}\frac{v_i(\lambda)}{z-z_i}=\frac{1}{u(z,\lambda)}。\tag{2}$$

证 利用(3.4)，仿引理4.11.2证明。

引理2 设$a_0\geqslant 0$，z正则。则

$$\lim_{i\to\infty}\frac{1-X_i^1(\lambda)-X_i^2(\lambda)}{z-z_i}=\lambda[X^2(\lambda)\mu,1]。\tag{3}$$

证 仿定理4.11.3即可证明。

定理3 设$a_0\geqslant 0$，z正则。则$\alpha\phi(\lambda)\in \mathbb{I}$即(1)成立的充要条件是

$$\sum_{i=0}^{\infty}\alpha_i(z-z_i)<\infty。\tag{4}$$

或等价地

$$\sum_{i=0}^{\infty}\alpha_i N_i<\infty。\tag{5}$$

这里N_i由(2.6)确定。

证 利用引理2，仿定理4.11.3即可证明。

定理4 设$a_0\geqslant 0$，z流出。则$\alpha\phi(\lambda)\in \mathbb{I}$的充要条件是(5)成立。

证 仿定理4.11.4即可证明。

定理5 设$\alpha_0\geqslant 0$，z流入或自然。则$\alpha\phi(\lambda)\in\mathbb{I}$的充要条件是

$$\sum_{i=0}^{\infty}\alpha_i<\infty. \tag{6}$$

证 必要性从

$$1-X_i^1(\lambda)\geqslant 1-X_0^1(\lambda)>0 \quad (i\geqslant 0)$$

得出。充分性是明显的。

第三篇　马亭边界及其在构造论中的应用

第六章　马亭边界和Q过程

§1. 引　　言

当Q过程不唯一时，为了解决Q过程的构造问题，必须对状态空间E进行紧化，即必须附加一些“边界点”到E中，使得满足某些要求。换言之，我们需要引进并研究Q过程的边界。最简单的紧化是将E单点紧化。例如对单边生灭过程就是这样，而且单点紧化就够了。然而，一般说来，仅仅用单点紧化是不够的。例如，对于双边生灭过程，我们显然要用两个“点”来紧化E。

Feller[2]中引进了一种边界，我们称为费勒边界。他的[3]中对保守的Q引进了流出边界和消极边界，并应用此边界在流出边界和流入边界都有限的情况下，构造了同时满足向后，向前方程组的全部Q过程。费勒使用的是纯分析方法。

Doob[2]中引进了马氏链的马亭边界，其特点是紧密地结合马氏链的轨道。杜勃证明了马氏链的轨道在马亭边界中的收敛性。在Дынкин[3]，*Hunt*[1]，*Watanabe*[1,2]中马氏链的马亭边界理论更进一步得到研究和发展。*Kunita*[1]中应用马亭边界来研究瞬返过程。但在上面的文献中，都是在马氏链上加了一些前提限制(例如，要求马氏链非常返，至少有一个“中心”)下展开讨论的。这

些前提条件虽然是非本质的，但却需要解除。*Hunt*[1] 中关于标准测度的引入可以解除有一个“中心”的限制。Дынкин[3，§9]中的式子

$$K(i,j)=\frac{G(i,j)}{\sum\limits_{s} r_s G(s,j)}=\frac{f_{ij}}{\sum\limits_{s} r_s f_{sj}}$$

提供了解除马氏链非常返的限制的途径。侯振挺、[1]对马氏链不加限制的情况下导出了马亭边界。

本章中，我们首先引进一般的马氏链的马亭边界，其详细证明基本上仿照Дынкин[3]。然后，我们紧密地结合 Q过程进行广泛的讨论。我们利用Feller[3]中引进的标准映象，它与定义1.11.2一致，但较狭窄。我们引进并详细讨论了λ映象。仿照 kunita[1]中的方法，我们导出了不加任何条件的Q 矩阵的马亭流出边界和消极边界，并应用这种边界对第一章§12中Q 过程的一般分析表达式作了进一步的刻划。

本章的内容无论对于用概率方法构造Q 过程，或者深刻理解Q过程的分析方法构造，都是非常必须的，而且是基本的。

§2. 马氏链

设$(\Omega,\mathscr{F},P)$ 为完备概率空间1)，β为定义在其上的取值非负整数及“∞”的随机变量。称$X_T(\omega)=\{x_n(\omega)$2)，$n\leqslant\beta\}$ $(\omega\in\Omega,n$为非负整数）为定义在概率空间（Ω，$\mathscr{F}$，P）3)上取值于可列指标集E 的齐次马氏链或简称马氏链，如果对任意整数$n\geqslant2$，非负整数$0\leqslant t_1<t_2<\cdots<t_{n+1}$和任意$i_1,i_2\cdots,i_{n+1}\in E$，只要 $P\{x(t_a)=i_a,1\leqslant a\leqslant n\}>0$，就有

$$P\{x(t_{n+1})=i_{n+1}\mid x(t_a)=i_a,1\leqslant a\leqslant n\}$$

1）以后考虑$\mathscr{F}$的子Borel域时，恒指按P完备化了的，不再声明.

2）我们视$x_n(\omega)=x(n,\omega)$，并且以后不强调ω时，略写ω.

3）如果在马氏链的定义中，（Ω，$\mathscr{F}$，P）放松为有限测度空间，X_T也称为马氏链.

$$=P\{x(t_{n+1})=i_{n+1}|x(t_n)=i_n\},\qquad(1)$$

而且右方的值与t_n无关，仅与$t_{n+1}-t_n$有关。

记 $P_{ij}^n=P\{x(n)=j|x(0)=i\}$。 (2)

称矩阵(P_{ij}^n)为链的n步转移概率矩阵。简记$p_{ij}=p_{ij}^1$。作为矩阵，

$$(p_{ij}^n)=(p_{ij})^n。\qquad(3)$$

$P=(p_{ij})$非负且行和不超过1。每一个这样的矩阵P都可以作为某个马氏链X_T的一步转移概率矩阵。因此，我们也称矩阵P为马氏链。称

$$d_i=1-\sum_j p_{ij},i\in E\qquad(4)$$

为链的中断量。称集合$H=\{i|d_i>0\}$为中断状态集。如H为空集，称链为不中断的，否则称为中断的。显然，链不中断的充要条件是

$$P_i\{\beta=\infty\}=1,\ i\in E。\qquad(5)$$

这里以及今后我们记$P_i\{\cdot\}=P\{\cdot|x_0=i\}$，$E_i\{\cdot\}$表示关于$P_i$而取的数学期望。例如

$$E_i\{f,\wedge\}=E_if1_\wedge=\int_\wedge fdp_i。\qquad(6)$$

$1_\wedge$表示$\wedge$的示性函数。

令 $\eta_i^*=inf\{n|1\leqslant n\leqslant\beta,\ x_n=i\}$， (7)

$$\eta_i=inf\{n|0\leqslant n\leqslant\beta,\ x_n=i\}。\qquad(8)$$

约定$inf\phi=\infty$，ϕ为空集。记

$$\left.\begin{aligned}&f_{ij}^{n*}=P_i\{\eta_j^*=n\},\ n\geqslant1。\\&f_{ij}^n=P_i\{\eta_j=n\},\ n\geqslant0。\\&f_{ij}^*=\sum_{n=1}^{\infty}f_{ij}^{n*},\ f_{ij}=\sum_{n=0}^{\infty}f_{ij}^n。\end{aligned}\right\}\qquad(9)$$

称f_{ij}^{n*}（或f_{ij}^n）为自i出发从第一（或零）步算起，于第n步首达j的概率。称f_{ij}^*（或f_{ij}）为自i 出发从第一（或零）步算起，经有穷步到达j的概率。显然

$$
\left.\begin{aligned}
&f_{ij}^{0}=0,\ f_{ij}^{*n}=f_{ij}^{n}(n\geqslant 1),\ f_{ij}^{*}=f_{ij},\ \text{如}i\neq j\\
&f_{ii}=f_{ii}^{0}=1,\ f_{ii}^{n}=0(n\geqslant 1),\\
&f_{ij}^{*}=\sum_{k}p_{ik}f_{kj},\ f_{ik}f_{kj}\leqslant f_{ij}.
\end{aligned}\right\}\tag{10}
$$

如果$f_{ij}^{*}>0(i\neq j)$，称自i可到达j，记为 $i\Rightarrow j$。约定$i\Rightarrow i$。如果$i\Rightarrow j$，$j\Rightarrow i$，称i与j互通，记为$i\Longleftrightarrow j$。设$C\subset E$，如果 C中任意二状态互通，且对任意$k\in E-C$有$p_{ik}=0(i\in C)$，则称C为不可约类。当$f_{i}^{*}\equiv f_{ii}^{*}=1$时，称状态$i$常返。

定理1 记

$$
G(i,j)=\sum_{n=0}^{\infty}p_{ij}^{n}.\tag{11}
$$

则

$$
G(i,j)=f_{ij}G(j,j),\ G(i,i)=\frac{1}{1-f_{i}^{*}}.\tag{12}
$$

特别地，i常返的充要条件是$G(i,i)=\infty$。

证. 记

$$
\delta_{j}(i)=\delta_{ij},\tag{13}
$$

则

$$
p_{ij}^{n}=E_{i}\delta_{j}(x_{n}),\ G(i,j)=E_{i}\sum_{n=0}^{\beta}\delta_{j}(x_{n}).\tag{14}
$$

记

$$
A_{m}=(x_{0}\neq j,\ x_{1}\neq j,\cdots,x_{m-1}\neq j,\ x_{m}=j).
$$

则

$$
P_{i}(A_{m})=f_{ij}^{m},\ P_{i}(A_{m},\ x_{m+k}=j)=f_{ij}^{m}p_{jj}^{k}.
$$

这样

$$
\begin{aligned}
G(i,j)=E_{i}\sum_{n=0}^{\beta}\delta_{j}(x_{n})&=E_{i}\sum_{m=0}^{\infty}1_{A_{m}}\sum_{n=m}^{\beta}\delta_{j}(x_{n})\\
&=\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}E_{i}\,1_{A_{m}}\delta_{j}(x_{n})\\
&=\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}P_{i}(A_{m},\ x_{m+k}=j)
\end{aligned}
$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} f_{ij}^{n} p_{jj}^{k} = f_{ij} G(j,j).$$

至于(12)第二式的证明可见王梓坤[1，§2.2，定理1]。

注．如 j 非常返，则 $G(i,j)<\infty$，因而 P_i 几乎必然地有 $\sum_{n=0}^{\beta}\delta_j(x_n)<\infty$，即以概率1，链在$j$停留的次数是有限的。

定理2 （状态空间的分解定理）马氏链的状态空间E有下列唯一分解：

$$E = E_0 \cup (\bigcup_{a\in\mathscr{A}} E_a). \tag{15}$$

其中E_0由一切非常返状态组成，可以不出现。$\mathscr{A}$是空集，有限集或可列无限集，且 $0\overline{\in}\mathscr{A}$。每个$E_a(a\in\mathscr{A})$为不可约常返类。

证明见王梓坤[1，§2.3，定理1]。我们指出，每个中断状态必非常返，即

$$H\equiv\{i;\ \sum_j p_{ij}<1\}\subset E_0. \tag{16}$$

链$X_T=\{x_n,\ n\leqslant\beta\}$的运动情况如下：如$x_0\in$某个$E_a(a\in\mathscr{A})$，则$\beta=\infty$且链将永远在$E_a$中运动并且无限次地回到$E_a$中每一个状态。如果$x_0\in E_0$，则可能$\beta<\infty$，此时必定$x_\beta\in H$；可能$\beta=\infty$，此时链或者永远在$E_0$中运动，或者在某一步进入某常返类$E_a$，然后永远在$E_a$中运动。至于对$E_0$的进一步考虑，有下面的Blackwell分解定理。

令 $\Omega_\infty=(\beta=\infty)$，$\Omega_F=(\beta<\infty)$。

设$A\subset E$。令 (17)

$$\left.\begin{aligned}\overline{\mathscr{L}}(A)&=\Omega_\infty\cap\limsup_{n\to\infty}\{x_n\in A\},\\ \underline{\mathscr{L}}(A)&=\Omega_\infty\cap\liminf_{n\to\infty}\{x_n\in A\}.\end{aligned}\right\} \tag{18}$$

如果$P_i\{\overline{\mathscr{L}}(A)\}=0(i\in E)$，称$A$为瞬时集；如果$P_i\{\underline{\mathscr{L}}(A)\}>0$对某 $i\in E$，称A为逗留集；如果A为逗留集，并且 $P_i\{\overline{\mathscr{L}}(A)\}=P_i\{\underline{\mathscr{L}}(A)\}(i\in E)$，称$A$为几乎闭集。当$A$为瞬时集或几乎闭集

时，我们用$\mathscr{L}(A)$表示与$\underline{\mathscr{L}}(A)$只相差$P_i$-零概率集（对一切$i$）的任一集合。

设A为几乎闭集。如果对任意$B\subset A$，或者B是瞬时集，或者$A-B$是瞬时集，则称A为原子几乎闭集。如果对任意$B\subset A$，B都不是原子几乎闭集，则称A为完全非原子几乎闭集。

下面是布勒克韦(Blackwell)分解定理。

定理3.（Chung[1，I § 17，定理4)

状态空间E有下列分解

$$E=A_0+\Big(\bigcup_{a\in\mathscr{A}}A_a.\Big) \tag{19}$$

其中A_0是完全非原子几乎闭集，可以不出现。$\mathscr{A}$是空集、有限集或可列无限集，且$0\bar{\in}\mathscr{A}$。$A_a(a\in\mathscr{A})$是原子几乎闭集。而且

$$P\{\mathscr{L}(A_0)\}+\sum_{a\in\mathscr{A}}P\{\mathscr{L}(A_a)\}=P(\Omega_\infty). \tag{20}$$

除去瞬时集的差别外，分解(19)是唯一的。

注意，分解(15)中每个不可约常返类都是原子几乎闭集。

§3. 马亭边界理论

为了研究马氏链$X_T=\{x_n,\ n\leqslant\beta\}$的轨道的终极性质，即当$n\uparrow\beta$时$x_n$的性质，产生了马氏链的马亭边界理论。应用边界理论还可以刻划与链相联系的一切过份函数和调和函数。设X_T的一步转移矩阵为$P=(p_{ij})$。如果中断状态集H非空，可取$\Delta\bar{\in}E$，并令

$$P_{\Delta\Delta}=1,\ P_{\Delta i}=0,\ P_{i\Delta}=1-\sum_j P_{ij},\ i\in E. \tag{1}$$

$$x_n=\Delta,\ 如果\beta<\infty,\ n>\beta. \tag{2}$$

则$\{x_n,\ n\geqslant0\}$是马氏链，一步转移概率为$(p_{ij},\ i,j\in E\cup\{\Delta\})$。今后将总是这么理解。

（一） 过份函数和过份测度

设u为E上的函数[1)]，函数pu如下给定：

1) 今后u_i也写为$u(i)$，类似地，p_{ij}也写为$p(i,j)$等等。

$$(pu)_i = \sum_j p_{ij} u_j.$$

设ν为E上的测度，测度νp如下给定：

$$(\nu p)_j = \sum_i \nu_i p_{ij}.$$

定义1 非负(包括$+\infty$)函数u称为P过份的，如果$pu \leqslant u$；称为P调和的，如果$Pu = u$。有界调和函数组成的类记为$\mathscr{M}^+$。囿于k的调和函数类记为$\mathscr{M}^+(k)$。

显然，如u过份，则$p^n u \leqslant u$对一切n。

定义2 非零过份(或调和)函数u称为极小的，如果$u = u^1 + u^2$，且u^1和u^2是过份(或调和)函数，则$u = c^a u^a (a = 1,2)$，c^a为常数。非零调和函数u称为完全非极小的，如果对任意非零调和函数$\nu \leqslant u$，ν都不是极小的。

定义3 (非负)测度μ称为p过份的，如果$\mu p \leqslant \mu$；称为调和的，如果$\mu p = \mu$。

设u是p过份函数。补定义$u(\Delta) = 0$。则$(u_i,\ i \in E \cup \{\Delta\})$关于$(p_{ij}, i, j \in E \cup \{\Delta\})$也是过份的。利用$X_T$的马氏性和$u$的过份性，

$$E[u(x_{n+1}) | x_0,\ x_1, \cdots, x_n] = E[u(x_{n+1}) | x_n]$$
$$= \sum_j P_{x_n j} u_j \leqslant u(x_n), \tag{3}$$

于是$\{u(x_n),\ n \geqslant 0\}$是上鞅，$\{-u(x_n),\ n \geqslant 0\}$是下鞅(即半鞅)，从而

$$Eu(x_0) \geqslant Eu(x_1) \geqslant Eu(x_2) \geqslant \cdots \tag{4}$$

设$0 < a < b$。令U_N为序列$u(x_0), \cdots,\ u(x_N)$向下穿过区间$[a,b]$的次数，即自区间$[a,b]$的右方到左方的次数，因而也就是$-u(x_0), \cdots,\ -u(x_N)$自区间$[-b,\ -a]$的左方到右方的次数，依王梓坤[1, § 1.4, 引理3]，

$$EU_N \leqslant \frac{E[-u(x_N) - (-b)]^+}{(-a) - (-b)}$$

$$= \frac{E[u(x_N) - b]^-}{b - a} \leqslant \frac{Eu(x_N)}{b - a}$$

$$\leqslant \frac{Eu(x_0)}{b-a}. \tag{5}$$

令ν为$u(x_0)$，$u(x_1)$，$\cdots,u(x_n),\cdots$向下穿过区间$[a, b]$的次数，则$\nu_N \uparrow \nu$，因而

$$E\nu \leqslant \frac{Eu(x_0)}{b-a}. \tag{6}$$

特别地

$$E_i\nu \leqslant \frac{u(i)}{b-a}. \tag{7}$$

因此，如果$u(i)<\infty$，则$E_i\nu<\infty$，故p_i几乎有$\nu<\infty$，即$\{u(x_n), n\geqslant 0\}$向下穿过$[a,b]$的次数有限。利用下列事实：一数列向下穿过任意有理端点的区间$[a,b]$的次数都有限，则此数列必存在有穷的或无穷的极限。因此，如$u(i)<\infty$，则存在极限

$$\xi=\lim_{n\to\infty}u(x_n),\ p_i\text{几乎} \tag{8}$$

但是

$$E_iu(x_n)=\sum_j p_{ij}^n u(j)\leqslant u(i),$$

故由法都引理，

$$E_i\xi\leqslant u(i).$$

从而p_i几乎有$\xi<\infty$。我们实际上已经证明了下面的定理。

定理1 设u为过份函数，$u(i)<\infty$。则在Ω_∞上，p_i几乎存在有穷极限$\lim\limits_{n\to\infty}u(x_n)$。

定理2 设马氏链的状态互通。则链为常返的充要条件是其一切过份函数都是常数。

证 设u过份。如果对某个j有$u(j)=\infty$，则对任意i，由互通性，存在n使$p_{ij}^n>0$，从而

$$u_i\geqslant\sum_k p_{ik}^n u(k)\geqslant p_{ij}^n u(j)=\infty.$$

于是$u\equiv\infty$。故只需考虑有限的过份函数u。

设X_T常返。由定理1，存在有限的极限

$$\lim_{n\to\infty}u(x_n)=\xi,\ p_i\text{几乎}.$$

由X_T的常返性及状态的互通性，对任意j，有

$p_i(x_n=j$对无穷多个$n)=1$，故$p_i(\xi=u(j))=1$。由于j任意，对状态k，$p_i(\xi=u(k))=1$，从而$u(j)=u(k)$，即$u\equiv$常数。

设X_T的一切过份函数都是常数。固定k，记$u_i=f_{ik}$。则按(2.7)的记号，

$$\sum_j p_{ij}u_j=p_i(\eta_k^+<\infty)=f_{ik}^+\leqslant f_{ik}=u_i. \tag{9}$$

即u过份，从而$u\equiv$常数。但显然$u_k=f_{kk}=1$，所以$u\equiv1$，即$f_{ik}\equiv1$对一切i和k。任取$i\neq j$，由(2.10)，

$$1=f_{ij}=\sum_m p_{im}f_{mj}=\sum_m p_{im},$$

$$f_{ii}^+=\sum_j p_{ij}f_{ji}=\sum_j p_{ij}=1.$$

即i常返，从而链常返。证完。

下面的系并不要求状态的互通性条件。

系1 当j固定时，f_{ij}作为i的函数是过份函数。

证 由(9)得出。

系2 设u是过份函数，则u在每一个常返类中取常数值。

定理3 对任意测度γ，γG是过份测度。特别地，当i固定时，$G(i,j)$是过份测度。

证 因为$G=\sum_{n=0}^{\infty}p^n$，故

$$GP=\sum_{n=1}^{\infty}p^n\leqslant G,(\gamma G)p\leqslant\gamma G.$$

定理3$_1$ 设测度γ使$\sum_i\gamma_i f_{ij}>0(j\in E)$，$h$是过份函数。如果$h$为$\gamma$可积，则$h$为有限函数；如果$\sum_i\gamma_i h_i=0$，则$h=0$。

证 由于$p^n h\leqslant h$，故

$$\gamma_i p_{ij}^n h\leqslant\sum_i\gamma_i\sum_j p_{ij}^n h_j\leqslant\sum_i\gamma_i h_i.$$

由关于γ的假设，对每个j，存在i及n使$\gamma_i p_{ij}^n>0$。故如果$\sum_i\gamma_i h_i<$

∞，则$h_j<\infty$；如果$\sum\limits_i \gamma_i h_i=0$，则$h_j=0$。证完。

（二）过份测度的密度函数

设马氏链X_T具有初始分布γ[1]，为了强调对于γ的依赖关系，宜记测度P为P_γ，记数学期望E为E_γ。于是链X_T在状态j停留的次数的数学期望为$\eta_j=\sum\limits_i \gamma_i G(i,j)$。测度

$$\eta=\gamma G \tag{10}$$

是P过份测度。

设D为E的有限子集，τ_D（或简记为τ）是链X_T末离D的时刻，即

$$\tau=sup\{n; \ 0\leqslant n\leqslant\beta, \ x_n\in D\} \tag{11}$$

如果对一切n：$0\leqslant n\leqslant\beta$，都有$x_n\overline{\in}D$，则认为$\tau$不确定。

记

$$L_D(i)=P_i(\tau=0)=P_i(x_0\in D, x_n\overline{\in}D\text{对}1\leqslant n\leqslant\beta), \tag{12}$$

故当$i\overline{\in}D$时，$L_D(i)=0$；当$i\in D$时，$L_D(i)$是从第一步就离开D而且永远再不回到D的概率。这样，当$i\in D$且i常返时，$L_D(i)=0$。

显然，对任意$j\in E$，

$$P_i(x_\tau=j)=\sum_{m=0}^{\infty}P_i(\tau=m, \ x_m=j)$$

$$=\sum_{m=0}^{\infty}P_{ij}^{m}L_D(j)=G(i,j)L_D(j), \tag{13}$$

$$\sum_j G(i, j)L_D(j)=\sum_j P_i(x_\tau=j)\leqslant 1. \tag{14}$$

由(13)可得

$$P_\gamma(x_\tau=j)=\eta(j)L_D(j). \tag{15}$$

其中$\eta=\gamma G$。

设n为非负整数。当$\tau<n$，或$\tau=\infty$，或τ不确定时，都约定$x_\tau=\Delta$。设$i_0, i_1, \cdots i_n\in E$，并记

1）γ的全测度可不必为1。

$$R(i_n,\cdots,i_0)=P(i_n,i_{n-1})\cdots P(i_1,i_0),$$

则

$$P_i(x_\tau=i_0,x_{\tau-1}=i_1,\cdots,x_{\tau-n}=i_n)$$

$$=\sum_{m=n}^{\infty}P_i(\tau=m,\ x_m=i_0,\ x_{m-1}=i_1,\cdots,x_{m-n}=i_n)$$

$$=\sum_{m=n}^{\infty}P^{m-n}(i,i_n)R(i_n,\cdots,i_0)L_D(i_0)$$

$$=G(i,i_n)R(i_n,\cdots,i_0)L_D(i_0),$$

乘以γ_i并对i求和得

$$P_\gamma(x_\tau=i_0,\cdots,x_{\tau-n}=i_n)=\eta(i_n)R(i_n,\cdots,i_0)L_D(i_0). \tag{16}$$

当$n=0$时，上式化为(15)。

引理4 设链X_T的每个状态都非常返，u是E上的非负函数，约定$u(\Delta)=0$。如果$u\eta=(u_j\eta_j,\ j\in E)$为过份测度，则关于测度$P_\gamma$，$\{u(x_{\tau-n}),n\geqslant 0\}$关于$\{x_{\tau-n},n\geqslant 0\}$是上鞅，即

$$E_\gamma[u(x_{\tau-n})\,|x_\tau,x_{\tau-1},\cdots,x_{\tau-(n-1)}]\leqslant u(x_{\tau-(n-1)}). \tag{17}$$

证 由于$u\eta$是过份测度，故

$$\sum_{i_n}u(i_n)\eta(i_n)P(i_n,i_{n-1})\leqslant u(i_{n-1})\eta(i_{n-1}).$$

因而

$$\sum_{i_n}\eta(i_n)R(i_n,\cdots,i_0)L_D(i_0)\,u(i_n)$$

$$\leqslant\eta(i_{n-1})R(i_{n-1},\cdots,i_0)L_D(i_0)u(i_{n-1}).$$

注意(16)，由上式得(17)。证完。

引理4中的函数u是过份测度$\mu=\mu\eta$关于η的密度。

记K_γ为满足下面条件(18)的非负函数u全体：

$$S\equiv sup\ \{E_\gamma u(x_{\tau_D}),\ 有穷D\subset E\}<\infty. \tag{18}$$

注意如$\tau_D=\infty$或τ_D不确定时，$x_{\tau_D}=\Delta$，且$u(\Delta)=0$。

定理5 设链X_T的每个状态都非常返，$u\in K_\gamma$且$u\eta$为过份测度，则在Ω_∞上，P_γ几乎存在有穷极限$\lim\limits_{n\to\infty}u(x_n)$。

证 令v_N为$u(x_\tau),\cdots,u(x_{\tau-N})$向下穿过$[a, b]$的次数，亦即$u(x_{\tau-N}),\cdots,u(x_\tau)$向上穿过$[a,b]$的次数。由(5)，

$$E_\gamma v_N\leqslant\frac{E_\gamma u(x_\tau)}{b-a}.$$

记v_D为$u(x_0),\cdots,u(x_\tau)$向上穿过$[a, b]$的次数。由非常返性，当D有限时，P_γ几乎必然$\tau<\infty$，因而$v_N\uparrow v_D(N\to\infty)$，故

$$E_\gamma v_D\leqslant\frac{E_\gamma u(x_{\tau_D})}{b-a}.$$

因 $u\in K_\gamma$，故

$$E_\gamma v_D\leqslant\frac{S}{b-a}$$

再用v表示$u(x_0),u(x_1),\cdots,u(x_n),\cdots$向上穿过$[a, b]$的次数，则$v_D\uparrow v(D\uparrow E)$，故

$$E_\gamma v\leqslant\frac{S}{b-a}. \tag{19}$$

因此$P_\gamma(v<\infty)=1$，从而P_γ几乎必然存在有穷或无穷的极限，$\xi=\lim\limits_{n\to\infty}u(x_n)$。剩下只需证$P_\gamma$几乎必然$\xi<\infty$。

设$v(c)$为$u(x_0),\cdots,u(x_n))\cdots$向上穿过$[c, 2c]$的数目。令$\varliminf\limits_{c\to\infty}v(c)=\overline{v}$。显然

$$(\xi=\infty)\subseteq(\overline{v}\geqslant1),$$

故 $$P_\gamma(\xi=\infty)\leqslant P_\gamma(\overline{v}\geqslant1)\leqslant E_\gamma\overline{v}.$$

由(19)及法都引理得

$$P_\gamma(\xi=\infty)\leqslant E_\gamma\varliminf_{c\to\infty}v(c)\leqslant\varliminf_{c\to\infty}E_\gamma v(c)$$

$$\leqslant \lim_{c\to\infty}\frac{S}{C}=0.$$

定理证完。

（三）马亭核

定义4 称测度$\gamma=(\gamma_i, i\in E)$为链$X_T$或$P$的标准测度，如果

$$\sum_i \gamma_i<\infty,\ 0<\sum_i \gamma_i f_{ij}. \tag{20}$$

从现在起，考虑马亭边界时，我们总是指定一个标准测度γ。由初始分布γ及矩阵$P=(p_{ij})$所产生的测度记为P_γ。因为$P_\gamma=\sum_i \gamma_i P_i$，故如果$\gamma_i>0(i\in E)$，则$P_\gamma$几乎一切$\omega$等价于对一切$i$，$P_i$几乎一切$\omega$；$P_\gamma$零集等价于对一切$i$，$P_i$零集。

显然

$$0<A_j\equiv\sum_i \gamma_i f_{ij}<\infty. \tag{21}$$

定义5 称

$$K(i,j)=\frac{f_{ij}}{A_j}=\frac{f_{ij}}{\sum_s \gamma_s f_{sj}} \tag{22}$$

为链的马亭核。

由定理1有 $A_j G(j,j)=(\gamma G)_j(j\in E_0)$，故

$$K(i,j)=\frac{G(i,j)}{\eta(j)},\ 如 j\in E_0, \tag{23}$$

这里$\eta=\gamma G$，E_0为非常返状态集。由定理2系1及定理3，当j固定时，$K(\cdot,j)$是过份函数。

按定义及（2.10）中最后一式，我们有

$$\left.\begin{aligned}&K(i,j)\leqslant\frac{1}{A_i},\ K(i,j)\leqslant\frac{1}{A_j},\\&\sum_i \gamma_i K(i,j)=1.\end{aligned}\right\} \tag{24}$$

定理6 设μ为全有限测度。则在Ω_∞上P_γ几乎必然存在有穷极限

$$\lim_{n\to\infty}\sum_i \mu(i)K(i,x_n). \tag{25}$$

特别地，对任意i，在Ω_∞上，P_γ几乎必然存在有限极限

$$\lim_{n\to\infty} K(i,\ x_n). \tag{26}$$

证 由(24)，我们有

$$\sum_i \mu(i)K(i,\ j)\leqslant \sum_i \mu(i)A_j^{-1}<\infty \tag{27}$$

设E_a为不可约常返类。当考虑E_a的类性质时，可以把E_a中的状态视为等同，例如E_a中的状态都等同一状态ξ_a。当$j\in E_a$时，f_{ij}与j无关，记为$f_i(\xi_a)$，因而$K(i,j)$也与$j\in E_a$无关，记为$K(i,\xi_a)$。因此，当链于某一步进入某不可约常返类E_a时，对一切充分大的n，

$$\sum_i \mu(i)K(i,\ x_n)=\sum_i \mu(i)K(i,\xi_a)<\infty.$$

因此极限(25)存在且有限。

剩下需要证明：当$x_n\in E_0$对一切$n\leqslant\beta=\infty$时，极限(25)存在且有限。此时只需考虑E_0上的非常返链$\widetilde{X}_T=\{x_n,n\leqslant\widetilde{\beta}\}$，其中$\widetilde{\beta}=\sup\{n:\ x_n\in E_0\}$，$\widetilde{X}_T$的初始分布为$(\gamma_i,\ i\in E_0)$，一步转移矩阵为$\widetilde{P}=(p_{ij},\ i,\ j\in E_0)$，对于$\widetilde{P}$，相应的$\widetilde{f}_{ij}=f_{ij},\widetilde{K}(i,\ j)=K(i,\ j)(i,j\in E_0)$，而且

$$\sum_i \mu(i)K(i,\ j)=\sum_{i\in E_0}\mu(i)\ \widetilde{K}(i,j),\ j\in E_0.$$

因此，不妨假定，链$P=(p_{ij},i,j\in E)$的每个状态都非常返，然后证明(25)极限存在且有限。

令 $u(j)=\sum_i \mu(i)K(i,j)$，注意(23)有$u\eta=\mu G$，而μG是过份测度。往证$u\in K_\gamma$。实际上，由(15)，

$$\begin{aligned}E_\gamma u(x_\tau)&=\sum_j P_\gamma(x_\tau=j)u(j)=\sum_j u(j)\eta(j)L_D(j)\\&=\sum_{i,j}\mu(i)G(i,j)L_D(j).\end{aligned}$$

由(14)，

$$E_{\gamma}u(x_{\tau})\leqslant\sum_{i}\mu(i)<\infty.$$

从而

$$sup\{E_{\gamma}u(x_{\tau_D}):\ 有穷D\subset E\}\leqslant\sum_{i}\mu(i)<\infty.$$

即$u\in K_{\gamma}$。利用定理5即得定理的结论。证完。

（四）马亭边界

在马亭边界理论中，我们将不可约常返类E_a中的每个状态都等同地视为同一状态ξ_a，并仍记

$$E=E_0\cup\{\xi_a,a\in\mathscr{A}\}. \tag{28}$$

将(28)中的E任意排序：$E=\{e_1,e_2,\cdots\}$，记$N(e_m)=m$。

令

$$d(i,j)=|2^{-N(i)}-2^{-N(j)}|+\sum_{s}|K(s,i)-K(s,j)|A_s2^{-N(s)}. \tag{29}$$

其中A_j由(21)确定。则d是E中的距离，d在E中导出离散拓扑，而且$d(i,j)\leqslant3$ $(i,j\in E)$。按距离d将E完备化而得完备距离空间E^*。

定义6 称$\partial E=E^*-E_0$为链P的马亭边界。E^*中开集所产生的波雷尔（Borel）域记为ε^*。ε^*中的元素称为E^*中的波雷尔集。定义于波雷尔集Γ上的ε^*可测函数，称为Γ上的波雷尔可测函数。

显然，

$$\left.\begin{aligned}&\partial E=(\partial E)_1\cup(\partial E)_2,\\&(\partial E)_1=\{\xi_a,a\in\mathscr{A}\},(\partial E)_2=(\partial E)-(\partial E)_1.\end{aligned}\right\} \tag{29$_1$}$$

从距离d的定义显然有

定理7 E中无穷序列$\{j_n\}$是距离空间E中的基本序列的充要条件是

(i) $\lim\limits_{n\to\infty}N(j_n)$存在(有穷或无穷)；

(ii) 对每个$i\in E$，$\{K(i,j_n)\}$是实数柯西基本序列。

由定理7，$N(i)$可以将定义域E开拓至E^*上，因此

$$N(\xi)<\infty(\xi\in(\partial E)_1),\quad N(\xi)=\infty(\xi\in(\partial E)_2). \tag{30}$$

对于每个i，$K(i,j)$作为j的函数可连续地开拓至E^*，即

$$K(i,\xi)=\lim_{j\xrightarrow{d}\xi}K(i,j).$$

由(24)及法都引理，$K(\cdot,\xi)$是过份函数，且

$$K(i,\xi)\leqslant\frac{1}{A_i},\quad \sum_i\gamma_i K(i,\xi)\leqslant 1,\xi\in E^*. \tag{31}$$

(24)作为 (31) 的特殊情形，上面第二式对$\xi\in E_0\cup(\partial E)_1$成立等号。

这样，(29)也可以开拓至$i\in E^*$，$j\in E^*$也成立，从而将定理7中的E改为E^*，$\{j_n\}$是E^*中的无穷序列时，定理7仍然正确。

定理8 E^*是列紧空间，也是紧空间。

证 对于距离空间，列紧性与紧性的概念是一致的[1]，故只需证列紧性。

设$\{\xi_n\}$是无穷序列，$\xi_n\in E^*$。则必可以取$\{\xi_n\}$的子序列，仍记为$\{\xi_n\}$，使得$\lim\limits_{n\to\infty}N(\xi_n)$存在(有穷或无穷)。由于(31)，对每个$i\in E$，有

$$K(i,\xi_n)\leqslant\frac{1}{A_i}.$$

利用对角线法则，可以选取 $\{\xi_n\}$ 的子列 $\{\xi_{nn}\}$，使对每个$i\in E$，$\{K(i,\xi_{nn})\}$都是实数柯西基本列，按照定理8前面的一段话，序列$\{\xi_{nn}\}$在E^*中必收敛于某点$\xi\in E^*$。此即列紧性。证完。

定理9 对P_γ几乎一切$\omega\in\Omega$，或者

$$x_\beta\in H,\text{ 如果}\beta<\infty \tag{32}$$

或者存在极限

$$d-\lim_{n\to\infty}x_n=x_\infty\in\partial E,\text{ 如果}\beta=\infty. \tag{33}$$

1) 关肇直：泛函分析讲义，第一章，§3，定理2.

这里 $H=\left\{i;\ \sum_j p_{ij}<1\right\}$为中断状态集。

证 在Ω_F上，$x_\beta\in H$ 是显然的。在Ω_∞上，定理6已指出，P_γ几乎必然存在有穷极限

$$\lim_{n\to\infty}K(i,x_n)\quad \text{对一切}i\in E.$$

从而存在极限

$$d-\lim_{n\to\infty}x_n=x_\infty\in E^*.$$

如果链进入某常返类E_α，则$x_\infty=\xi_\alpha\in\partial E$. 否则，链将总是在$E_0$中运动，由于链在每个非常返状态 $i\in E_0$ 中停留的次数总是有限的，因而$x_\infty\bar{\in}E_0$，所以$x_\infty\in E^*-E_0=\partial E$. 证完。

（五）终极状态的分布

依定理9，终极状态x_β是P_γ几乎确定的。记x_β的分布为

$$\mu(\Gamma)\equiv\mu_1(\Gamma)=P_\gamma(x_\beta\in\Gamma),\Gamma\in\varepsilon^*. \tag{34}$$

则μ的质量分布在$H\cup(\partial E)$上。

定理10 设u为E^*上的连续函数或非负Borel函数。则

$$E_i u(x_\beta)=\int_{H\cup(\partial E)}K(i,\xi)u(\xi)\mu(d\xi), \tag{35}$$

$$E_i u(x_\beta)1_{\beta<\infty}=\sum_j G(i,j)u_j\left(1-\sum_s p_{js}\right), \tag{36}$$

$$E_i u(x_\beta)1_{\beta=\infty}=\int_{\partial E}K(i,\xi)u(\xi)\mu(d\xi). \tag{37}$$

特别地

$$P_i(x_\beta\in\Gamma)=\int_\Gamma K(i,\xi)\mu(d\xi),\ \Gamma\subset H\cup(\partial E) \tag{38}$$

$$\begin{aligned}P_i(x_\beta=j)&=P_i(x_\beta=j,\beta<\infty)\\&=G(i,j)\left(1-\sum_s P_{js}\right),j\in E_0,\end{aligned} \tag{39}$$

$$\begin{aligned}P_i(x_\beta\in\Gamma)&=P_i(x_\beta\in\Gamma,\beta=\infty)\\&=\int_\Gamma K(i,\xi)\mu(d\xi),\ \Gamma\subset\partial E,\end{aligned} \tag{40}$$

$$\mu(j)\equiv P_\gamma(x_\beta=j)=\eta(j)\Big(1-\sum_s p_{js}\Big),\ j\in E_0. \tag{41}$$

证 当$\xi_a\in(\partial E)_1$时

$$P_i(x_\beta=\xi_a)=f_i(\xi_a),$$

$$\mu(\xi_a)\equiv P_\gamma(x_\beta=\xi_a)=\sum_s \gamma_s f_s(\xi_a).$$

故

$$P_i(x_\beta=\xi)=K(i,\xi)\mu(\xi),\xi\in(\partial E)_1.$$

于是

$$E_i u(x_\beta)1_{(x_\beta\in(\partial E)_1)}=\int_{(\partial E)_1}K(i,\xi)u(\xi)\mu(d\xi). \tag{42}$$

其次，当$j\in E_0$时，

$$P_i(x_\beta=j)=P_i(x_\beta=j,\beta<\infty)$$

$$=\sum_{n=0}^{\infty}P_i(x_n=j,\ n=\beta)=\sum_{n=0}^{\infty}p_{ij}^{n}\Big(1-\sum_s p_{js}\Big).$$

由此得(39)。由(39)得(36)和(41)，并且由(39)及(41)得

$$E_i u(x_\beta)1_{(\beta<\infty)}=\int_H K(i,\xi)u(\xi)\mu(d\xi). \tag{43}$$

再次，比较(13)(15)(22)，我们得

$$P_i(x_\tau=j)=K(i,j)P_\gamma(x_\tau=j),j\in E. \tag{44}$$

于是

$$E_i u(x_\tau)=\sum_j u(j)P_i(x_\tau=j)=\sum_j u(j)K(i,j)P_\gamma(x_\tau=j)$$

$$=E_\gamma K(i,x_\tau)u(x_\tau). \tag{45}$$

令$D\uparrow E$。则当$x_\beta\in H\cup(\partial E)_2$时，$\tau_D\uparrow\beta$，$x_{\tau_D}\to x_\beta$；当$x_\beta\in(\partial E)_1$时，只要$D$充分大即$D$与$(\partial E)_1$有非空交时，$\tau_D=\infty$，按约定$x_{\tau_D}=\Delta$，$u(\Delta)=0$。这样，当$u$是$E^*$上的连续函数时，注意(31)第一式，从(45)得

$E_i u(x_\beta)1_{(x_\beta\in H\cup(\partial E)_2)}$

$$= E_r K(i,\ x_\beta) u(x_\beta) 1_{(x_\beta \in H \cup (\partial E)_r)}$$

$$= \int_{H \cup (\partial E)_r} K(i,\ \xi) u(\xi) \mu(d\xi). \tag{46}$$

从（42）(46）得（35）对$H \cup (\partial E)$上的连续函数u成立，从而得出对非负Borel函数u也成立。由（35）(43）得（37）。证完。

定理11 设u为E^*上的连续函数，则

$$\int_{E^*} u(\xi) \mu(d\xi) = E_r u(x_\beta) = \sum_j u(j) \eta(j) (1 - \sum_s p_{js})$$

$$+ \lim_{n \to \infty} \sum_{i,j} r(i) p_{ij}^n u(j). \tag{47}$$

证。 当u是E^*上的连续函数时，

$$E_i u(x_\infty) = \lim_{n \to \infty} E_i u(x_n) = \lim_{n \to \infty} (p^n u)_i,$$

由此及(39)得(47)。

（六）h-链和过份函数的马事表现

设h为γ可积的P过份函数，即$\sum_i \gamma_i h_i < \infty$。由于定理$3_1$，$h$是有限值的。记

$$E^h = \{i;\ h_i > 0\},\quad p_{ij}^h = \frac{p_{ij} h_j}{h_i},\quad i,\ j \in E^h. \tag{48}$$

则由h的过份性易得

$$\left.\begin{aligned} p_{ij} &= 0,\quad i \overline{\in} E^h,\ j \in E^h, \\ f_{ij} &= 0,\quad i \overline{\in} E^h,\ j \in E^h. \end{aligned}\right\} \tag{49}$$

而$P^h = (p_{ij}^h,\ i,\ j \in E^h)$是非负且行和不超过壹的矩阵。以$P^h$为一步转移矩阵的马氏链$X_T = \{x_n, n \leqslant \beta\}$称为$h$-链。当$h \equiv 1$或常数时，$h$-链就化为以$P$为一步转移矩阵的马氏链。以后$h$-链的一切特征均冠以$h$。例如$G^h = \sum_{n=0}^{\infty} P^{hn}$，测度$P_i^h$等等。

显然测度$\gamma_i^h = \gamma_i h_i$（$i \in E^h$）对于$h$-链是标准测度，而且

$$P_{ij}^{hn} = \frac{P_{ij}^n h_j}{h_i},\quad i,\ j \in E^h \tag{50}$$

从而

$$G^h(i, j) = \frac{G(i, j)h_j}{h_i}, \quad i, j \in E^h \tag{51}$$

$$\eta^h(j) = \eta(j)h_j, \quad j \in E^h \tag{52}$$

其中$\eta^h = \gamma^h G^h$，$\eta = \gamma G$。

由（51）可见，对于h-链，定理2.2中状态分解定理取下面形式：

$$E^h = E_0^h \cup \left(\bigcup_{a \in \mathscr{A}} E_a^h\right). \tag{53_1}$$

其中

$$E_0^h = E^h \cap E_0, \quad E_a^h = E^h \cap E_a, \quad a \in \mathscr{A}. \tag{53_2}$$

记h-链的中断状态集为H^h，即

$$H^h = \left\{i: \ i \in E^h, \sum_{j \in E^h} P_{ij}^h < 1\right\}$$

$$= \left\{i: \ i \in E^h, \sum_{j \in E} P_{ij} h_j < h_i\right\}. \tag{53_3}$$

则有

$$H^h \subset E_0^h \subset E_0. \tag{53_4}$$

设i，$j \in E^h$。由(49)，

$$f_{ij}^{h\,n} = \sum_1 P_{ij_1}^h P_{j_1 j_2}^h \cdots P_{j_{n-1} j}^h$$

$$= \frac{1}{h_i} \sum_2 P_{ij_1} P_{j_1 j_2} \cdots P_{j_{n-1} j} h_j$$

$$= \frac{f_{ij}^n h_j}{h_i}, \tag{53}$$

其中$\sum_1$表示对$j_l \neq j$，$j_l \in E^h$，$1 \leqslant l \leqslant n-1$求和，而$\sum_2$表示对$j_l \neq j$，$j_l \in E$，$1 \leqslant l \leqslant n-1$求和。从（52）得

$$f_{ij}^h = \frac{f_{ij} h_j}{h_i}, \quad i, j \in E^h. \tag{54}$$

从而

$$A_j^h = \sum_{i \in E^h} \gamma_i^h f_{ij}^h = \sum_{i \in E} \gamma_i f_{ij} h_j = A_j h_j, \quad j \in E^h. \tag{55}$$

于是P^h对应于γ^h的马亭核为

$$K^h(i, j) = \frac{f_{ij}^h}{A_j^h} = \frac{K(i, j)}{h_i}, \quad i, j \in E^h. \tag{56}$$

由上式看出，如果$\{K^h(i, j_n)\}$是实数柯西序列，则$\{K(i, j_n)\}$也是。因此由定理7看出，h-链的马亭拓扑与1-链的马亭拓扑一致，因而h-链的马亭列紧空间E^{h*}可以看作是E^*的闭子空间，即E^{h*}是E^h在E^*中的闭包。而h-链的马亭边界∂E^h就是E^h在E^*中的边界，即

$$\partial E^h = (\partial E) \cap E^{h*}.$$

设$\{x_n, n \leqslant \beta\}$是$h$-链，初始分布为$\gamma^h$。记终极状态$x_\beta$的分布为$\mu_h$，$\mu_h$的支集含于$H^h \cup \partial E^h$，但$\mu_h$可视为$E^{h*}$上，更可视为$E^*$上的测度，即

$$\mu_h(\Gamma) = P_{\gamma^h}^h(x_\beta \in \Gamma), \quad \Gamma \in \mathscr{E}^*. \tag{57}$$

则

$$\mu_h(E^*) = \sum_{i \in E^h} \gamma_i^h P_i^h(x_\beta \in E^*) = \sum_{i \in E^h} \gamma_i^h = \sum_i \gamma_i h_i. \tag{58}$$

$$\mu_h(E^* - E^{h*}) = 0. \tag{59}$$

定理12 设u是E^*上的连续函数或非负Borel函数，则当$i \in E^h$时，

$$E_i^h u(x_\beta) = \frac{1}{h_i} \int_{E^*} K(i, \xi) u(\xi) \mu_h(d\xi), \tag{60}$$

$$E_i^h u(x_\beta) L_{\beta < \infty} = \frac{1}{h_i} \sum_j G(i, j) u(j) (h_j - \sum_s p_{js} h_s), \tag{61}$$

$$E_i^h u(x_\beta) L_{\beta = \infty} = \frac{1}{h_i} \int_{\partial E} K(i, \xi) u(\xi) \mu_h(d\xi). \tag{62}$$

特别地，$i \in E^h$时

$$p_i^h(x_\beta \in \Gamma) = \frac{1}{h_i} \int_\Gamma K(i, \xi) \mu_h(d\xi), \tag{63}$$

$$p_i^h(x_\beta=j)=p_i^h(x_\beta=j,\ \beta<\infty)$$
$$=\frac{1}{h_i}G(i,\ j)\Big(h_i-\sum_s p_{js}h_s\Big),\ j\in E_0, \tag{64}$$
$$p_i^h(x_\beta\in\Gamma)=p_i(x_\beta\in\Gamma,\ \beta=\infty)$$
$$=\frac{1}{h_i}\int_\Gamma K(i,\ \xi)\mu_h(d\xi),\ \Gamma\subset\partial E, \tag{65}$$
$$\mu_h(j)=\eta(j)\Big(h_j-\sum_s p_{js}h_s\Big),\ j\in E_0 \tag{66}$$

如u是$\bar{E}^*$上的连续函数，则

$$\int_{E^*}u(\xi)\mu_h(d\xi)=\sum_j u_j\eta_j\Big(h_j-\sum_s p_{js}h_s\Big)$$
$$+\lim_{n\to\infty}\sum_{i,j}\gamma_i p_{ij}^n h_j u_j. \tag{67}$$

证. 应用定理10和11的结论于h-链，并注意（56）及$E_0^h\subset E_0$即可。

定理13 设h是γ可积过份函数。则

$$h_i=\int_{E^*}K(i,\ \xi)\mu_h(d\xi) \tag{68}$$
$$=\sum_j G(i,\ j)\Big(h_j-\sum_s p_{js}h_s\Big)+\int_{\partial E}K(i,\ \xi)\mu_h(d\xi) \tag{68_1}$$

证. 在（60）中令$u\equiv1$得（68）对$i\in E^h$成立。设$i\bar{\in}E^h$，即$h_i=0$。由（49），有$K(i,\ j)=0$（$j\in E^h$），从而$K(i,\ \xi)=0$（$\xi\in E^{h*}$）。注意(59)，

$$\int_{E^*}K(i,\ \xi)\mu_h(d\xi)=\int_{E^{h*}}K(i,\ \xi)\mu_h(d\xi)=0.$$

于是（68）对$i\bar{\in}E^h$也成立。

利用（66），从(68)得(68_1)。证完。

注. 测度μ_h称为h的谱测度，（68）称为过份测度h的马亭表现。

（七）本质马丁边界

设$j\in E$，则$K(\cdot, j)$是过份测度，因而可以确定测度$\mu_{K(\cdot, j)}$。由(58)及(24)，

$$\mu_{K(\cdot;j)}(E^*) = \sum_i r_i K(i, j) = 1。 \tag{69}$$

定理14 设δ_j为集中于j的单位测度，则

$$\mu_{K(\cdot;j)} = \delta_j, \quad j\in E_0 \cup (\partial E)_1。 \tag{70}$$

证。 记$h = K(\cdot, j)$。当$j\in E_0$时，则

$$(ph)_i = \sum_s p_{is} K(s, j) = \frac{G(i, j) - \delta_j(i)}{\eta(j)}$$

$$= h_i - \frac{\delta_j(i)}{\eta(j)}。$$

$$h_j - (ph)_j = \frac{1}{\eta(j)}。$$

于是按(66)，$\mu_{K(\cdot;j)}(j) = 1$。注意(69)得(70)。

当$j = \xi_a$（$a\in \mathscr{A}$）时，$K(i, \xi_a) = \dfrac{f_i(\xi_a)}{A(\xi_a)}$，

$$\mu_h(\xi_a) = \sum\nolimits_1 \gamma_i^h p_i^h(x_\beta = \xi_a)$$

$$= \sum\nolimits_1 \gamma_i^h f_i^h(\xi_a)。$$

其中$\sum_1$表示在E^h上求和。考虑到(54)及$h(\xi_a) = K(\xi_a, \xi_a) = 1/A(\xi_a)$，故

$$\mu_h(\xi_a) = \sum\nolimits_1 \gamma_i h_i \frac{f_i(\xi_a) h(\xi_a)}{h_i}$$

$$= \sum_i \gamma_i f_i(\xi_a) \frac{1}{A(\xi_a)} = 1。 \tag{71}$$

再由(69)得$\mu_{K(\cdot;\xi_a)} = \delta_{\xi_a}$。证完。

定义7 令

$$B = \{\xi:\ \xi\in \partial E,\ \mu_{K(\cdot;\xi)} = \delta_\xi\}。 \tag{72}$$

称B为本质马亭边界。如果$\xi\in B$且$\mu(\xi)>0$，称ξ为原子边界点。一切原子边界点组成的集合B_1称为链的原子边界，$B_2=B-B_1$称为链的非原子边界。$B=B_1\cup B_2$。

由定理14，$\xi_a\in B$。又因为

$$\mu(\xi_a)=p_\gamma(x_\beta=\xi_a)=\sum_i \gamma_i p_i(x_\beta=\xi_a)$$

$$=\sum_i \gamma_i f_i(\xi_a)=A(\xi_a)>0,$$

故每个常返边界点ξ_a是原子边界点，即

$$(\partial E)_1\subset B_1. \tag{73}$$

定理15 设$\xi\in B$，则$K(\cdot,\ \xi)$是调和函数，且

$$\sum_i \gamma_i K(i,\ \xi)=1,\ \xi\in B. \tag{74}$$

证. 记$h=K(\cdot,\ \xi)$，因$\mu_h=\delta_\xi$，故由（68_1）得$Gf=0$，其中$f=h-ph$。于是

$$\sum_j \eta(j)f(j)=\sum_j \gamma_j(Gf)_j=0,$$

从而$f=0$，即$K(\cdot,\ \xi)(\xi\in B)$调和。又由（58）及（31），当$\xi\in B$时，

$$1=\mu_{K(\cdot,\xi)}=\sum_i \gamma_i K(i,\ \xi)\leqslant 1.$$

从而得(74)。证完。

定理16 B是∂E的Borel子集。对任何γ可积过份函数h，有

$$\mu_h(\partial E-B)=0.$$

证. 因$\xi\in\partial E$时，由(58)及(31)，

$$\mu_{K(\cdot,\xi)}(E^*)=\sum_i \gamma_i K(i,\ \xi)\leqslant 1,$$

因此

$$B=\{\xi:\ \xi\in\partial E,\ \mu_{K(\cdot,\xi)}(\xi)=1\}. \tag{75}$$

对$\xi\in\partial E$，由(67)，

$$\mu_{K(\cdot,\xi)}(\xi) = \lim_{m\to\infty}\int_{E^*} e^{-md(\zeta,\xi)}\mu_{K(\cdot,\xi)}(d\zeta)$$

$$= \lim_{m\to\infty}\sum_j e^{md(j,\xi)}\eta_j\Big[K(j,\ \xi)-\sum_s p_{js}K(s,\ \xi)\Big]$$

$$+\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i,j}\gamma_i p_{ij}^n K(j,\ \xi)e^{-md(\xi,j)}. \qquad (75_1)$$

由此知B是Borel集。

设$\xi\in\partial E$，简记$K=K(\cdot,\ \xi)$。设φ和ψ均为E^*上的连续函数，$m\geqslant 0$，$n\geqslant 0$。则

$$E_\gamma^h\varphi(x_n)\psi(x_{n+m}) = \sum_{i,j,s}\gamma_i h_i p_{ij}^{h\,n}\varphi(j)p_{js}^{h\,m}\psi(s)$$

$$= \sum_{i,j}\gamma_i p_{ij}^n h_j\varphi(j)E_j^h\psi(x_m).$$

令$m\to\infty$，利用(62)我们有

$$E_\gamma^h\varphi(x_n)\psi(x_\infty)$$

$$= \sum_{i,j}\gamma_i p_{ij}^n h_j\varphi(j)\int_{\partial E}K^h(j,\ \xi)\psi(\xi)\mu_h(d\xi)$$

$$= \int_{\partial E}\sum_{i,j}\gamma_i K(i,\ \xi)P_{ij}^K(n)\varphi(j)\psi(\xi)\mu_h(d\xi)$$

$$= \int_{\partial B}E_\gamma^K\varphi(x_n)\psi(\xi)\mu_h(d\xi). \qquad (76)$$

考虑到

$$E_\gamma^K\varphi(x_\infty) = \int_{\partial E}\varphi(\zeta)\mu_K(d\zeta),$$

在(76)中令$n\to\infty$得

$$\int_{\partial E}\varphi(\xi)\psi(\xi)\mu_h(d\xi)$$

$$= \int_{\partial E}\Big[\int_{\partial B}\varphi(\zeta)\mu_K(d\zeta)\Big]\psi(\xi)\mu_h(d\xi).$$

由于连续函数ψ可任意选取，由上式得

$$\varphi(\xi)=\int_{\partial E}\varphi(\zeta)\mu_{K(\cdot,\xi)}(d\zeta).$$

对μ_h几乎一切$\xi\in\partial E$成立。特别地，取φ为

$$\varphi_m(\zeta)=e^{-md(\zeta,\xi)},\quad m=1,\ 2,\ 3,\ \cdots$$

得

$$1=\int_{\partial E}e^{-md(\zeta,\xi)}\mu_{K(\cdot,\xi)}(d\zeta),\quad m=1,\ 2,\ \cdots$$

对μ_h几乎一切$\xi\in\partial E$成立。在上式中令$m\to\infty$得

$$1=\mu_{K(\cdot,\xi)}(\xi)$$

对μ_h几乎一切$\xi\in\partial E$成立。由(75)即得$\mu_h(\partial E-B)=0$。证完。

系 记h-链的本质马亭边界为B^h，则

$$B^h=B\cap E^{h*}.$$

证 类似于(75)，

$$B^h=\{\xi:\ \xi\in\partial E^h,\ \mu_{K^h(\cdot,\xi)}(\xi)=1\}.$$

而$\partial E^h=(\partial E)\cap E^{h*}$。当$\xi\in\partial E^h$时，按$(75_1)$可计算$\mu_{K^h(\cdot,\xi)}(\xi)$：

$$\begin{aligned}\mu_{K^h(\cdot,\xi)}(\xi)=\lim_{m\to\infty}\sum_{j\in E^h}e^{md(j,\xi)}\eta_j^h\Big[K^h(j,\ \xi)\\-\sum_{s\in E^h}p_{js}^hK^h(s,\ \xi)\Big]\\+\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i,j\in E^h}\gamma_i^hp_{ij}^{h\,n}K^h(j,\ \xi)e^{-md(j;\xi)}.\end{aligned}$$

注意到（49）——（52）和（56），上式右方化为(75_1)右方，因而$\mu_{K(\cdot,\xi)}(\xi)=\mu_{K^h(\cdot,\xi)}(\xi)$，由此可得证$B^h=B\cap E^{h*}$。证完。

由于定理16，定理9可如下加强。

定理17 设$X_T=\{x_n,\ n<\beta\}$是1-链，则对p_r几乎一切$\omega\in\Omega$，或者

$$x_\beta\in H,\ \text{如果}\beta<\infty \tag{77}$$

或者存在极限

$$d-\lim_{n\to\infty}x_n=x_\infty\in B,\ \text{如果}\beta=\infty. \tag{78}$$

(60)(62)可加强为

$$E_i^h u(x_\beta) = \frac{1}{h_i}\int_{E_0\cup B} K(i,\ \xi)u(\xi)\mu_h(d\xi), \tag{79}$$

$$E_i^h u(x_\beta)1_{\beta<\infty} = \frac{1}{h_i}\int_B K(i,\ \xi)u(\xi)\mu_h(d\xi). \tag{80}$$

特别地

$$E_i u(x_\beta) = \int_{E_0\cup B} K(i,\ \xi)u(\xi)\mu(d\xi), \tag{81}$$

$$E_i u(x_\beta)1_{\beta<\infty} = \int_B K(i,\ \xi)u(\xi)\mu(d\xi). \tag{82}$$

(八) 马丁表现的唯一性

定理18 每个γ可积过份函数h可唯一地表示为

$$h_i = \int_{E_0\cup B} K(i,\ \xi)\lambda(d\xi) \tag{83}$$

其中λ是Borel集$E_0\cup B$上的全有限测度，因而$\lambda=\mu_h$。反之，任给$E_0\cup B$上的全有限测度λ，(83) 确定γ可积过份函数。此函数调和当且仅当$\lambda(E_0)=0$

证。 在(79)中令$u=1$得h有表现(83)，其中$\lambda=\mu_h$是$E_0\cup B$上的测度，μ_h的全有限性由(58)得出。

现设h有表现(83)，往证$\lambda=\mu_h$。对E^*上的连续函数u，分别应用h和$K(\cdot,\ \xi)$于(67)，注意(83)，我们得

$$\int_{E^*} u(\xi)\mu_h(d\xi) = \int_{E_0\cup B}\left[\int_{E^*} u(\xi)\mu_{K(\cdot,\zeta)}(d\xi)\right]\lambda(d\zeta).$$

但$\mu_{K(\cdot,\zeta)}=\delta_\zeta$ $(\zeta\in E_0\cup B)$，故

$$\int_{E^*} u(\xi)\mu_h(d\xi) = \int_{E_0\cup B} u(\xi)\lambda(d\xi).$$

因μ_h的质量全部集中于$E_0\cup B$上，由u的任意性，从上式得$\lambda=\mu_h$。

因$K(\cdot,\ \xi)$是过分函数，且 (31) 成立，因此(83) 确定γ可积的过份函数。由定理15，$K(\cdot,\ \xi)$ $(\xi\in B)$ 调和，故如$\lambda(E_0)=0$，则(83)中的h是调和的。反之，设(83)中的h是调和的，由(66) 得$\mu_h(j)=0$对一切$j\in E_0$，即$\lambda(E_0)=\mu_h(E_0)=0$。证完。

（九）极小过份函数

回忆极小过份函数的定义2。我们有：

定理19 γ可积极小过份函数的一般形式为$CK(\cdot,\xi)$，$\xi\in E_0\cup B$，C是常数。

证 由(68)看出，$\mu_{h_1+h_2}=\mu_{h_1}+\mu_{h_2}$。设$\xi\in E_0\cup B$且$K(\cdot,\xi)=h_1+h_2$，而且$h_1$和$h_2$均为过份函数。则$\mu_{h_1+h_2}=\mu_h(\cdot,\xi)=\delta_\xi$。故

$$\mu_{h_1}(E^*-\xi)+\mu_{h_2}(E^*-\xi)=\delta_\xi(E^*-\xi)=0,\text{从而}$$

$$\mu_{h_e}(E^*-\xi)=0(e=1,2)。$$

按h_e的谱表现(68)，

$$h_e(i)=\int_{E^*}K(i,\zeta)\mu_{h_e}(d\zeta)$$

$$=K(i,\xi)\mu_{h_e}(\xi)=C_eK(i,\xi)。$$

其中$C_e=\mu_{h_e}(\xi)$是常数。因此$K(\cdot,\xi)(\xi\in E_0\cup B)$为极小过份函数。

其次，设h为γ可积极小过份函数，并设μ_h为其谱测度。不妨设$h\neq0$。由(58)及定理3，$\mu_h(E_0\cup B)>0$。故存在$\xi\in E_0\cup B$使μ_h在ξ的任何邻域中有正测度。令$U_n=\{\zeta:\ d(\zeta,\xi)<\frac{1}{n}\}$，

$h_n=\int_{U_n}K(\cdot,\zeta)\mu_h(d\zeta)$。则$h_n$和

$$h-h_n=\int_{(E_0\cup B)-U_n}K(\cdot,\zeta)\mu_h(d\zeta)$$

都是过份函数。由h的极小性，$h_n=C_nh$（C_n为常数），从而$\sum_i\gamma_ih_n(i)=C_n\sum_i\gamma_ih(i)$。又由$h_n$的表现的唯一性得

$$\mu_{h_n}(\Gamma)=\mu_h(U_n\cap\Gamma)。$$

考虑到(58)，

$$\mu_h(U_n)=\mu_{h_n}(E^*)=\sum_i\gamma_ih_n(i),$$

$$\mu_h(E^*)=\sum_i\gamma_ih(i)。$$

这样

$$h_i=\frac{\mu_h(E^*)}{\mu_h(U_n)}\int\limits_{U_n}K(i,\zeta)\mu_h(d\zeta)。$$

令$n\to\infty$得$h_i=CK(i,\xi)$，其中$C=\mu_h(E^*)$。证完。

定理20 本质马亭边界

$$B=\{\xi: \xi\in\partial E,\ K(\cdot,\xi)\text{是极小调和函数},$$

$$\sum_i \gamma_i K(i,\xi)=1\}。\tag{84}$$

证 记右方的集合为C。由定理15和19，$B\subset C$。

设$\xi\in C$。则$h\equiv K(\cdot,\xi)$为γ可积。记其谱测度为μ_h。显然h有表现：

$$h_i=\int\limits_{E_1\cup B}K(i,\zeta)\delta_\xi(d\zeta)$$

由唯一性定理18，$\mu_h=\delta_\xi$，即$\xi\in B$，$C\subset B$。证完。

（十） 终极域和终极随机变量

无穷维函数$f(j_0,j_1,j_2,\cdots)(j_k\in E,\ k=0,1,2,\cdots)$称为不变的，如果对任意$j_k\in E$，$k=0,1,2,\cdots$，有$f(j_0,j_1,j_2,\cdots)=f(j_1,j_2,j_3,\cdots)$。

定义8 定义在Ω_∞上的函数Φ称为链的终极随机变量，如果存在不变函数f使得

$$\Phi(\omega)=f\{x_n(\omega),x_{n+1}(\omega),\cdots\},\ \omega\in\Omega_\infty,$$

对$n=0$从而对一切$n\geqslant0$成立。约定终极随机变量Φ在$\Omega_F=\Omega-\Omega_\infty$上取值为0。称集合$\Lambda\subset\Omega_\infty$为终极集，如果示性函数$1_\Lambda$是终极随机变量。由终极集组成的$\Omega_\infty$上的波雷尔域记为$\mathscr{B}_\infty$，称为终极域。

设$\Lambda_1,\Lambda_2\in\mathscr{F}$，如果$P_\gamma\{(\Lambda_1-\Lambda_2)+(\Lambda_2-\Lambda_1)\}=0$，则记$\Lambda_1\doteq\Lambda_2$。如果$\Phi_1,\Phi_2$为随机变量，且$P_\gamma(\Phi_1\neq\Phi_2)=0$，则记$\Phi_1\doteq\Phi_2$。

定理21 非零终极集Λ和几乎闭集A（除瞬时集不计外）在下列条件下一一对应：

$$\wedge \doteq \mathscr{S}(A). \tag{85}$$

A可以取为$A=\{i:\ P_i(\Lambda)>\varepsilon\}$，$0<\varepsilon<1$。

证 见Chung[1, I § 17, 定理1]。

定理22 非负有界P调和函数u与非负有界终极随机变量Φ在下列条件下一一对应：

$$u_i=E_i\phi,\quad \phi=\lim_{n\to\infty}u(x_n) \tag{86}$$

证 见Chung[1, I § 17, 定理5]。

定义9 非零终极集$\Lambda\in\mathscr{B}_\infty$称为原子的，如果$\Lambda$不能分成两个非零终极集的并，非零终极集$\Lambda\in B_\infty$称为完全非原子的，如果对任意$\Lambda_1\subset\Lambda$，$\Lambda_1\in\mathscr{B}_\infty$，则$\Lambda_1$都不是原子的。

定理23 Ω_∞有下列分解

$$\Omega_\infty=\wedge_0+\left(\bigcup_{a\in\mathscr{A}}\wedge_a\right). \tag{87}$$

其中$\wedge_0$为完全非原子终极集，它可以不出现。$\mathscr{A}$为空集，有限集或可列无限集，且$0\overline{\in}\mathscr{A}$。$\wedge_a(a\in\mathscr{A})$为原子终极集。分解是唯一的。

证 利用定理2.3和将要证明的定理5.6而获证。

(十一)马亭流入边界

前面论及的马亭边界实际上是马亭流出边界，即描述过程是如何从有限状态跑向无穷的。至于描述过程如何从无穷来到有穷状态，则需要马亭流入边界。

设马氏链$P=(P_{ij},i,j\in E)$的状态分解为(2.15)，E_0是非常返状态集，记

$$E_{00}=\{i:\ i\in E_0,\ 存在j\in E-E_0使i\Longrightarrow j\},$$

$$E_1=E-E_{00},\quad P_1=(P_{ij},i,j\in E_1)$$

侯[1，第八章]指出：P的任意有限过份测度$(a_j,\ j\in E)$在E_{00}上必定为零，即$a_j=0(j\in E_{00})$，而且存在在E_1上为正的过份测度，我们指定一个这样的P过份测度$(a_j,\ j\in E)$，则测度$(V_j,\ j\in E)$是P的有限过份（有限调和）测度的充要条件是$V_j=0$ $(j\in$

E_{00})及$\left(\frac{V_j}{\alpha_j},\ j\in E_1\right)$是$\widetilde{P}$的有限过份(有限调和)函数。这里

$$\widetilde{P}=(\widetilde{P}_{ij},\ i,j\in E_1),\quad \widetilde{P}_{ij}=\frac{\alpha_j P_{ji}}{\alpha_i}.$$

选取$\widetilde{P}$的标准测度$\widetilde{\gamma}$。根据$\widetilde{\gamma}$和$\widetilde{P}$，我们可以确定$\widetilde{P}$的马亭边界$\widetilde{\partial E_1}$，本质马亭边界$\widetilde{B}$，原子边界$\widetilde{B}_1$和非原子边界$\widetilde{B}_2$。分别称$\widetilde{\partial E_1}$，$\widetilde{B}$，$\widetilde{B}_1$和$\widetilde{B}_2$为P的马亭流入边界，本质马亭流入边界，原子流入边界，非原子流入边界，上述诸边界当然与α及$\widetilde{\gamma}$有关。因此，对于流入边界及过份测度的研究，可以借助于对流出边界及过份函数的研究而得到。我们不详述了。

§4. 中断位势的概率表现

定义1 称非负(包括$+\infty$)函数u为链P的位势，如果存在非负函数v使$u=Gv=\sum_{n=0}^{\infty}P^n v$，即

$$u_i=\sum_j G(i,j)v_j,\quad i\in E \tag{1}$$

非零位势u称为极小的，如果$u=u^1+u^2$，u^1,u^2为位势，则$u=C^a u^a$ $(a=1,\ 2,\ )$，c^a为常数。

如果$u=Gv$，则$Pu=PGv=(G-I)v=u-v$。因此，如$u_i<\infty$，则$v_i=u_i-(Pu)_i$由u唯一决定。

定义2 称u为P的中断位势，如果$u=Gv$，并且$v_i=0$ $(i\in E-H)$，H为中断状态集。

定理1 u是P的中断位势的充要条件是：存在非负函数f_a $(a\in H)$使得

$$u_i=E_i\{f_{x(\beta)},\ \beta<\infty\}=\sum_{a\in H}f_a u_i^a,\quad i\in E. \tag{2}$$

其中

$$u_i^a=P_i\{x_\beta=a\}=G(i,a)d_a,\quad a\in H, \tag{3}$$

是极小位势，而d_a是状态a的中断量。

如果u有限，则u与f的对应是一一的，并且$f_a(a\in H)$也是有限的。

证 充分性。由(3.77)，

$$u_i=E_i\{f_{x(\beta)},\ \beta<\infty\}=E_i\{f_{x(\beta)},\ \beta<\infty,\ x(\beta)\in H\}$$

$$=\sum_{n=0}^{\infty}E_i\{f_{x(n)},\ \beta=n,x(n)\in H\}$$

$$=\sum_{n=0}^{\infty}\sum_{a\in H}P_{ia}^{n}d_af_a=\sum_{a\in H}G(i,a)d_af_a. \tag{4}$$

故由(2)确定的u是中断位势。特别，(3)成立。

往证u^a是极小位势。设$u^a=u_1+u_2$，$u_b=Gv_b$ $(b=1,2)$ 是位势。

故

$$d_a\delta_a(i)=(u^a-pu^a)_i=\sum_{b=1}^{2}(u_b-pu_b)_i$$

$$=v_1(i)+v_2(i),$$

从而得$v_b=c^b(d_a\delta_a)(b=1,\ 2,\ c^b\geqslant 0$为常数$)$，于是$u_b=c^bu^a\ (b=1,2)$。

必要性。设$u=Gv$是中断位势。取$f_a=\dfrac{v_a}{d_a}(a\in H)$。由(4)，知(2)成立。

对应的一一性由定义2前面的一段话得出，证毕。

§5. 逗留解，终极集，几乎闭集和边界

定理1 （非负）有界调和函数h与B上的非负有界波雷尔函数f按下列关系一一对应：

$$h_i=\int_B K(i,\xi)f(\xi)\mu(d\xi), \tag{1}$$

$$\lim_{n\to\infty} h(x_n) = f(x_\infty),\quad E_i f(x_\infty) = h_i. \tag{2}$$

其中测度μ是终极状态的分布，由(3.34)确定

证 依定理3.18，由(1)确定有界调和函数h。反之，设h为有界调和函数，不妨设其界为1。则h和$g=1-h$均过份，按定理3.18，$\mu_h(E_0)=0$，且

$$1=\int_{E_0\cup B} K(i,\xi)u(d\xi)=\int_{E_0\cup B} K(i,\xi)(\mu_h+\mu_g)(d\xi).$$

故$\mu=\mu_h+\mu_g$，μ_h关于μ有界为1的密度函数f，$\mu_h(d\xi)=f(\xi)\mu(d\xi)$。故(3.83)成为(1)。

改写(1)为$h_i=E_i\{f(x_\beta),\ \Omega_\infty\}$。因而(2)式由定理3.22得出，证毕。

定理2 记$\mathscr{B}_{x(\infty)}$为由集合$\{x(\infty)\in\Gamma\}$(Borel集合$\Gamma\subset B$)所产生的Ω_∞上的Borel域，则$\mathscr{B}_\infty=\mathscr{B}_{x(\infty)}$。

证 因为非负有界$\mathscr{B}_{x(\infty)}$可测函数Φ可以写成$f(x_\infty)$的形式，其中f是B上的非负有界波雷尔函数。故由(2)知$\Phi=f(x_\infty)$是$\mathscr{B}_\infty$可测的。即$\mathscr{B}_{x(\infty)}\subset\mathscr{B}_\infty$。

设Φ为非负有界的$\mathscr{B}_\infty$可测函数。依定理3.22，$h_i=E_i\Phi$为有界调和且$\Phi=\lim\limits_{n\to\infty}h_{x(n)}$。由定理1，存在$B$上的有界Borel可测函数$f$使(2)成立。因而$\Phi=f(x_\infty)$。于是$\Phi$为$\mathscr{B}_{x(\infty)}$可测，即$\mathscr{B}_\infty\subset\mathscr{B}_{x(\infty)}$，证毕。

定理3 对任意几乎闭集A，存在波雷尔集$\Gamma\subset B$，使

$$\mathscr{L}(A)\doteq\{x_\infty\in\Gamma\}. \tag{3}$$

除μ零集不计外，Γ是唯一确定的。

证 因$\mathscr{L}(A)\in\mathscr{B}_\infty$，利用定理2便得证本定理。

定义1 称非零的$u\in\mathscr{M}^+(1)$为逗留解，如果对任意$v\in\mathscr{M}^+(1)$，$v\leqslant u$，$v\leqslant\overline{u}-u$有$v=0$。这里

$$\overline{u}_i=P_i(\Omega_\infty)=P_i(\mathscr{L}(E))\in\mathscr{M}^+(1). \tag{4}$$

定理4 逗留解u与几乎闭集A(除瞬时集不计外)在下列条件

下一一对应：

$$u_i = P_i\{\mathscr{L}(A)\} \tag{5}$$

证 设u为逗留解，因而u调和，依定理3.22，存在非零的终极随机变量Φ：$0 \leqslant \Phi \leqslant 1$使$u_i = E_i\Phi$。显然

$$\psi = \min\{\Phi,\ (1-\Phi)1_{(\Phi<1)}\}$$

是终极随机变量，故$v_i = E_i\psi \in \mathscr{M}^+(1)$，而且$v \leqslant u$，$v \leqslant \overline{u} - u$。于是$v = 0$，从而$\psi \doteq 0$。由此得$\Phi \doteq 1_{(\Phi=1)}$。因$(\Phi = 1)$是非零的终极集，依定理3.21，存在几乎闭集$A$使$(\Phi = 1) \doteq \mathscr{L}(A)$，这样，

$$u_i = E_i\Phi = E_i 1_{(\Phi=1)} = E_i 1_{\mathscr{L}(A)} = P_i(\mathscr{L}(A))。$$

今设u由(5)确定。设$v \in \mathscr{M}^+(1)$，$v \leqslant u$，$v \leqslant \overline{u} - u$。依定理3.21和3.22，

$$1_{\mathscr{L}(E-A)} = \lim_{n\to\infty} u_{x(n)},\quad 1_{\mathscr{L}(E-A)} = \lim_{n\to\infty}\left[\overline{u}_{x(n)} - u_{x(n)}\right]。$$

且存在终极随机变量Φ：$0 \leqslant \Phi \leqslant 1$，使

$$v_i = E_i\Phi,\quad \Phi = \lim_{n\to\infty} v_{x(n)}。$$

因$v \leqslant \min(u,\ \overline{u} - u)$，故$\Phi \leqslant \min(1_{\mathscr{L}(A)},\ 1_{\mathscr{L}(E-A)}) = 0$。

所以$v_i = E_i\Phi = 0$，证毕。

定理5 逗留解u，非零的终极集$\wedge$，几乎闭集A(不计瞬时集之差别)，以及μ-非零的波雷尔集$\Gamma(\subset B$，除μ零集不计外)，在下列条件下一一对应：

$$u_i = P_i(\mathscr{L}(A)),\quad \wedge \doteq \mathscr{L}(A) \doteq \{x_\infty \in \Gamma\}。 \tag{6}$$

证 综合定理3和4以及定理3.21即可。

定义2 称逗留解u为极小的，如果u作为调和函数是极小的，逗留解u称为完全非极小的，如果对任意逗留解$v \leqslant u$，v都不是极小的。

显然，如果布勒克韦分解中完全非原子A_0出现，则$u_i^0 = P(\mathscr{L}(A_0))$是最大的完全非极小逗留解。

作为定理5的特款，我们有下面的定理6。

定理6 完全非极小逗留解u（$\leqslant u^0$），完全非原子终极集$\wedge$（$\subset \wedge_0$，见(3.87)），完全非原子几乎闭集$A(\subset A_0$，并不计瞬时

集之差别)，μ-非零的波雷尔集$\Gamma(\subset B_2$，并不计μ零集之差别) 按(6)一一对应。

极小逗留解u，原子终极集$\wedge$，原子几乎闭集A(不计瞬时集之差别)和原子边界点$\xi\in B_1$按下列关系一一对应：

$$\wedge \doteq \mathscr{L}(A) \doteq \{x_\infty = \xi\},\quad u_i = P_i\{\mathscr{L}(A)\}. \tag{7}$$

定理7 有界调和函数u，非负有界终极随机变量Φ，非负有界波雷尔函数f(定义在B上并不计μ零集上函数值之差别) 按下列关系一一对应：

$$u_i = E_i\Phi,\quad \Phi \doteq \lim_{n\to\infty} u_{x(n)},\quad f(x_\infty) \doteq \Phi. \tag{8}$$

证 综合定理3.22和定理1而得。

§6. 典范过程

设E具有离散拓扑。用单点"∞" ($\infty\in E$) 紧化可列集E 而得$\overline{E} = E\cup\{\infty\}$。设$\sigma$是定义在完备概率空间$(\Omega,\mathscr{F},P)$上的非负(包括无穷)随机变量，称$X(\omega)=\{x_t(\omega), t<\sigma(\omega)\}$[1] $(\omega\in\Omega)$为齐次马氏过程，或简称过程，如果对任意$\omega\in\Omega$和$t<\sigma(\omega)$有$x(t,\omega)\in\overline{E}$，但

$$P\{x(t)=\infty\}=0,\quad t\geqslant 0. \tag{1}$$

而对任意$l\geqslant 2$，$0\leqslant t_1<t_2<\cdots<t_{l+1}$，$i_1, i_2, \cdots, i_{l+1}\in E$，只要$P\{x(t_a)=i_a, 1\leqslant a\leqslant l\}>0$，就有

$$\begin{aligned}P\{x(t_{l+1}) = i_{l+1} \mid x(t_a)=i_a,\ 1\leqslant a\leqslant l\}\\ = P\{x(t_{l+1})=i_{l+1}\mid x(t_l)=i_l\}.\end{aligned} \tag{2}$$

而且上式的值与t_l无关，仅与$t_{l+1}-t_l$有关。称

$$P_{ij}(t) = P\{x(s+t)=j \mid x(s)=i\} \tag{3}$$

为过程的转移概率，其转移概率矩阵 $P(t)=\{p_{ij}(t)\}$必满足(1.2.A，B)，我们假定还满足(1.2.C)。按照第一章§1的记号，即$P(t)\in\mathscr{T}$。如果两个过程X和$\overline{X}$有相同的转移概率矩阵$P(t)$我们说X与$\overline{X}$是同一过程。

---

1) 视$x_t(\omega)=x(t,\omega)$，并常略写ω。

反之，对于每个$P(t)\in\mathscr{T}$，存在过程$X=\{x(t),\ t<\sigma\}$，以$P(t)$为转移概率。根据王梓坤[2]，我们还可以要求X完全可分，波雷尔可测，而且右下半连续，即对每个$\omega\in\Omega$，

$$\varliminf_{s\downarrow t} x(s,\omega)=x(t,\omega)\in\overline{E}\text{ 对一切 }t<\sigma(\omega)。\tag{4}$$

称这样的过程为典范过程。如果对过程X，(4)对几乎一切$\omega\in\Omega$成立，对例外的ω，$X(\omega)$甚至可以无定义。此时我们可以对例外的ω，修改或补充定义为

$$x(t,\ \omega)=i_0\in E,\ \ t<\sigma(\omega)=\infty。\tag{5}$$

记典范过程X组成的类为$\mathscr{X}$。显然，$\mathscr{X}$与$\mathscr{T}$一一对应。按照第一章§1的记号，对应于$P(t)\in\mathscr{T}_s$的过程$X\in\mathscr{X}$组成的类记为$\mathscr{X}_s$。对应于$P(t)\in\mathscr{T}_s(Q)$的过程$X\in\mathscr{X}_s$组成的类记为$\mathscr{X}_s(Q)$，并称$\mathscr{X}_s(Q)$中的X为Q过程。

当$P_i(\sigma=\infty)=1\ (i\in E)$时，过程$X$称为不中断的。当且仅当(1.2.$D$)成立时，$X$是不中断的。

对于中断过程$X=\{x(t),\ t<\sigma\}$，可以任取$\triangle\,\overline{\in}\,\overline{E}$，然后令

$$\tilde{x}(t)=\begin{cases}x(t), & \text{如果 }t<\sigma,\\ \triangle, & \text{如果 }t\geqslant\sigma.\end{cases}\tag{6}$$

则$\tilde{X}=\{\tilde{x}(t),\ t<\infty\}$是不中断过程，其转移概率矩阵$\tilde{P}(t)=\{\tilde{p}_{ij}(t)\}(i,j\in E\cup\{\triangle\})$由(1.2.3)确定。

设$X\in\mathscr{X}$。记$\mathscr{F}_t^0$为由集合$\{x(s)=i\}(s\leqslant t,\ i\in E)$所产生的Borel域。记$\mathscr{F}_{t+0}^0=\bigcap\limits_{s>t}\mathscr{F}_s^0$。

定理1 $\mathscr{F}_{t+0}^0=\mathscr{F}_t^0$。

证明见*Chung*[1，*II*§8，定理1]。

称非负随机变量β为过程X的马氏时刻或简称马时，如果$\beta\leqslant\sigma$，且对任意$t\geqslant0$，

$$(\beta\leqslant t<\sigma)\in\mathscr{F}_t^0。\tag{7}$$

或等价地

$$(\beta<t<\sigma)\in\mathscr{F}_t^0。\tag{8}$$

对马时 β，令$\mathscr{F}_\beta$为满足$\wedge \subset \Omega_\beta \equiv (\beta < \sigma)$及$\wedge \cap (\beta \leqslant t < \sigma) \in \mathscr{F}_t^0$ $(t \geqslant 0)$ 的$\wedge$集系，$\mathscr{F}_\beta$称为β前域。令$\mathscr{F}_\beta'$为形如 $\{x(\beta+t)=i\}$ $(i \in E,\ t \geqslant 0)$的集合所产生的$\Omega_\beta$上的波雷尔域，称为$\beta$后域。

对X的马时β，可以定义$\mathscr{F}_\infty^0$到$\mathscr{F}_\infty^0$中集合推移算子θ_β，使保持并、交、补运算，并且

$$\theta_\beta\{x(t)=i\}=\{x(\beta+t)=i\}. \tag{9}$$

从而可以对$\mathscr{F}_\infty^0$可测函数ξ定义$\theta_\beta\xi$，特别地

$$\theta_\beta x(t)=x(\beta+t),\ (t \geqslant 0) \tag{10}$$

详见*Дынкин*[1，第121—144页]。

定理2 $X \in \mathscr{X}$具有强马氏性，即它具有下列性质：对任何马时β，假定$P(\Omega_\beta)>0$。

(i) 设$M \in \mathscr{F}_\infty^0$，$\wedge \in \mathscr{F}_\beta$，则

$$P\{\theta_\beta M \mid \wedge,\ x(\beta)=i\}=P_i\{M\},\ i \in E. \tag{11}$$

(ii) $P\{x(\beta+t)=\infty\}=0,\ t>0.$ (12)

$\{X(\beta+t),\ 0<t<\sigma-\beta\}$是$(\Omega_\beta,\ \Omega_\beta F,\ p(\cdot \mid \Omega_\beta))$上的开马氏链，其转移概率为$(p_{ij}(t))$。

(iii)令$\triangle'=\{x(\beta)\neq\infty\}$，$\sigma=\sigma-\beta$，$x'(t)=x(\beta+t)$。则$X'(\omega)=\{x'(t,\ \omega),\ t<\sigma'(\omega)\}$，$(\omega \in \triangle')$ 是定义在概率空间$(\triangle',\ \triangle'\mathscr{F},\ P(\cdot \mid \triangle'))$上的典范过程，$X'$ 的状态空间E'含于E，其转移概率矩阵是X的转移概率矩阵在$E' \times E'$上的约束。

称X'为β后过程。

证明见王梓坤[2，§3.3]或*Chung*[1，*II* §9]。

定理3 设$X \in \mathscr{X}$，β是X的马时，α是β后过程X' 的马时，则$\beta+\alpha$是X的马时。

证明见*Chung*[1，*II* §15，定理1]。

§7. 概率的Q过程

概率的Q过程$X \in \mathscr{X}_s(Q)$和分析的Q过程$P(t) \in \mathscr{P}_s(Q)$是一致的。我们说$Q$过程$X$满足柯氏方程组，当然是指和$X$对应的$P(t) \in$

$\mathscr{P}_s(Q)$满足柯氏方程组等等。

对于Q过程$X\in\mathscr{X}_s(Q)$，右下半连续性(6.4)成为在$\bar{E}$中的右连续性：

$$\lim_{s\downarrow t}x(s,\ \omega)=x(t,\ \omega)\in\bar{E},\ 对一切t<\sigma(\omega)。\tag{1}$$

如果Q过程$X\in\mathscr{X}_s(Q)$满足更强的条件：

$$\lim_{s\downarrow t}x(s,\ \omega)=x(t,\ \omega)\in E,\ 对一切t<\sigma(\omega)。\tag{2}$$

称X为D型过程。D型过程组成的类记为$\mathscr{X}_D$。当Q确指时，D型Q过程X组成的类记为$\mathscr{X}_D(Q)$。

定义1 设$X\in\mathscr{X}$，$q_i<\infty$。称$[a,\ b)$为$X(\omega)$的一个i区间，如果

$$x(t,\ \omega)=i,\ 对一切t\in[a,\ b),$$

但在任意以$[a,\ b)$为真子集的区间$[c,\ d)$上，$x(t,\ \omega)\not\equiv i$。当$i$没有指时，$i$区间称为常值区间。

定理1 设$X\in\mathscr{X}$，$q_i<\infty$。则对几乎一切$\omega\in\Omega$，$X(\omega)$在任意有限区间中只有有限个i区间。

证明见王梓坤[2，§3.1，定理2]或Chung[1，II§5，定理7]。

设$X\in\mathscr{X}_s(Q)$。称

$$\tau_1=\begin{cases}\inf\{t\mid 0<t<\sigma,\ x(t)\neq x(0)\},\\ \sigma,\ 如果上面的集合为空集。\end{cases}\tag{3}$$

为X的第一个间断点。

定理2 设$X\in\mathscr{X}_s(Q)$，则

$$P_i\{\tau_1>t\}=e^{-q_it}。\tag{4}$$

当$q_i>0$时，

$$P_i\{x(\tau_1)=j\}=\Pi_{ij},\ j\in E。\tag{5}$$

$$P_i\{x(\tau_1)=\infty\}=p_i\{\tau_1<\sigma\}-\sum_j\Pi_{ij}。\tag{6}$$

其中$\Pi=(\Pi_{ij})$由(1.9.7)确定。

证明见王梓坤[2，§2.2定理5和6]。

定理3 设$X\in\mathscr{X}$，$0<q_i<\infty$，β为X的马时，$p\{x(\beta)=i\}=$

0，α为β后的第一个间断点，令$\rho=\alpha-\beta$。则对$\wedge\in\mathscr{F}_\beta, M\in\mathscr{F}'_\alpha$，

$$P\{\wedge, \ \rho>t, \ \ M\}=P\{\wedge\}e^{-q_i t}P\{M|\Omega_\beta\}.$$

特别地，$\mathscr{F}_\alpha$和$\mathscr{F}'_\alpha$关于Ω_β条件独立。更特别地，$x(\tau_1)$与τ_1关于测度P_i条件独立。

证明见王梓坤[2，§3.3定理1系2]或Chung[1，II§15，定理2]。

设 $X\in\mathscr{X}_s(Q)$。称

$$\eta_i^*=\begin{cases}\inf\{t\,|\,\tau_1<t<\sigma, \ x(t)=i\},\\ \sigma，\text{如果上面的集合为空集。}\end{cases} \tag{7}$$

为第一个间断点后首回i的时刻。如果$P_i\{\eta_i^*<\sigma\}=1$，称i为常返的，如果i常返，并且

$$m_{ii}=E_i\eta_i^*<\infty, \tag{8}$$

称i为遍历的。如果一切i都常返或遍历，称过程X常返或遍历。

定理4

(i) i常返的充要条件是对几乎一切$\omega\in\{x(0)=i\}$，$X(\omega)$有无穷多个i区间，或等价地，

$$\int_0^\infty p_{ii}(t)dt=\infty.$$

(ii) 设i常返，则$P_i\{\sigma=\infty\}=1$，并且i遍历的充要条件是

$$\lim_{t\to\infty}p_{ii}(t)=\pi_i>0. \tag{9}$$

如果i遍历，C为含i的不可约常返类，则

$$\lim_{t\to\infty}p_{ji}(t)=\frac{1}{q_i m_{ii}}, \quad j\in C. \tag{10}$$

证明见王梓坤[2，§4.2，定理1]或Chung[1，II§10，定理4及其系1，II§12第(9)式]。

§8. 概率的最小过程

定义1 设$X\in\mathscr{X}_s$。称$t\in(0, \sigma(\omega)]$为$X(\omega)$的跳跃点，如果$t=\sigma(\omega)<\infty$且存在$i\in E$及$\varepsilon>0$使$x(u, \omega)=i$对$u\in(t-\varepsilon, t)$，或

者$t<\sigma(\omega)$且存在不同的i，$j\in E$及$\varepsilon>0$使对$u\in(t-\varepsilon,\ t)$有$x(u,\ \omega)=i$，对$u\in(t,\ t+\varepsilon)$有$x(u,\ \omega)=j$。称$t\in(0,\ \sigma(\omega)]$为$X(\omega)$的飞跃点，如果$t=\sigma(\omega)<\infty$，或者$t<\sigma(\omega)$且对任意$\varepsilon>0$，$X(\omega)$在$(t-\varepsilon,\ t+\varepsilon)$中有无穷多个跳跃点。约定：$t=0$是跳跃点和飞跃点，称为第0个跳跃点和第0个飞跃点。

每个Q过程$X\in\mathscr{X}_s(Q)$都有第一个飞跃点τ：

$$\tau=\begin{cases}\inf\{t\mid 0<t<\sigma(\omega),\ \lim\limits_{s\to t}x(s,\ \omega)=\infty\},\\ \sigma(\omega),\ \text{如果上面的集合为空集。}\end{cases}\tag{1}$$

X的第一个间断点τ_1未必是跳跃点。但Q保守时，τ_1是跳跃点。如果$\tau_1(<\sigma)$不是跳跃点，依定理7.2，则τ_1是第一个飞跃点。

定理1 设$X\in\mathscr{X}_s(Q)$，$q_i>0$。则

$$P_i\{\tau_1=\tau\}=\frac{d_i}{q_i}\tag{2}$$

其中d为Q的非保守量，由（1.2.6）确定。

证。由（7.6），

$$P_i\{\tau_1=\tau\}=P_i\{x(\tau_1)=\infty\}+P_i\{\tau_1=\sigma\}$$

$$=1-\sum_j \Pi_{ij}=\frac{d_i}{q_i}.$$

定理2 设$X\in\mathscr{X}_s(Q)$，集合$\wedge$和非负随机变量ξ均$\mathscr{F}^0_\infty$可测。

(i) 如果

$$P\{\wedge=\theta_{\tau_1}\wedge\}=P_i\{\xi=\tau_1+\theta_{\tau_1}\xi\}=1,\tag{3}$$

则 $u_j=E_j\{\xi,\ \wedge\}$满足方程

$$\sum_j q_{ij}u_j=-P_i\{\wedge\}.\tag{4}$$

$u_j(\lambda)=E_j\{e^{-\lambda\xi},\ \wedge\}$ $(\lambda>0)$ 满足方程

$$\lambda u_i-\sum_j q_{ij}u_j=0.\tag{5}$$

(ii) 如果

$$P_i\{\tau_1=\tau,\ \wedge\}=P_i\{\tau_1=\tau\},\tag{6}$$

$$P_i\{\tau_1<\tau,\ \theta_{\tau_1}\wedge\}=P_i\{\tau_1<\tau,\ \wedge\},\tag{7}$$

则 $u_j = P_j\{\wedge\}$ 满足方程

$$\sum_j q_{ij} u_j = d_i. \tag{8}$$

证 当$q_i = 0$时平凡。设$q_i > 0$。

由强马氏性，定理7.2及τ_1与$x(\tau_1)$的独立性，

$$E_i\{\xi, \ \wedge\} = E_i\{\tau_1 + \theta_{\tau_1}\xi, \ \theta_{\tau_1}\wedge\}$$

$$= \sum_j \Pi_{ij} E_i\{\tau_1 + \theta_{\tau_1}\xi, \ \theta_{\tau_1}\wedge \mid x(\tau_1) = j\}$$

$$= \sum_j \Pi_{ij}(E_i\{\tau_1, \theta_{\tau_1}\wedge \mid x(\tau_1) = j\} + E_i\{\theta_{\tau_1}\xi, \theta_{\tau_1}\wedge \mid x(\tau_1)$$

$$= j\})$$

$$= \sum_j \Pi_{ij}(E_i\{\tau_1\} P_j\{\wedge\} + E_j\{\xi, \ \wedge\})$$

$$= \frac{1}{q_i} P_i\{\wedge\} + \sum_j \Pi_{ij} E_j\{\xi, \ \wedge\}.$$

由此得（4）。其次，

$$E_i\{e^{-\lambda\xi}, \ \wedge\} = \sum_j \Pi_{ij} E_i\{e^{-\lambda\tau_1}\theta_{\tau_1} e^{-\lambda\xi}, \theta_{\tau_1}\wedge \mid x(\tau_1) = j\}$$

$$= \sum_j \Pi_{ij} E_i\{e^{-\lambda\tau_1} \mid x(\tau_1) = j\} E_i\{\theta_{\tau_1} e^{-\lambda\xi}, \ \theta_{\tau_1}\wedge \mid x(\tau_1) = j\}$$

$$= \sum_j \Pi_{ij} E_i\{e^{-\lambda\tau_1}\} E_j\{e^{-\lambda\xi}, \ \wedge\}$$

$$= \frac{q_i}{\lambda + q_i} \sum_j \Pi_{ij} E_j\{e^{-\lambda\xi}, \ \wedge\},$$

得证（5）。再次，

$$P_i\{\wedge\} = P_i\{\wedge, \ \tau = \tau_1\} + P_i\{\theta_{\tau_1}\wedge, \ \tau_1 < \tau\}$$

$$= P_i\{\tau = \tau_1\} + \sum_j \Pi_{ij} P_j\{\wedge\}.$$

注意（2），得证（8），证毕。

定理3 设$X \in \mathscr{X}_s(Q)$。X满足向后方程组的充要条件是：对几乎一切$\omega \in \Omega$，$X(\omega)$有第一个间断点，而且是跳跃点。X满足向前方程组的充要条件是：对任意指定$t > 0$，对几乎一切$\omega \in (t < \sigma)$，

$X(\omega)$在$[0, t]$中有最后一个间断点，而且是跳跃点。

证明见王梓坤[2，§2.3定理1和2]。

定理4 设$X\in\mathscr{X}_s(Q)$，τ为第一个飞跃点，则$X=\{x(t), t<\tau\}\in\mathscr{X}_s(Q)$，其转移概率是第一章§9中的最小解$f(t)=\{f_{ij}(t)\}$。

证明见王梓坤[2，§2.3，定理5]。

定义2 称$X=\{x(t), t<\sigma\}\in\mathscr{X}_s(Q)$为最小$Q$过程，如果$P\{\tau=\sigma\}=1$，其中$\tau$为第一个飞跃点。

设$X\in\mathscr{X}_s(Q)$，τ_1是第一个间断点，$\tau_1\leqslant\tau$。如果$\tau_1=\infty$，此时$\tau=\infty$，并令$\tau_n=\infty(n>1)$。如果$\tau_1<\infty$，则或者$\tau_1=\tau<\infty$，此时$\tau_n(n>2)$不确定，或者$\tau_1<\tau$，此时τ_1是跳跃点，因而τ_1后有第一个间断点$\tau_2\leqslant\tau$。如果$\tau_2=\infty$，此时$\tau=\infty$，并令$\tau_n=\infty(n>2)$。如果$\tau_2<\infty$，则或者$\tau_2=\tau<\infty$，此时$\tau_n(n>2)$不确定，或者$\tau_2<\tau$，此时τ_2是跳跃点，因而τ_2后有第一个间断点$\tau_3\leqslant\tau$。如此继续。

记$\tau_0=0$，$\Omega=\Omega_F\cup\Omega_\infty$：

$$\left.\begin{aligned}\Omega_F&=\bigcup_{n=1}^{\infty}(\tau_n=\tau<\infty),\\ \Omega_\infty&=\bigcap_{n=1}^{\infty}(\tau_n<\tau)+\bigcup_{n=1}^{\infty}(\tau_n=\tau=\infty)\end{aligned}\right\}\tag{9}$$

$$\beta=\begin{cases}+\infty, & \text{如}\omega\in\Omega_\infty,\\ \sup\{n;\ n\geqslant 0,\ \tau_n<\tau\}, & \text{如}\omega\in\Omega_F.\end{cases}\tag{10}$$

$$\tau_\beta=\lim_{n\to\beta}\tau_n\tag{11}$$

则在Ω_F上有$\tau_\beta<\tau$，在Ω_∞上有$\tau_\beta=\tau$。

当$\tau_n=\infty$时，约定$x(\tau_n)=x(\tau_k)$，其中$k=\max\{m;\ \tau_m<\infty\}$。这样，当$n\leqslant\beta$即$\tau_n\leqslant\tau_\beta$时，$x(\tau_n)$有定义。

由强马氏性及定理7.2得

定理5 $X_T=\{X(\tau_n), n\leqslant\beta\}$或记作$X_T=\{X(\tau_n),\ \tau_n\leqslant\tau_\beta\}$是马氏链，它的一步转移概率矩阵$\Pi=(\Pi_{ij})$由(1.9.7)确定。

称马氏链X_T为过程$\{X(t),\ t<\tau\}$的嵌入链，称矩阵Π为矩阵

Q的嵌入矩阵。

§9. 预解过程和导出过程

设$X=\{x(t),\ t<\sigma\}\in\mathscr{X}$，$\rho$为与$X$独立的随机变量，其分布是参数为 1 的指数分布：

$$P\{\rho>t\}=e^{-t},\quad t\geqslant 0. \tag{1}$$

对$\lambda>0$，令

$$\left.\begin{aligned}&\rho^{\lambda}=\frac{\rho}{\lambda},\quad \sigma^{\lambda}=min(\sigma,\ \rho^{\lambda}).\\&x^{\lambda}(t)=x(t),\quad t<\sigma^{\lambda}.\end{aligned}\right\} \tag{2}$$

定理1 $X^{\lambda}=\{x^{\lambda}(t),\ t<\sigma^{\lambda}\}\in\mathscr{X}$，其转移概率为

$$p_{ij}^{\lambda}(t)=e^{-\lambda t}p_{ij}(t). \tag{3}$$

其中$p_{ij}(t)$是X的转移概率。

证 因X^{λ}的轨道是X的轨道的前面一部分。我们只需证X^{λ}的齐次马氏性即可。

设$0\leqslant t_1<t_2<\cdots<t_{n+1}$，$i_1,\ i_2,\cdots,i_{n+1}\in E$。由$X$的齐次马氏性和$\rho$与$X$的独立性，

$$\begin{aligned}&P\{x^{\lambda}(t_a)=i_a,\ 1\leqslant a\leqslant n+1\}\\&=P\{x(t_a)=i_a,\ 1\leqslant a\leqslant n+1,\ t_{n+1}<\rho^{\lambda}\}\\&=P\{x(t_1)=i_1\}\prod_{a=1}^{n}P_{i_a i_{a+1}}(t_{a+1}-t_a)\cdot e^{-\lambda t_{n+1}}\\&=P\{x(t_1)=i_1\}p\{t_1<\rho^{\lambda}\}\prod_{a=1}^{n}p^{\lambda}_{i_a i_{a+1}}(t_{a+1}-t_a)\\&=P\{x^{\lambda}(t_1)=i_1\}\prod_{a=1}^{n}p^{\lambda}_{i_a i_{a+1}}(t_{a+1}-t_a).\end{aligned}$$

称X^{λ}为X的预解过程。

定理2 设$X=\{x(t),\ t<\tau\}$为最小Q过程，嵌入矩阵为Π。则预解过程X^{λ}是最小Q^{λ}过程，其中

$$q_{ij}^{\lambda}=q_{ij}(i\neq j),\quad q_i^{\lambda}=\lambda+q_i. \tag{4}$$

嵌入矩阵$\Pi(\lambda)$由（1.9.8）确定。

证 只需说明Q^{λ}形如(4)即可，其余明显。实际上，设τ_1^{λ}为X^{λ}的第一个间断点，则$\tau_1^{\lambda}=min(\tau_1, \rho^{\lambda})$，故

$$exp(-q_i^{\lambda}t)=P_i\{\tau_1^{\lambda}>t\}=P_i\{\tau_1>t, \rho^{\lambda}>t\}$$

$$=P_i\{\tau_1>t\}P_i\{\rho^{\lambda}>t\}=e^{-q_it}e^{-\lambda t}=exp[-(\lambda+q_i)t],$$

故$q_i^{\lambda}=\lambda+q_i$。其次，对$j\neq i$，

$$\frac{q_{ij}^{\lambda}}{q_i^{\lambda}}=P_i\{x^{\lambda}(\tau_1^{\lambda})=j\}=P_i\{x(\tau_1)=j, \tau_1<\rho^{\lambda}\}$$

$$=P_i\{x(\tau_1)=j\}P\{\tau_1<\rho^{\lambda}\}=\frac{q_i}{\lambda+q_i}\Pi_{ij},$$

故$q_{ij}^{\lambda}=q_{ij}$，证毕。

定理3 设$X=\{x(t), t<\tau\}$是最小Q过程，A为嵌入链X_T的终极集，$P(A)>0$。则约束在A上的过程$X^A(\omega)=\{x(t, \omega), t<\tau(\omega)\}$（$\omega\in A$）是概率空间（$A$, $A\mathscr{F}$, $P\{\cdot|A\}$）上的最小Q^A过程，其状态空间为

$$E^A=\{i|P_i(A)>0\}. \tag{5}$$

转移概率为

$$P_{ij}^A(t)=\frac{f_{ij}(t)p_j(A)}{P_i(A)}. \tag{6}$$

Q矩阵$Q^A=(q_{ij}^A)(i, j\in E^A)$保守，且

$$q_{ij}^A=\frac{q_{ij}D_j(A)}{P_j(A)}. \tag{7}$$

证 首先注意，对任意$t\geqslant 0$，在（$t<\tau$）上有$A=\theta_tA$，θ_t为推移算子。实际上，因A为终极集，存在无穷维不变函数g使

$$1_A=g(x(\tau_n), x(\tau_{n+1}), \cdots)\ 对一切n. \tag{8}$$

于是由Дынкин[1，第122页]，

$$1_{\theta_tA}=g(\theta_tx(\tau_n), \theta_tx(\tau_{n+1}), \cdots)$$

$$=g(x(\theta_t\tau_n), x(\theta_t\tau_{n+1}), \cdots). \tag{9}$$

在（$t<\tau$）上，必定存在l使$\tau_l\leqslant t<\tau_{l+1}$，从而$\theta_t\tau_m=\tau_{m+l}$，于是由(8)(9)，

$$1_{\theta_t A} = g(x(\tau_{n+l}),\ x(\tau_{n+l+1}),\ \cdots) = 1_A$$

即在$(t<\tau)$上有$A=\theta_t A$。

设$0\leqslant t_1<t_2<\cdots<t_{n+1}$，$i_1$，$i_2$，$\cdots$，$i_{n+1}\in E^A$。

由X的齐次马氏性

$$\begin{aligned}
P\{x^A(t_{n+1}) &= i_{n+1}\mid A,\ x^A(t_a)=i_a 1\leqslant a\leqslant n\} \\
&= P\{x(t_{n+1})=i_{n+1}\mid A,\ x(t_a)=i_a,\ 1\leqslant a\leqslant n\} \\
&= \frac{P\{x(t_a)=i_a,\ 1\leqslant a\leqslant n+1,\ A\}}{P\{x(t_a)=ia,\ 1\leqslant a\leqslant n, A\}} \\
&= \frac{P\{x(t_a)=i_a,\ 1\leqslant a\leqslant n+1,\ \theta_{t_{n+1}}A\}}{P\{x(t_a)=i_a,\ 1\leqslant a\leqslant n, \theta_{t_n}A\}} \\
&= \frac{P\{x(t_a)=i_a\ 1\leqslant a\leqslant n\} f_{i_n i_{n+1}}(t_{n+1}-t_n) P_{i_{n+1}}(A)}{P\{x(t_a)=i_a,\ 1\leqslant a\leqslant n\} P_{i_n}(A)} \\
&= \overset{A}{P}_{i_n i_{n-1}}(t_{n+1}-t_n).
\end{aligned}$$

剩下只需证Q^A保守，而这可以从

$$P_i(A) = P_i(\theta_{\tau_1} A) = \sum_j \Pi_{ij} P_j(A)$$

得出，证毕。

系 设X为最小Q过程，$P(\Omega_\infty)>0$。则X^{Ω_∞}是$(\Omega_\infty,\ \Omega_\infty\mathscr{F}$，$P(\cdot\mid\Omega_\infty))$上的最小$Q^{\Omega_\infty}$保守过程。

定义1 称过程X^A为最小Q过程X在终极集A上的导出过程。

§10. $\Pi(\lambda)$的位势的概率表现

设$X=\{x(t),\ t<\tau\}$为最小Q过程，H为Q的非保守状态，d为Q的非保守量，$\Pi(\lambda)$为X的预解过程X^λ的嵌入链。为了方便，视$f_i=f(i)$。

定理1 对于链$\Pi(\lambda)$，$u(\lambda)$是在E-H上为零的非负函数v的位势的充要条件是$u(\lambda)$有表现：

$$u_i(\lambda) = E_i\{e^{-\lambda\tau} f(x_{\tau-0}),\ \Omega_F\} = [\phi(\lambda) df]_i, \tag{1}$$

其中$f(a)\geqslant 0\ (a\in H)$。

证 充分性. 由（1），

$$u_i(\lambda)=\sum_{n=1}^{\infty}E_i\{e^{-\lambda\tau_n}f(x_{\tau_n-0}),\quad \tau_n=\tau\}=\sum_{n=1}^{\infty}T_i^n,$$

其中 $$T_i^1=E_i\{e^{-\lambda\tau_1}f(x_{\tau_1-0}),\quad \tau_1=\tau\}=f_i\frac{d_i}{q_i}\cdot\frac{q_i}{\lambda+q_i}$$

$$=\frac{1}{\lambda+q_i}d_if_i.$$

$$T_i^n=E_i\{e^{-\lambda\tau_n}f(x_{\tau_n-0}),\quad \tau_n=\tau\}$$

$$=\sum_j \Pi_{ij}\frac{q_i}{\lambda+q_i}T_j^{n-1}=\sum_j \Pi_{ij}(\lambda)T_j^{n-1},$$

$$T^n=\Pi(\lambda)T^{n-1}=\cdots=\Pi^{n-1}(\lambda)(\lambda+q)^{-1}df,$$

$$u(\lambda)=\sum_{n=1}^{\infty}\Pi^{n-1}(\lambda)(\lambda+q)^{-1}df=\phi(\lambda)df.$$

故$u(\lambda)$ 是函数$v_j=\dfrac{d_jf_j}{\lambda+q_j}$的位势，而$v$在$E-H$上为0.

必要性. 取$f_a=\dfrac{(\lambda+q_a)v_a}{d_a}(a\in H)$，由充分性的证明知$u(\lambda)$有表现（1），证毕.

定理2 设 $G(\lambda)=\sum\limits_{n=0}^{\infty}\Pi^n(\lambda)=\phi(\lambda)(\lambda+q)$，

列矢量$v\geqslant 0$. 则

$$[\Pi^n(\lambda)v]_i=E_i\{e^{-\lambda\tau_n}v(x_{\tau_n})\},\tag{2}$$

$$[G(\lambda)v]_i=\sum^{\infty}E_i\{e^{-\lambda\tau_n}v(x_{\tau_n})\}.\tag{3}$$

证 仿定理1证（2），从而得(3).

系 $\Pi_{ij}^n(\lambda)=E_i\{e^{-\lambda\tau_n},\ x(\tau_n)=j\}$.

§11. λ映象与标准映象

回忆调和函数非负. 记有限的Π（或$\Pi(\lambda)$）调和函数类为$\hat{\mathscr{M}}^+$（或$\hat{\mathscr{M}}_\lambda^+$），记有界的$\Pi$（或$\Pi(\lambda)$）调和函数类为$\mathscr{M}^+$（或$\mathscr{M}_\lambda^+$），记

有上界为K的Π（或$\Pi(\lambda)$）调和函数类为$\mathscr{M}^+(K)$（或$\mathscr{M}^+_\lambda(K)$）。

设 $u\in\hat{\mathscr{M}}^+$，则 $\Pi(\lambda)u\leqslant\Pi u=u$，$\Pi^{n+1}(\lambda)u\leqslant\Pi^n(\lambda)u\leqslant u$，故 $\Pi^n(\lambda)u\downarrow u(\lambda)\leqslant u$，且 $u(\lambda)\in\hat{\mathscr{M}}^+_\lambda$。如果 $u\in\mathscr{M}^+(K)$，则 $u(\lambda)\in\mathscr{M}^+_\lambda(K)$。

定义1 设$u\in\hat{\mathscr{M}}^+$，λ指定称$u(\lambda)=\lim\limits_{n\to\infty}\Pi^n(\lambda)u\in\hat{\mathscr{M}}^+_\lambda$为$u$的$\lambda$映象.

定理1 $u\in\hat{\mathscr{M}}^+$与其λ象$u(\lambda)\in\hat{\mathscr{M}}^+_\lambda$有关系。

$$\lambda\phi(\lambda)u=u-u(\lambda)。\tag{1}$$

证 显然

$$\sum_{a=0}^{n}\Pi^a(\lambda)=\sum_{a=0}^{n-1}\Pi^a(\lambda)(\lambda I+q)^{-1}q\Pi+I,\tag{2}$$

故如令$\phi^n(\lambda)=\sum\limits_{a=0}^{n-1}\Pi_a(\lambda)(\lambda I+q)^{-1}$，则(2)成为

$$\phi^{n+1}(\lambda)(\lambda+q)=\phi^n(\lambda)q\Pi+I$$

$$\lambda\phi^{n+1}(\lambda)+\phi^{n+1}(\lambda)q=\phi^n(\lambda)q\Pi+I。\tag{3}$$

右乘u并注意$\Pi u=u$得

$$\lambda\phi^{n+1}(\lambda)u+\phi^{n+1}(\lambda)q\Pi u=\phi^n(\lambda)q\Pi u+u,$$

$$\lambda\phi^{n+1}(\lambda)u+\Pi^n(\lambda)(\lambda I+q)^{-1}q\Pi u=u,$$

$$\lambda\phi^{n+1}(\lambda)u+\Pi^{n+1}(\lambda)u=u。\tag{4}$$

令$n\to\infty$得(1)。证毕。

设$u(\lambda)$是$\Pi(\lambda)$调和函数，未必有限。则$\Pi u(\lambda)\geqslant\Pi(\lambda)u(\lambda)=u(\lambda)$，$\Pi^{n+1}u(\lambda)\geqslant\Pi^n u(\lambda)$。故$\Pi^n u(\lambda)\uparrow u\geqslant u(\lambda)$，并且$\Pi u=u$，即$u$是$\Pi$调和函数，但未必有限。

定义2 设$u(\lambda)$为$\Pi(\lambda)$调和函数，称Π调和函数$u=\lim\limits_{n\to\infty}\Pi^n u(\lambda)$为$u(\lambda)$的标准映象。

定理2 设$u(\lambda)\in\hat{\mathscr{M}}^+_\lambda$，则其标准映象为[1)]

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1) 当$q_i=0$时，约定$\frac{1}{q_i}=\infty$。但由$\Pi(\lambda)u(\lambda)=u(\lambda)$知$u_i(\lambda)=0$，因而$\frac{1}{q_i}u_i(\lambda)=\infty\cdot 0=0$。

$$u=u(\lambda)+\lambda\Gamma u(\lambda). \tag{5}$$

其中$\Gamma=\lim\limits_{\lambda\downarrow 0}\phi(\lambda)=\sum\limits_{n=0}^{\infty}\Pi^n\frac{1}{q}$.

证 改写$\Pi(\lambda)u(\lambda)=u(\lambda)$为

$$u(\lambda)+\lambda q^{-1}u(\lambda)=\Pi u(\lambda).$$

左乘Π^n并对$0\leqslant n\leqslant a-1$求和得

$$u(\lambda)+\lambda\sum_{n=0}^{a-1}\Pi^n q^{-1}u(\lambda)=\Pi^a u(\lambda).$$

令$a\to\infty$得(5)。证毕。

注 由(5)可见，列协调族$u(\lambda)(\lambda>0)$的标准映象与定义1.10.2相容。但定义2中λ可以是指定的。

定理3 设$u(\lambda)\in\hat{\mathscr{M}}_\lambda^+$，$u(\lambda)$的标准映象$u\in\hat{\mathscr{M}}^+$。则(1)成立。

证 已知(3)成立。在(3)中右乘$u=u(\lambda)+\lambda\Gamma u(\lambda)\in\hat{\mathscr{M}}^+$得

$$\lambda\phi^{n+1}(\lambda)u+\phi^{n+1}(\lambda)q\Pi u=\phi^n(\lambda)q\Pi u+u,$$

$$\lambda\phi^{n+1}(\lambda)u+\sum_{a=0}^{n}\Pi^a(\lambda)(\lambda+q)^{-1}q\Pi[u(\lambda)+\lambda\Gamma u(\lambda)]$$

$$=\sum_{a=0}^{n-1}\Pi^a(\lambda)(\lambda+q)^{-1}q\Pi[u(\lambda)+\lambda\Gamma u(\lambda)]+u,$$

$$\lambda\phi^{n+1}(\lambda)u+\Pi^{n+1}(\lambda)[u(\lambda)+\lambda\Gamma u(\lambda)]=u.$$

注意 $u(\lambda)\in\hat{\mathscr{M}}_\lambda^+$，即

$$\lambda\phi^{n+1}(\lambda)u+u(\lambda)+\Pi^{n+1}(\lambda)\lambda\Gamma u(\lambda)=u. \tag{6}$$

而因$u\in\hat{\mathscr{M}}_\lambda^+$，

$$\Pi^{n+1}(\lambda)\lambda\Gamma u(\lambda)\leqslant\Pi^{n+1}\lambda\Gamma u(\lambda)=\Pi^{n+1}[u-u(\lambda)]$$

$$=u-\Pi^{n+1}u(\lambda)\downarrow u-u=0,$$

在(6)中令$n\to\infty$得(1)。证毕。

系 设$u(\lambda)\in\hat{\mathscr{M}}_\lambda^+$，其标准象映$u\in\hat{\mathscr{M}}^+$。则

$$\phi(\lambda)u=\Gamma u(\lambda). \tag{7}$$

证 从定理2和3得出。

令$\overline{\mathscr{M}}_\lambda^+=\{u(\lambda)\in\hat{\mathscr{M}}_\lambda^+\,|\,u(\lambda)\text{的标准象映}u\in\hat{\mathscr{M}}^+\}$. (8)

$$\overline{\mathscr{M}}^+ = \{u \in \overline{\mathscr{M}}^+ \mid u\text{是某个}u(\lambda) \in \overline{\overline{\mathscr{M}}}{}_\lambda^+\text{的标准映象}\}. \tag{9}$$

定理 4 在$\overline{\mathscr{M}}{}_\lambda^+$和$\overline{\mathscr{M}}^+$之间，按标准映象和$\lambda$映象建立了一一对应，并且标准映象和$\lambda$映象互为逆映象。

证 显然，标准映象将$\overline{\mathscr{M}}{}_\lambda^+$映象在$\overline{\mathscr{M}}^+$上。由定理 3，标准映象是一对一的。今设$u(\lambda) \in \overline{\mathscr{M}}{}_\lambda^+$的标准映象为$u \in \overline{\mathscr{M}}^+$，而$u$的$\lambda$映象为$v(\lambda)$。由定理1和3，$v(\lambda) = u(\lambda)$，即$u(\lambda)$是$u$的$\lambda$映象。

定理 5 在标准映象或λ映象下，$\Pi(\lambda)$极小的$u(\lambda) \in \overline{\mathscr{M}}{}_\lambda^+$ 与Π极小的$u \in \overline{\mathscr{M}}$一一对应。

证 依极小性定义，利用定理4可得。

§ 12. 最小Q过程的边界

设$X = \{x(t),\ t < \tau\}$为最小Q过程。$X_T = \{x(\tau_n),\ n \leqslant \beta\}$为其嵌入链，其嵌入矩阵为$\Pi$。根据 § 3，对链$\Pi$和$X_T$，我们可以导出它的马亭边界$\partial E$，本质马亭边界$B$，原子边界$B_1$和非原子边界$B_2$，以及终极波雷尔域$\mathscr{B}_\infty$等概念。我们沿用本章第 3 至 5 节的记号。但现在的情形，(2.17)中的Ω_F和Ω_∞应按(8.9)(8.10) 理解；H应为Q的非保守状态集；(3.11)成为

$$x(\tau - 0) = \lim_{\tau_n \uparrow \tau} x(\tau_n) \in H \cup B. \tag{1}$$

或更精确些，在Ω_F上，$x(\tau - 0) \in H$，在 Ω_∞上，$x(\tau - 0) \in B$；而(3.34)的测度μ成为

$$\mu(\Gamma) = P_\gamma\{x(\tau - 0) \in \Gamma\},\quad \Gamma \subset H \cup B. \tag{2}$$

显然，当$\xi \in B$ 时，$K(\cdot, \xi) \in \hat{\mathscr{M}}^+$。假定 $K(\cdot,\ \xi)$ 的λ映象为$K_\lambda(\cdot,\ \xi)$。由定理11.1有

$$\lambda\phi(\lambda)K(\cdot,\ \xi) = K(\cdot,\ \xi) - K_\lambda(\cdot,\ \xi),\quad \xi \in B. \tag{3}$$

由$\Phi(\lambda)$的预解方程，从上式得

$$K_\lambda(\cdot,\ \xi) - K_\nu(\cdot,\ \xi) + (\lambda - \nu)\Phi(\lambda)K_\nu(\cdot,\ \xi) = 0,$$

$$(\xi \in B,\ \lambda,\ \nu > 0.) \tag{4}$$

即$K_\lambda(\cdot,\ \xi)$是列协调族。因此或者对一切$\lambda > 0$，$K_\lambda(\cdot,\ \xi) = 0$ 或

者$\neq 0$。于是我们可以令

$$B_e=\{\xi\in B\mid K_\lambda(\cdot,\ \xi)\neq 0\},\tag{5}$$

$$B_p=\{\xi\in B\mid K_\lambda(\cdot,\ \xi)=0\}.\tag{6}$$

定义 1 分别称B_e和B_p为最小Q过程X或矩阵Q的马亭流出边界和消极边界。

定理 1 B_e和B_p是波雷尔集。

证 因B是波雷尔集，且对每个$i\in E$,$K(i,\xi)(\xi\in B)$是波雷尔可测函数，从而$K_\lambda(i,\ \xi)(\xi\in B)$也是波雷尔可测函数。于是

$$B_e=\bigcup_{i\in E}\{\xi\mid K_\lambda(i,\ \xi)\neq 0\}$$

是波雷尔集，$B-B_e=B_p$也是，证毕。

定理 2 设$\xi\in B_e$，则$K(\cdot,\ \xi)\in\overline{\mathscr{M}}^+$，$K_\lambda(\cdot,\ \xi)\in\overline{\mathscr{M}}_\lambda^+$，并且$K_\lambda(\cdot,\ \xi)$的标准映象是$K(\cdot,\ \xi)$，$K(\cdot,\ \xi)$的$\lambda$映象是$K_\lambda(\cdot,\xi)$。

证 由(3)，$K_\lambda(\cdot,\ \xi)\leqslant K(\cdot,\ \xi)$，$\Pi^n K_\lambda(\cdot,\ \xi)\leqslant \Pi^n K(\cdot,\ \xi)=K(\cdot,\ \xi)$。于是$K_\lambda(\cdot,\ \xi)$的标准映象$u=\lim\limits_{n\to\infty}\Pi^n K_\lambda(\cdot,\xi)\leqslant K(\cdot,\ \xi)$，即$K_\lambda(\cdot,\ \xi)\in\overline{\mathscr{M}}_\lambda^+$，$u\in\overline{\mathscr{M}}^+$。按定理11.4，$u=K(\cdot,\ \xi)$。

系 $\{u(\lambda)\in\overline{\mathscr{M}}_\lambda^+\mid u(\lambda)$是$\Pi(\lambda)$极小的，$u(\lambda)$的标准映象$\gamma$可积$\}=\{cK_\lambda(\cdot,\ \xi)\mid \xi\in B_e$，常数$c>0\}$。

证 从定理 2 和定理8.19得出。

对任意波雷尔集$\Gamma\subset B$，令

$$X_i^\Gamma\equiv P_i\{x(\tau-0)\in\Gamma\}=\int_\Gamma K(i,\ \xi)\mu(d\xi),\tag{7}$$

$$X_i^\Gamma(\lambda)\equiv E_i\{e^{-\lambda\tau},\ x(\tau-0)\in\Gamma\}.\tag{8}$$

定理 3 对任意波雷尔集$\Gamma\subset B$，

$$\lambda\phi(\lambda)X^\Gamma=X^\Gamma-X^\Gamma(\lambda).\tag{9}$$

证 由X的齐次马氏性及$\wedge=\{x(\tau-0)\in\Gamma\}$是终极集，

$$X_i^\Gamma(\lambda)=\int_0^\infty e^{-\lambda t}dP_i\{\wedge,\ \tau\leqslant t\}$$

$$=\int_0^\infty e^{-\lambda t}\frac{d}{dt}P_i\{\wedge,\ \tau\leqslant t\}dt$$

$$=\lambda\int_0^\infty e^{-\lambda t}P_i\{\wedge,\ \tau\leqslant t\}dt$$

$$=\lambda\int_0^\infty e^{-\lambda t}[P_i\{\wedge\}-P_i\{\wedge,\ \tau>t\}]dt$$

$$=X_i^\Gamma-\lambda\sum_j\int_0^\infty e^{-\lambda t}P_i\{t<\tau,\ x(t)=j,\ \theta_t\wedge\}dt$$

$$=X_i^\Gamma-\lambda\sum_j\int_0^\infty e^{-\lambda t}f_{ij}(t)P_j\{\wedge\}dt$$

$$=X_i^\Gamma-\lambda\sum_j\phi_{ij}(\lambda)X_j^\Gamma.$$

定理 4 对任意波雷尔集$\Gamma\subset B$,

$$X_i^\Gamma(\lambda)=\int_\Gamma K_\lambda(i,\ \xi)\mu(d\xi)=\int_{\Gamma\cap Be}K_\lambda(i,\ \xi)\mu(d\xi)$$
$$=X_i^{\Gamma\cap Be}(\lambda).\tag{10}$$

$X^\Gamma(\lambda)$的标准映象是$X^{\Gamma\cap Be}\in\overline{\mathscr{M}}^+$。而$X^\Gamma$的λ映象为$X^\Gamma(\lambda)$。

证 由B_p之定义得(10)第二等号。

因$\Pi^n(\lambda)X^\Gamma=\int_\Gamma \Pi^n(\lambda)K(\cdot,\ \xi)\mu(d\xi)$, 令$n\to\infty$得

X^Γ的λ映象为$\int_\Gamma K_\lambda(\cdot,\ \xi)\mu(d\xi)$. 由定理11.1及定理3得(10) 中第一等号. 于是由(10),

$$\Pi^n X^\Gamma(\lambda)=\int_{\Gamma\cap Be}\Pi^n K_\lambda(\cdot,\ \xi)\mu(d\xi),$$

取极限知$X^\Gamma(\lambda)$的标准映象为$\int_{\Gamma\cap Be}K(\cdot,\ \xi)\mu(d\xi)=X^{\Gamma\cap Be}$, 证毕。

定理 5 对一切$i\in E$,

$$P_i\{\tau<\infty\,|\,x(\tau-0)\in B_e\}=1,\tag{11}$$
$$P_i\{\tau=\infty\,|\,x(\tau-0)\in B_p\}=1.\tag{12}$$

证 由(10),

$$X_i^{B_p}(\lambda)=E_i\{e^{-\lambda\tau},\ x(\tau-0)\in B_p\}$$
$$=E_i\{e^{-\lambda\tau},\ \tau<\infty,\ x(\tau-0)\in B_p\}$$

$$=\int_{B_p} K_\lambda(i,\ \xi)\mu(d\xi)=0.$$

由此得(12)。其次，

$$\Pi^* X_i^\Gamma(\lambda)=E_i\{E_{x(\tau_n)}[e^{-\lambda\tau},\ \tau<\infty,\ x(\tau-0)\in\Gamma]\}$$

$$=E_i\{\theta_{\tau_n}[e^{-\lambda\tau},\ \tau<\infty,\ x(\tau-0)\in\Gamma]\}$$

$$=E_i\{e^{-\lambda(\tau-\tau_n)},\ \tau<\infty,\ x(\tau-0)\in\Gamma\}.$$

取极限知$X^\Gamma(\lambda)$的标准映象为 $P_i\{\tau<\infty,\ x(\tau-0)\in\Gamma\}$。特别地，$X^{B_e}(\lambda)$的标准映象为$P_i\{\tau<\infty, x(\tau-0)\in B_e\}$。但依定理4，$X^{B_e}(\lambda)$的标准映象为$X^{B_e}$。故$P_i\{\tau<\infty, x(\tau-0)\in B_e\}=P_i\{x(\tau-0)\in B_e\}$。由此得(11)，证毕。

定理 6 设f为B_e上的非负波雷尔函数。则

$$u_i(\lambda)=E_i\{e^{-\lambda\tau}f[x(\tau-0)],\ \Omega_\infty\}$$

$$=\int_{B_e} K_\lambda(i,\ \xi)f(\xi)\mu(d\xi) \tag{13}$$

的标准映象为

$$u_i=E_i\{f[x(\tau-0)],\ x(\tau-0)\in B_e\}$$

$$=\int_{B_e} K(i,\ \xi)f(\xi)\mu(d\xi). \tag{14}$$

反之，如果上式确定的$u_i<\infty(i\in E)$，或者f为B上的非负波雷尔函数，而且

$$\bar{u}_i=E_i\{f[x(\tau-0)],\ \Omega_\infty\}=\int_B K(i,\ \xi)f(\xi)u(d\xi)$$

$$<\infty. \tag{15}$$

则u或$\bar{u}$的λ映象为$u(\lambda)$。

证 仿定理 4 证明。

定理 7 $(\partial E)_1=\{\xi_a,\ a\in\mathscr{A}\}\subset B_p$.

证 指定$j\in E_a$，设最小Q过程X第n次到达j后停留在j的时间为ρ_j^n。故关于P_i，$\rho_j^n(n\geqslant 1)$相互独立，$\tau\geqslant\sum_{n=1}^{\infty}\rho_j^n=\infty$，即$P_i(\tau=$

$\infty)=1$。从而对任意i，$P_i\{\tau=\infty \mid x(\tau-0)=\xi_0\}=1$。依定理5，$\xi_0\in B_p$。

定理 8 对任意$i\in E$，

$$P_i\{\tau<\infty \mid x(\tau-0)\in H\}=1. \tag{16}$$

证 注意$j\in H$时$q_j>0$，即j非吸引。如果$x(\tau-0)=j$，且$\tau=\infty$，则存在t_0使对一切$t\geqslant t_0$有$x(t,\omega)=j$。因j非吸引，除概率为0的ω集外，这是不可能的。证毕。

结合定理5和8，P_γ几乎有

$$x(\tau-0)\in H\cup B_e,\quad \text{如果}\tau<\infty \tag{17}$$

$$x(\tau-0)\in B_p,\quad \text{如果}\tau=\infty. \tag{18}$$

§13. $\mathscr{A}_\lambda^+$的概率表现

定理 1 设f为B上的非负有界波雷尔函数，$u(\lambda)$按（12.13）确定，则P_γ几乎有

$$\lim_{n\to\infty} u_{x(\tau_n)}(\lambda)=\begin{cases} f[x(\tau-0)], & \text{当}x(\tau-0)\in B_e\text{时}, \\ 0 & \text{当}x(\tau-0)\in B_p\text{时}. \end{cases} \tag{1}$$

证 因在Ω_∞上，

$$\begin{aligned} u_{x(\tau_n)}(\lambda) &= E_i\{\theta_{\tau_n}(e^{-\lambda\tau}f[x(\tau-0)],\ x(\tau \\ &\qquad -0)\in B_e) \mid \mathscr{F}_{\tau_n}\} \\ &= E_i\{e^{-\lambda(\tau-\tau_n)}f[x(\tau-0)],\ x(\tau-0)\in B_e \mid \mathscr{F}_{\tau_n}\} \\ &= e^{\lambda\tau_n}E_i\{e^{-\lambda\tau}f[x(\tau-0)],\ x(\tau-0)\in B_e \mid \mathscr{F}_{\tau_n}\}, \end{aligned}$$

当$n\to\infty$时，利用鞅收敛定理得

$$\lim_{n\to\infty} u_{x(\tau_n)}(\lambda)=e^{\lambda\tau}\cdot e^{-\lambda\tau}f[x(\tau-0)]1_{(x(\tau-0)\in B_e)}.$$

由此利用定理12.5而得(1)，证毕。

定理 2 设波雷尔集$\Gamma\subset B$，X^Γ按(12.7)确定。则$P\gamma$几乎有

$$\lim_{n\to\infty} X^\Gamma_{x(\tau_n)}=\begin{cases} 1, & \text{如}x(\tau-0)\in\Gamma, \\ 0, & \text{如}x(\tau-0)\in B-\Gamma. \end{cases} \tag{2}$$

证 这是定理3.22的特款。

设f是在B_p上为0在B_e上非负的有界波雷尔函数．这样的f组成的类记为$\mathscr{F}_e^+$．如果$f_1, f_2 \in \mathscr{F}_e^+$且$\mu\{f_1 \neq f_2\} = 0$，视$f_1$和$f_2$为同一函数。

定理 3 $\mathscr{M}_\lambda^+$中元$u(\lambda)$的一般形式是

$$u_i(\lambda) = E_i\{e^{-\lambda\tau} f[x(\tau-0)]\} = \int_{B_e} K_\lambda(i, \xi) f(\xi)\mu(d\xi),$$
$$f \in \mathscr{F}_e^+. \tag{3}$$

$\mathscr{M}^+ \cap \overline{\overline{\mathscr{M}}}^+$中元$u$的一般形式为

$$u_i = E_i\{f[x(\tau-0)]\} = \int_{B_e} K(i,\xi) f(\xi)\mu(d\xi), \quad f \in \mathscr{F}_e^+. \tag{4}$$

$u(\lambda) \in \mathscr{M}_\lambda^+$，$u \in \mathscr{M}^+ \cap \overline{\overline{\mathscr{M}}}^+$，$f \in \mathscr{F}_e^+$在下列条件下是一一对应的：

$$u_i(\lambda) = E_i\{e^{-\lambda\tau} f[x(\tau-0)]\} = E_i\{e^{-\lambda\tau} \lim_{n\to\infty} u_{x(\tau_n)}\}, \tag{5}$$

$$u_i = E_i f[x(\tau_\infty - 0)] = E_i\{\lim_{n\to\infty} u_{x(\tau_n)}(\lambda)\}, \tag{6}$$

$$f[x(\tau-0)] \doteq \lim_{n\to\infty} u_{x(\tau_n)}(\lambda) \doteq \lim_{n\to\infty} u_{x(\tau_n)}. \tag{7}$$

证 由定理12.6，如$f \in \mathscr{F}_e^+$，则由(5)(6)确定的$u(\lambda) \in \mathscr{M}_\lambda^+$，$u \in \mathscr{M}^+ \cap \overline{\overline{\mathscr{M}}}^+$。

设$u(\lambda) \in \mathscr{M}_\lambda^+$，其标准映象为$u \in \mathscr{M}^+ \cap \overline{\overline{\mathscr{M}}}^+$．依定理5.7，存在$B$上的非负有界波雷尔函数$g$使$u_i = E_i g[x(\tau-0)]$．依定理12.6，$u$的$\lambda$映象为$u_i(\lambda) = E_i\{e^{-\lambda\tau} g[x(\tau-0)]\}$．取$f = g \cdot 1_{B_e} \in \mathscr{F}_e^+$便知$u(\lambda)$有形式(3)。

由定理1得(7)第一等式．再由(4)及定理5.7得$f[x(\tau-0)] = \lim_{n\to\infty} u_{x(\tau_n)}$，从而得(7)第二等式，证毕。

§14．最小Q过程的原子流出边界和非原子流出边界

定义 1 分别称$B_{ea} = B_e \cap B_a (a=1, 2)$为最小$Q$过程（或矩

阵Q）的原子流出边界和非原子流出边界。类似可定义原子消极边界和非原子消极边界$B_{pa}=B_p\cap B_a(a=1,\ 2)$。

定理 1 当$\xi\in B_{e1}$时，$K_\lambda(\cdot,\xi)$有界；当$\xi\in B_{e2}$时，$K_\lambda(\cdot,\xi)$无界。

证 由标准映象及λ映象的定义以及定理12.2，当$\xi\in B_e$时，$K(\cdot,\ \xi)$与$K_\lambda(\cdot,\ \xi)$同时有界或同时无界。

设 $\xi\in B_{e1}$，则$\mu(\xi)>0$。由(3.40)，

$1\geqslant p_i(x(\tau-0)=\xi)=K(i,\ \xi)\mu(d\xi)$，

$$K(i,\ \xi)\leqslant\frac{1}{\mu(\xi)}<\infty,$$

即$K(\cdot,\ \xi)$有界。

设 $\xi\in B_e$，$K(\cdot,\ \xi)$有界。于是$K(\cdot,\ \xi)$是 $K_\lambda(\cdot,\ \xi)$ 的标准映象，而且是有界Π 调和函数，即 $K(\cdot,\ \xi)\in\mathscr{M}^+\cap\bar{\mathscr{M}}^+$。依定理13.3，存在$B_e$上的非负有界Borel函数$f$使

$$K(\cdot,\ \xi)=\int_{B_e}K(\cdot,\ \zeta)f(\zeta)\mu(d\zeta).$$

由唯一性定理3.18，必定$f(\zeta)\mu(d\zeta)=\delta_\xi(d\zeta)$，从而 $f(\xi)\mu(\xi)=1$。而f有界，故$\mu(\xi)>0$，即 $\xi\in B_{e1}$。证完。

定义 1 称$\hat{\mathscr{M}}_\lambda^+$中的元为方程

$$\lambda u-Qu=0,\ \lambda>0 \tag{1}$$

的非负解。称$u(\lambda)\in\hat{\mathscr{M}}_\lambda^+$为方程(1)的极小解或完全非极小解，如果$u(\lambda)$作为$\Pi(\lambda)$调和函数是极小的或完全非极小的。

定理 2 $B_{e1}=\phi$的充要条件是方程（1）不存在有界的非负极小解；$B_{e2}=\phi$的充要条件是方程（1）不存在有界的非负完全非极小解，或等价地，方程(1)不存在这样的非负极小解 $u(\lambda)$：$u(\lambda)$无界而$u(\lambda)$的标准映象γ可积。

证 由定理1，定理12.2的系和定理13.3得出。

§15. 流出的几乎闭集和最小Q过程的布勒克韦分解

设 X 为最小 Q 过程。依定理 5.6，根据嵌入矩阵Π的布勒克

韦分解(1.2.19)中，指标集$\mathscr{A}$的势与原子边界B_1的势相等，因此我们可以认为$\mathscr{A}=B_1$。于是有

$$\mathscr{L}(A_a)=\{x(\tau-0)=a\},a\in B_1. \tag{1}$$

$$\mathscr{L}(A_0)=\{x(\tau-0)\in B_2\}. \tag{2}$$

定义1 设A为几乎闭集，称A为流出的，如果

$$P\{\tau<\infty\,|\,\mathscr{L}(A)\}=1. \tag{3}$$

称A为消极的，如果

$$P\{\tau=\infty\,|\,\mathscr{L}(A)\}=1. \tag{4}$$

显然，如果A是原子几乎闭集，如果A不是流出的，就必定是消极的。但如果A是非原子几乎闭集，上述结论不真。

对于完全非原子几乎闭集A_0，依定理5.6和12.5，可以有分解$A_0=A_{0e}\cup A_{0p}$，其中A_{0e}和A_{0p}都是几乎闭集，使得

$$\mathscr{L}(A_{0e})=\{x(\tau-0)\in B_{e2}\}=\{x(\tau-0)\in B_2,\tau<\infty\},$$

$$\mathscr{L}(A_{0p})=\{x(\tau-0)\in B_{p2}\}=\{x(\tau-0)\in B_2,\tau=\infty\}.$$

因此A_{0e}是流出的，A_{0p}是消极的。这样，我们得到最小Q过程的布勒克韦分解。

定理1 设$X=\{x(t),\ t<\tau\}$为最小Q过程，则它的状态空间E有下列分解

$$E=A_{0e}\cup\Big(\bigcup_{a\in B_{e1}}A_a\Big)\cup A_{0p}\cup\Big(\bigcup_{a\in B_{p1}}A_a\Big). \tag{5}$$

其中A_{0e}和A_{0p}分别为流出的和消极的完全非原子几乎闭集，可以不出现。$A_a(a\in B_{e1})$为流出的原子几乎闭集，$A_a(a\in B_{p1})$为消极的原子几乎闭集。除瞬时集不计外，分解是唯一的。

§16. 有限流出的条件

定理1 设有限或可列无限个流出的几乎闭集A_a互不相交。记

$$X_i^a(\lambda)=E_i\{e^{-\lambda\tau},\ \mathscr{L}(A_a)\}. \tag{1}$$

如果 $\sum_{a}|c_a|<\infty$，$\sum_{a}c_aX^a(\lambda)=0$，则$c_a=0$。

证 由定理5.6，存在B_e中互不相交的波雷尔集Γ_a使$\mathscr{L}(A_a)=\{x(\tau-0)\in\Gamma_a\}$而且$\mu(\Gamma_a)>0$。依定理12.4，$X^a(\lambda)$的标准映象为$X_i^a=P_i\{x(\tau-0)\in\Gamma_a\}$。于是由$\sum_{a}C_aX^a(\lambda)=0$得$\sum_{a}C_aX^a=0$。

依定理3.22，$\sum_{a}C_a1_{(x(\tau-0)\in\Gamma_a)}=0$，即 $\sum_{a}C_a1_{\Gamma_a}=0$($\mu$几乎)，从而$C_a=0$。证毕。

定理 2 设a，$b\in B_{e1}$。则

$$X_i^a(\lambda)=E_i\{e^{-\lambda\tau},\ x(\tau-0)=a\}\longrightarrow\begin{cases}1 \text{ 当 } i\to a,\\ 0 \text{ 当 } i\to b\neq a.\end{cases}\tag{2}$$

证 依王梓坤［2，§0.2，定理4］，对任意 $b\in B_{e1}$，极限$\lim\limits_{i\to b}X_i^a(\lambda)$存在，故只需计算其极限，而这可由定理13.2得出。

定理 3 设$0\leqslant n<\infty$。下列条件等价

(i) 分解(15.5)中，A_{0e}不出现，集合B_{e1}的势为n。

(ii) $\mathscr{M}_\lambda^+$的维数为n。

(iii) B_e仅由n个原子边界点组成。

证 (i)与(iii)等价是显然的。

(i)⇒(ii)。设A_1，…，A_n为流出的原子。由定理1，$X^a(\lambda)$ $(1\leqslant a\leqslant n)$相互独立。其次，由定理13.3，对$u(\lambda)\in\mathscr{M}_\lambda^+$，有$f$使

$$u_i(\lambda)=E_i\{e^{-\lambda\tau}f[x(\tau-0)]\}=\sum_{a=1}^{n}E_i\{e^{-\lambda\tau}f[x(\tau-0)],$$

$$\mathscr{L}(A_a)\}=\sum_{a=1}^{n}f(a)X_i^a(\lambda).$$

即$\mathscr{M}_\lambda^+$的维数为n。

(ii)⇒(i)。先说明$P_i\{\mathscr{L}(A_{0e})\}=0$ $(i\in E)$。不然，存在i使$P_i\{\mathscr{L}(A_{0e})\}>0$。由于$A_{0e}$完全非原子，故存在无限多个不相交

的几乎闭集，从而由定理1，$\mathscr{M}_{\lambda}^{+}$有多于$n$个线性独立解。这与(ii)冲突。故$A_{0e}$不出现。由(i)⇒(ii)的证明过程知，流出的原子个数与$\mathscr{M}_{\lambda}^{+}$的维数一致，证毕。

§17．一个条件独立定理

设$X=\{x(t),\ t<\sigma\}\in\mathscr{X}_{s}(Q)$为非最小$Q$过程。$\alpha$为$X$的马时，$P\{x(\alpha)=\infty\}=0$，$P\{\alpha<\sigma\}>0$。记$\alpha$后的第一个飞跃点为$\tau_{\alpha}$，在$(\alpha<\sigma)$上，$\theta_{\alpha}\tau=\tau_{\alpha}$，令$\tau_{\alpha}^{n}=\theta_{\alpha}\tau_{n}$。记$\mathscr{F}_{\tau_{\alpha}^{n}}$为前$\tau_{\alpha}^{n}$域，$\mathscr{F}_{\tau_{\alpha}^{-}}$为含一切$\mathscr{F}_{\tau_{\alpha}^{n}}(n\geqslant 1)$的$\Omega_{\tau_{\alpha}}$上的最小波雷尔域。

定理1 设$A\in\mathscr{F}_{\infty}^{0}$，$\wedge\in\mathscr{F}_{\tau_{\alpha}^{-}}$。存在定义在$H\cup B_{e}$上的，与$x(\alpha)$关于$(\alpha<\sigma)$的条件分布无关的波雷尔函数$f$，使在$\{x(\tau_{\alpha}-0)\in H\cup B_{e}\}$上，$P_{\gamma}$几乎成立

$$P\{\theta_{\tau_{\alpha}}A\mid\wedge,\ x(\tau-0)\}=f\{x(\tau_{\alpha}-0)\}. \tag{1}$$

(1)式左方的条件理解为$|_{\wedge}$，$x(\tau_{\alpha}-0)$。

为了证明定理1，我们需要引用Дынкин[1，第782页]中的一个定理，叙述如下。

定理2 设$(\Omega,\ \mathscr{F},\ P)$是概率空间，$E|\xi|<\infty$，$\Omega_{n}\in\mathscr{F}$，$\Omega_{n}\downarrow\Omega'(n\uparrow\infty)$，$\mathscr{A}_{n}\subseteq\mathscr{F}$，$\mathscr{A}_{n}$是$\Omega_{n}$上的$\sigma$代数，且$\mathscr{A}_{m}\cap\Omega_{n}\subseteq\mathscr{A}_{n}(m<n)$。则在$\Omega'$上几乎有

$$\lim_{n\to\infty}E(\xi\mid\mathscr{A}_{n})=E(\xi\mid\mathscr{A}).$$

其中$\mathscr{A}$是Ω上含一切$\mathscr{A}_{n}$的最小σ代数。

定理1的证明 先证(1)在$M\equiv\{X(\tau_{\alpha}-0)\in H\}$上几乎处处成立。为此，令

$$\beta=\begin{cases}\tau_{\alpha}^{n-1}, & \text{如}\omega\in M,\ \tau_{\alpha}=\tau_{\alpha}^{n},\ \tau_{\alpha}^{n-1}<\tau_{\alpha}^{n}\\ \infty, & \text{如}\omega\overline{\in}M.\end{cases}$$

注意$A\in\mathscr{F}_{\tau_{\alpha}-}$时，$A(X(\beta)=i)\in\mathscr{F}_{\beta}$，又$\{X(\tau_{\alpha}-0)=i\}=\{X(\beta)=i\}$。由于定理6.2(i)，当$X(\tau_{\alpha}-0)=i\in H$时，(1)是成立的，而且$f$在$H$上的限制$f(a)$，$a\in H$。与$X(\alpha)$关于$\alpha<\sigma$的条件分

布无关。故(1)在M上几乎处处成立。

次证(1)在$N\equiv\{X(\tau_\alpha-0)\in B_e\}$上几乎处处成立，即要证

$$1_N P\{\theta_{\tau_\alpha}A\mid A, X(\tau_\alpha-0)\}=1_N f\{X(\tau_\alpha-0)\}. \tag{2}$$

依条件概率的定义，在N上几乎有

$$P\{\theta_{\tau_\alpha}A,\ M\mid A,\ X(\tau_\alpha-0)\}=0,$$

即

$$1_N P\{\theta_{\tau_\alpha}A,\ M\mid A,\ X(\tau_\alpha-0)\}=0,$$

故(2)左方等于

$$1_N P\{\theta_{\tau_\alpha}A,\ M\mid A,\ X(\tau_\alpha-0)\}. \tag{3}$$

现在令$\Omega_n=(\tau_\alpha^n<\tau_\alpha)$，$\mathscr{A}_n=\mathscr{F}_{\tau_\alpha^n}$，$\xi=1_{N\cap\theta_{\tau_\alpha}A}$。则定理2的条件满足。显然地，$N\subset\Omega'=\bigcap_n\Omega_n$，在$\Omega_n$上有$\xi=\theta_{\tau_\alpha^n}\eta$，其中$\eta=1_{(\theta_{\tau}A)\cap(X(\tau-0)\in B_e)}$。而且易见

$$u(i)\equiv E_i\eta\in\mathscr{A}^+.$$

这样，由强马氏性及定理2，在Ω'上几乎有

$$E(\xi\mid\mathscr{A})=\lim_{n\to\infty}E(\xi\mid\mathscr{A}_n)=\lim_{n\to\infty}E_{X(\tau_\alpha^n)}\eta$$
$$=\lim_{n\to\infty}u\{X(\tau_\alpha^n)\}.$$

从而有

$$1_N E(\xi\mid\mathscr{A})=\lim_{n\to\infty}1_N u\{X(\tau_\alpha^n)\}.$$

易见$\Phi=\lim\limits_{n\to\infty}1_N u\{X(\tau_\alpha^n)\}$是非负的终极随机变量，因而存在$B$上的非负Bord可测函数$f$使$\Phi=f\{X(\tau_\alpha-0)\}=1_N f\{X(\tau_\alpha-0)\}$.

于是

$$1_N E(\xi\mid\mathscr{A})=1_N f\{x(\tau_\alpha-0)\}.$$

因$A\in\mathscr{F}_{\tau_\alpha-}\subset\mathscr{A}$，$x(\tau_\alpha-0)$为$\mathscr{A}$可测，故(2)式左方即(3)式等于

$$1_N E\{\xi\mid A,\ x(\tau_\alpha-0)\}=1_N E\{E(\xi\mid\mathscr{A})\mid A,\ x(\tau_\alpha-0)\}$$
$$=E\{1_N E(\xi\mid\mathscr{A})\mid A,\ x(\tau_\alpha-0)\}$$
$$=E\{1_N f[x(\tau_\alpha-0)]\mid A,\ x(\tau_\alpha-0)\}$$
$$=1_N f\{x(\tau_\alpha-0)\}.$$

得证(2)。

往证f在B_e上的限制$f(\alpha)$，$\alpha\in B_e$与$x(\alpha)$关于$(\alpha<\sigma)$的条件分布无关，但需视f与μ-几乎处处相等的函数为同一函数，μ是终极状态分布．为此，只需应用下述事实：设(x_n)是马氏链，$u(i)$ $(i\in E)$为实值函数。如果对任意初始分布，以概率1极限$\lim\limits_{n\to\infty}u(x_n)$存在，则此极限与初始分布无关．在上述事实中，取定义在概率空间$((\alpha<\sigma),(\alpha<\sigma)\mathscr{F},\ P(\cdot|\alpha<\sigma))$上的马氏链$x_n=x(\tau_\alpha^n)$，便得所欲证。

上述事实的证明：设分布$v=(v_i)$满足$v_i>0$，一切i。由$P_v(u(x_n)\to A(v))=1$得$P_i(u(x_n)\to A(v))=1$，一切i。另一方面由假定$P_i(u(x_n)\to A(\delta_i))=1$，$\delta_i$表示集中于$i$的单点分布．故$P_i(A(u)=A(\delta_i))=1$。今设$v'$为任一分布，当$v_i'>0$时，重复上述论证得$P_i(A(v)=A(\delta_i)=A(v'))=1$，$P_i(A(v)=A(v'))=1$，从而有$P_{v'}(A(v)=A(v'))=1$。由对称性，$P_v(A(v)=A(v'))=1$。如果$\tilde{v}$又是一分布，由上述$P_v(A(v)=A(\tilde{v}))=1$，从而$P_v(A(v')=A(\tilde{v}))=1$，$P_i(A(v')=A(\tilde{v}))=1$（一切$i$），$P_{v'}(A(v')=A(\tilde{v}))=1$。

依定理5.7，存在非负Borel函数$f_{v'}$及$f_{\tilde{v}}$使$A(v')=f_{v'}(x_\infty)$，$A(\tilde{v})=f_{\tilde{v}}(x_\infty)$。故$P_{v'}(f_{v'}(x_\infty)=f_{\tilde{v}}(x_\infty))=1$，即$f_{v'}=f_{\tilde{v}}$ μ几乎处处成立。证完

系　设$\wedge\in\mathscr{F}_{\tau_\alpha-}$，$A\in\mathscr{F}_\infty^0$，则在$\{\tau<\infty,\ x(\tau-0)\}$之下，$\wedge$与$\theta_\tau A$条件独立．即在$\{x(\tau-0)\in H\cup B_e\}$上几乎成立

$$P\{\wedge\theta_\tau A|x(\tau-0)\}=P\{\wedge|x(\tau-0)\}P\{\theta_\tau A|x(\tau-0)\}. \tag{4}$$

§18．Q过程的一般形式的进一步刻划

设$X=\{x(t),\ t<\sigma\}\in\mathscr{X}_s(Q)$为非最小Q过程，其预解算子为$\psi(\lambda)$。

定理1　$\psi(\lambda)$有下列表现：

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+\sum_{a\in H\cup B_{e1}} X_i^a(\lambda)F_j^a(\lambda)$$

$$+\int_{B_{e2}} X_i(\lambda,da)F_j(\lambda,\ a). \tag{1}$$

其中 $X_i^a(\lambda)=E_i\{e^{-\lambda\tau},\ x(\tau-0)=a\}=\phi_{ia}(\lambda)d_a,\ a\in H.$

(2)

$$X_i^a(\lambda)=E_i\{e^{-\lambda\tau},\ x(\tau-0)=a\},\ a\in B_{e1}, \tag{3}$$

$$X_i(\lambda,\ \Gamma)=E_i\{^{-\lambda\tau},\ x(\tau-0)\in\Gamma\},\ \Gamma\subset B_{e2}.$$

(4)

而 $F^a(\lambda)\geqslant 0,\ \lambda[F^a(\lambda),\ 1]\leqslant 1,\ a\in H\cup B_{e1}.$ (5)

$F(\lambda,a)\geqslant 0,\ \lambda[F(\lambda,\ a),\ 1]\leqslant 1,$ 对μ几乎一切$a\in B_{e2}$.

(6)

证

$$\psi_{ij}(\lambda)=E_i\int_0^\sigma e^{-\lambda\tau}1_j[x(t)]dt$$

$$=E_i\int_0^\tau e^{-\lambda\tau}1_j[x(t)]dt+E_i\int_\tau^{\sigma-\tau} e^{-\lambda t}1_j[x(t)]dt$$

$$=\phi_{ij}(\lambda)+\ E_ie^{-\lambda\tau}\int_\tau^{\sigma-\tau} e^{-\lambda(t-\tau)}1_j[x(t)]dt.$$

由定理17.1的系，上式第二项等于

$$E_i\{e^{-\lambda\tau}1_{(\tau<\infty)}\theta_\tau\int_0^\sigma e^{-\lambda\tau}1_j[x(t)]dt\}$$

$$=\int_{H\cup B_e} E_i\{e^{-\lambda\tau},\ x(\tau-0)\in da\}E_i\Big\{\theta_\tau\int_0^\sigma e^{-\lambda t}1_j[x(t)]dt$$

$$\Big|_{x(\tau-0)=a}\Big\}=\int_{H\cup B_e} X_i(\lambda,\ da)F_j(\lambda,\ a).$$

由此便得(1)及(5)(6)。至于(2)第二等号，是由于定理10.1，证毕。

§19. 瞬返过程及其边界

回忆§7中定义的D型过程类$\mathscr{X}_D$及跳跃点和飞跃点的定义

8.1. 约定最小过程为零阶瞬返过程。

定义1 设$X=\{x(t),\ t<\sigma\}\in\mathscr{X}_D$为非最小过程。如果对几乎一切$\omega$，对任意的$t<\sigma(\omega)$，$X(\omega)$在$[0,t)$中最多只有有穷多个飞跃点，称$X$为一阶瞬返过程，简称一阶过程。一阶过程(或一阶$Q$过程)组成的类记为$\mathscr{X}_1$(或$\mathscr{X}_1(Q)$)。

设$X\in\mathscr{X}_1$。则X的不连续点可以排序：

$$0=\tau_0,\ \tau_1,\ \tau_2,\ \cdots;\ \tau_\omega,\ \tau_{\omega+1},\ \cdots;\ \tau_{\omega 2},\ \tau_{\omega 2+1},\ \cdots,\ \tau_{\omega n},\ \tau_{\omega n+1},\ \cdots\cdots \tag{1}$$

其中或者对某个序数$a<\omega^2$有$\tau_a=\sigma$，或者对一切序数$a<\omega^2$有$\tau_a<\sigma$，且$\lim\limits_{n\to\infty}\tau_{\omega n}=\sigma$。

如果$X=\{x(t),\ t<\sigma\}\in\mathscr{X}_D$，则$X$的不连续点仍可排序：

$$0=\tau_0,\ \tau_1,\ \tau_2,\ \cdots\cdots;\ \tau_\omega,\ \tau_{\omega+1},\ \cdots\cdots;\ \tau_{\omega 2},\ \cdots\cdots,\ \tau_{\omega n},\ \cdots\cdots,\ \tau_{\omega^l},\ \cdots\cdots,\ \tau_{\omega^l+\omega m+n},\ \cdots\cdots. \tag{2}$$

我们用归纳法定义k阶瞬返过程

定义2 称$X=\{x(t),\ t<\sigma\}\in\mathscr{X}_D$为$k$阶瞬返过程，如果$X$不是$n(0\leqslant n\leqslant k-1)$阶瞬返过程，而且或者存在某序数$a<\omega^{k+1}$使$\tau_a=\sigma$，或者对一切序数$a<\omega^{k+1}$有$\tau_a<\sigma$，且$\tau_{\omega^{k+1}}=\sigma$。$k$阶瞬返过程组成的类记为$\mathscr{X}_k$。$k$阶瞬返$Q$过程组成的类记为$\mathscr{X}_k(Q)$。

设$X\in\mathscr{X}_k(Q)$。称嵌入链$X_T^0=\{x(\tau_n),\ \tau_n\leqslant\tau_\beta\}$为$X$的零阶嵌入链，其中$\tau_\beta$按(8.9)—(8.11)定义，$X_T^0$的一步转移概率为

$$_0\Pi_{ij}=P_i\{x(\tau_1)=j\}. \tag{3}$$

称$X_T^1=\{x(\tau_{\omega n}),\tau_{\omega n}\leqslant\tau_{\omega\beta}\}$为一阶嵌入链，其中$\tau_{\omega\beta}$可仿(8.9)—(8.11)定义。$X_T^1$的一步转移概率为

$$_1\Pi_{ij}=P_i\{x(\tau_\omega)=j\}. \tag{4}$$

$X_T^l=\{x(\tau_{\omega^l n}),\tau_{\omega^l n}\leqslant\tau_{\omega^l\beta}\}$为$l$阶嵌入链，其中$\tau_{\omega^l\beta}$可仿(8.9)—(8.11)定义。$X_T^l$的一步转移概率为

$$_l\Pi_{ij}=P_i\{x(\tau_{\omega^l})=j\} \tag{5}$$

类似地，可以确定

$$_0\Pi_{ij}(\lambda)=E_i\{e^{-\lambda\tau_1},\ x(\tau_1)=j\},\ \lambda>0. \tag{6}$$

$$_l\Pi_{ij}(\lambda)=E_i\{e^{-\lambda\tau_{\omega^l}},\ x(\tau_{\omega^l})=j\},\ \lambda>0. \tag{7}$$

象§12§14根据$_0\Pi$，$_0\Pi(\lambda)$确定马亭边界∂E，本质马亭边界B，马亭流出边界B_e，马亭消极边界B_p，原子流出边界B_{e1}，非原子流出边界B_{e2}，原子消极边界B_{p1}，非原子消极边界B_{p2}等等一样，根据$_l\Pi$和$_l\Pi(\lambda)$可以确定l阶马亭边界$_l(\partial E)$，l阶本质马亭边界$_lB$，l阶流出边界$_lB_e$，l阶消极边界$_lB_p$，l阶原子流出边界$_lB_{e1}$，l阶非原子流出边界$_lB_{e2}$，l阶原子消极边界$_lB_{p1}$和l阶非原子消极边界$_lB_{p2}$等等。最小过程即零阶过程的边界理论，很多都可以移植到k阶瞬返过程中来。例如，对于k阶瞬返过程$X=\{x(t),\ t<\sigma\}\in\mathscr{X}_k$，如果令

$$_k\Omega_F=\{存在序数\alpha<\omega^{k+1}使\tau_\alpha=\sigma<\infty\},$$

$$_k\Omega_\infty=\{对一切序数\alpha<\omega^{k+1}有\tau_\alpha<\sigma\}+\{存在序数$$
$$\alpha<\omega^{k+1}使\tau_\alpha=\sigma=\infty\}.$$

$$H={}_0H\cup{}_1B_e\cup{}_1B_e\cup\cdots\cup{}_{k-1}B_e,\quad {}_0H=H,\quad {}_0B_e=B_e.$$

则在$_k\Omega_F$上有$x(\sigma-0)\in{}_kH$，在$_k\Omega_\infty$上有$x(\sigma-0)\in{}_kB$。类似定理12.5和12.8有

定理1 设$X=\{x(t),\ t<\sigma\}\in\mathscr{X}_k$，则对一切$i\in E$有

$$P_i\{\sigma<\infty\,|\,x(\sigma-0)\in{}_kB_e\}=1,\tag{8}$$

$$P_i\{\sigma=\infty\,|\,x(\sigma-0)\in{}_kB_p\}=1,\tag{9}$$

$$P_i\{\sigma<\infty\,|\,x(\sigma-0)\in{}_kH\}=1.\tag{10}$$

第七章　有限非保守有限流出 Q过程的构造

§1. 引言

对于一般情形的Q过程的构造，通常都假定Q保守。在此假设下，Feller[3]对有限流出和有限流入情形，构造了满足向前方程组的全部Q过程。杨向群[1]在同样假设下，构造了全部Q过程。Williams[1]和Chung[2]在Q保守且有限流出的情形下，求出了全部Q过程。杨向群[13]中，对有限非保守和有限流出（简称双有限）情形，构造了全部Q过程，但其结果与Williams[1]中的结果的联系不是很明显。为此，熊大国[3]仿照Williams[1]的办法，构造了双有限的全部Q过程，但其结果与Feller[3]杨向群[1]的结果联系仍不明显。本章中，我们将在双有限条件下，按Feller[3]杨向群[1]的方式构造出全部Q过程，并指出与Williams[1]、熊大国[3]结果的联系。本章内容取自杨向群[14]。

§2. 基本假定及$F^a(\lambda)$满足的条件

给定$Q=(q_{ij})$满足(1,2,6)，$d=Q1$是非保守量，$H_e=\{i:d_i>0\}$是非保守状态集，B_e是由Q导出的马亭流出边界。

基本假定　设$A_e=H_e\cup B_e$是有限集。H_e或B_e可以是空集。但B_e是空集时，或者$\mathscr{L}_1^+$非空，或者

$$\inf_i \lambda \sum_j \phi_{ij}(\lambda)=0,\ \lambda>0. \tag{1}$$

在基本假定下，Q过程不唯一。

当$a\in H_e$时，$X_i^a(\lambda)=\phi_{ia}(\lambda)d_a$是列协调族，且

$$X_i^a(\lambda)\uparrow X_i^a=\Gamma_{ia}d_a,\ \lambda\downarrow 0,\tag{2}$$

$$\lambda X_i^a(\lambda)\to\delta_{ia}d_a,\ \lambda\to\infty,\tag{3}$$

记$X(\lambda)$是$\mathscr{M}_\lambda^+(1)$的最大解，$X(\lambda)\uparrow X(\lambda\downarrow 0)$。如果$B_e$非空有限，依第六章§13—§16的讨论，可以选取列协调族$X^a(\lambda)$ ($a\in B_e$)使$X(\lambda)=\sum\limits_{a\in B_e}X^a(\lambda)$，而且

$$X_i^a(\lambda)\longrightarrow\delta_{ab},\ a,\ b\in B_e,\ i\longrightarrow b,\tag{4}$$

$$X^a(\lambda)\uparrow X^a,\ \lambda\downarrow 0,\ a\in B_e.\tag{5}$$

$$\lambda X^a(\lambda)\longrightarrow 0,\ a\in B_e,\ \lambda\longrightarrow\infty.\tag{6}$$

$$X_i^a(\lambda)\longrightarrow 0,\ a\in H_e,\ i\longrightarrow b\in B_e.\tag{7}$$

这样，在基本假设下，我们可以选取列协调族$X^a(\lambda)$ ($a\in A_e$)，其标准映象为$X^a(a\in A_e)$，使

$$\lambda\phi(\lambda)\mathbf{1}=\mathbf{1}-Z(\lambda),\ \ Z(\lambda)=\sum_{a\in A_e}X^a(\lambda),\tag{8}$$

$$Z(\lambda)\uparrow Z=\sum_{a\in A_e}X^a,\ \ \lambda\downarrow 0.\tag{9}$$

$X=\sum\limits_{a\in B_e}X^a$和$X^0=\mathbf{1}-Z$分别是$Q$的最大流出解和最大消极解。

$X^a(\lambda)$ ($a\in A_e$)是线性独立的。实际上，设$\sum\limits_{a\in A_e}c^aX^a(\lambda)=\mathbf{0}$。由(7)得$\sum\limits_{a\in B_e}c^aX^a(\lambda)=\mathbf{0}$，从而由(4)得$c^a=0(a\in B_e)$。于是$\sum\limits_{a\in H_e}c^aX^a(\lambda)=\mathbf{0}$。由引理1.11.6，$c^a=0$ ($a\in H_e$)。

依定理6.18.1，任何Q过程$\psi(\lambda)$具有下列形式

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+\sum_{a\in A_e}X_i^a(\lambda)F_j^a(\lambda),\tag{10}$$

其中当$a\in A_e$时，

$$F^a(\lambda)\geqslant 0,\qquad \lambda[F^a(\lambda),\ 1]\leqslant 1. \tag{11}$$

引理1 对于(10)中的 $\psi(\lambda)$，范条件等价于对每个 $a\in A_e$，(11)成立；预解方程等价于对每个$a\in A_e$，有

$$F^a(\lambda)A(\lambda,\ \mu)=F^a(\mu)+(\mu-\lambda)\sum_{b\in A_e}[F^a(\lambda),\ X^b(\mu)]\cdot F^b(\mu), \tag{12}$$

其中$A(\lambda,\mu)$见(1.10.7)，(12)也等价于

$$F^a(\lambda)-F^a(\mu)=(\mu-\lambda)F^a(\lambda)\psi(\mu); \tag{13}$$

Q条件等价于对每个$a\in H_e$，有

$$\lim_{\lambda\to\infty}\lambda F^a(\lambda)=0. \tag{14}$$

证 关于范条件的必要性条件(11)已在前面指出。充分性由

$$\lambda\psi(\lambda)1=1-\sum_{a\in A_e}X^a(\lambda)+\sum_{a\in A_e}X^a(\lambda)\lambda[F^a(\lambda),\ 1]$$

及(11)得出。

将(10)中的$\psi(\lambda)$代入预解方程，由于$\phi(\lambda)$满足预解方程，$X^a(\lambda)$ $(a\in A_e)$是列协调族而且是线性独立的，我们知$\psi(\lambda)$的预解方程等价于(12)。将(10)中的$\psi(\lambda)$代入(13)便知(13)与(12)等价。

注意$\phi(\lambda)$满足Q条件及A_e的有限性，我们得$\psi(\lambda)Q$的条件等价于

$$\lim_{\lambda\to\infty}\lambda X_i^a(\lambda)\lambda F_j^a(\lambda)=0,\quad a\in A_e.$$

由(3)和(6)，上式等价于对每个$a\in H_e$有(14)。

§3. 问题的简化

定义1 设a与$b\in A_e$，称a与b为不可辨的，如果对一切$\lambda>0$，有$F^a(\lambda)=F^b(\lambda)$。

显然，不可辨关系是一个等价关系。按不可辨关系，可以将A_e分解为互不相交的等价类a_1，a_2，…，我们直接地把等价类a_i视为一个新的边界点，并且记

$$A=\{a_1,\ a_2,\ \cdots\},\ A=H\cup B, \tag{1}$$

$$H=\{a: a\in A,\ a\cap H_e\neq\phi\},\quad B=\{a: a\in A,\ a\cap H_e=\phi\}. \tag{2}$$

记 $$Y^a(\lambda)=\sum_{b\subset a}X^b(\lambda)\uparrow Y^a=\sum_{b\in a}X^b,\ a\in A,\ \lambda\downarrow 0. \tag{3}$$

则(2.10)成为

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+\sum_{a\in A}Y^a(\lambda)F^a(\lambda). \tag{4}$$

A中的边界点都是可辨的。

引理1 存在子集$J\subset A$及$A\times J$矩阵$G=(G^{ab})$，G的每一列都不是零列，而且满足

$$G^{ab}\geqslant 0,\quad \sum_{b\in J}G^{ab}\leqslant 1\ (a\in A),\ G^{aa}=1(a\in J), \tag{5}$$

使得

$$F^a(\lambda)=\sum_{b\in J}G^{ab}F^b(\lambda),\ a\in A,\ \lambda>0. \tag{6}$$

而对于$a\in J$，$F^a(\lambda)$不能表示成下面的形式：

$$F^a(\lambda)=\sum_{b\in J}h^{ab}F^b(\lambda),\ \lambda>0 \tag{7}$$

其中

$$h^{aa}=0,\ h^{ab}\geqslant 0,\quad \sum_{b\subset J}h^{ab}\leqslant 1. \tag{8}$$

特别地有$F^a(\lambda)\neq 0(a\in J)$。

证 存在子集$J\subset A$及$A\times J$矩阵G使(5)(6)成立是明显的，例如取A为J，G为单位矩阵即可。但此时可能(7)(8)对某$a\in J$成立。

假定存在$J\subset A$及$A\times J$矩阵G使(5)(6)成立，而且存在$a_0\in J$，使(7)(8)对$a=a_0$成立。则由(6)(7)得

$$F^a(\lambda)=\sum_{b\in J_1}G_0^{ab}F^b(\lambda),\ a\in A,\ \lambda>0,$$

其中$J_0=J-\{a_c\}$，$A\times J_0$矩阵$G_0=(G_0^{ab})$的元素为

$$G_0^{ab}=G^{ab}+G^{aa_c}h^{a_c b},\quad a\in A,\ b\in J_0$$

由（5）（8），G_0满足

$$G_0^{ab}\geqslant 0,\quad \sum_{b\in J_0}G_0^{ab}\leqslant 1(a\in A),\quad G_0^{aa}=1\ (a\in J_0)$$

这样，用J_0及$A\times J_0$矩阵G_0代替J和G后，（5）（6）仍成立。按此手续继续，总可以得到子集J及$A\times J$矩阵G使（5）（6）成立，而对每个$a\in J$，（7）（8）不能同时成立。最后，如果G中有的列为零列，则可用$J-\{b\in J:\ G$的第b列为零列$\}$代替J而达到引理的要求。证完。

引理2 每个Q过程$\psi(\lambda)$有形式

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+\sum_{a\in J}Z_i^a(\lambda)F_j^a(\lambda),\tag{9}$$

其中子集$J\subset A$，J中的边界点都是可辨的，而

$$Z^b(\lambda)=\sum_{a\in A}Y^a(\lambda)G^{ab},\quad b\in J\tag{10}$$

是线性独立的列协调族。$A\times J$矩阵$G=(G^{ab})$的每一列都不为零，满足(5)，使（6）成立，对每个$a\in J$，（7）（8）不能同时成立。特别地，$F^a(\lambda)\neq 0(a\in J)$。

证 将（8）代入（4）即得（9）。由于$X^a(\lambda),a\in A$是线性独立的列协调族，因而可得（10）中的$Z^b(\lambda)$（$b\in J$）是线性独立的列协调族。剩下的结论由引理1得出。

引理3 设Q过程$\psi(\lambda)$具有引理2中的形式(9)。则$\psi(\lambda)$的范条件等价于

$$F^a(\lambda)\geqslant 0,\ \lambda[F^a(\lambda),\ 1]\leqslant 1,\ a\in J;\tag{11}$$

预解方程等价于

$$F^a(\lambda)A(\lambda,\mu)=F^a(\mu)+(\mu-\lambda)\sum_{b\in J}[F^a(\lambda),\ Z^b(\mu)]F^b(\mu),\quad a\in J;\tag{12}$$

Q条件等价于

$$\lim_{\lambda\to\infty}\lambda F^a(\lambda)=0,\quad a\in J_H,\tag{13}$$

其中

$$J_H = J \cap H = \{a \in J;\ a \cap H_e \neq \phi\}, \tag{14}$$

$$J_B = J \cap B = \{a \in J;\ a \cap H_e = \phi\}, \tag{15}$$

$$J = J_H \cup J_B. \tag{16}$$

证 利用引理2.1及引理3.2得出。

记

$$Z^b = \sum_{a \in A} Y^a G^{ab},\ b \in J, \tag{17}$$

$$Z^*(\lambda) = Z(\lambda) - \sum_{b \in J} Z^b(\lambda) = \sum_{a \in A} Y^a(\lambda)\Big(1 - \sum_{b \in J} G^{ab}\Big), \tag{18}$$

$$Z^* = Z - \sum_{b \in J} Z^b = \sum_{a \in A} Y^a\Big(1 - \sum_{b \in J} G^{ab}\Big), \tag{19}$$

则显然

$$Z^b(\lambda) \uparrow Z^b\ \ (b \in J),\ \ Z^*(\lambda) \uparrow Z^*,\ \lambda \downarrow 0, \tag{20}$$

$$X^0 + Z^* + \sum_{a \in J} Z^a = 1. \tag{21}$$

§4. $F^a(\lambda)$的一般形式

引理1 设$g \in \mathscr{l}$，则

$$C_{\nu\lambda}^{-1}\|g\| \leqslant \|gA(\lambda,\ \nu)\| \leqslant C_{\lambda\nu}\|g\|. \tag{1}$$

这里$\|g\|$表示巴拿赫空间$\mathscr{l}$中的范数，

$$C_{\lambda\nu} = 1 + \frac{|\lambda - \nu|}{\nu}. \tag{2}$$

证 由$A(\lambda,\ \nu)$的定义

$$\|gA(\lambda,\nu)\| \leqslant \|g\| + \frac{|\lambda - \nu|}{\nu}[\,|g|,\ \nu\phi(\nu)1]$$

$$\leqslant \|g\| + \frac{|\lambda - \nu|}{\nu}[\,|g|,\ 1] = C_{\lambda\nu}\|g\|,$$

得证(1) 右边的不等式。左边不等式由

$$\|g\| = \|gA(\lambda,\ \nu)A(\nu,\ \lambda)\| \leqslant C_{\nu\lambda}\ \|gA(\lambda,\ \nu)\|$$

得出。

引理2 设非零的$F^a(\lambda)(a\in J)$ 满足（3.11）和（3.12）。则存在非负数$M^{ab}(\lambda)(a,\ b\in J)$ 及行协调族$\eta^a(\lambda)(a\in J)$使得

$$F^a(\lambda) = \sum_{b\in J} M^{ab}(\lambda)\eta^b(\lambda),\quad a\in J. \tag{3}$$

而且还可以选取$M^{ab}(\lambda)$和$\eta^b(\lambda)$具有下列性质：

(*i*) $\eta^b(\lambda)=0$ 当且仅当$M^{ab}(\lambda)=0(a\in J)$；

(*ii*) 存在$J\times J$非负矩阵$H=(H^{ab})$使

$$\eta^a(\lambda) = \sum_{b\in J} H^{ab}\eta^b(\lambda),\quad a\in J; \tag{4}$$

(*iii*) 存在正数$\nu_a(a\in J)$ 使当μ沿某一子列$\mu_n\to\infty$时有

$$H^{ab}(\mu) = \frac{M^{ab}(\mu)}{\|F^a(\mu)A(\mu,\nu_a)\|} \to H^{ab},\quad a,\ b\in J. \tag{5}$$

如果$\eta^a(\lambda)\neq 0$，则在强收敛意义下，当$\mu=\mu_n\to\infty$时，

$$\eta^a(\lambda,\ \mu) = \frac{F^a(\mu)A(\mu,\lambda)}{\|F^a(\mu)A(\mu,\nu_a)\|} \longrightarrow \eta^a(\lambda)\ \text{（强收敛）}. \tag{6}$$

证 首先注意，对一切$\lambda,\ \mu>0$有 $F^a(\lambda)A(\lambda,\ \mu)$ 非负。实际上，由$A(\lambda,\ \mu)$ 的定义，当 $\lambda\geqslant\mu$时，结论显然。当 $\lambda<\mu$时，由于（3.12）右方非负，从而左方也非负。因$F^a(\lambda)$非零，由(1)知$\|F^a(\lambda)A(\lambda,\ \mu)\|>0$。

右乘$A(\mu,\ \lambda)$ 到(3.12) 得

$$F^a(\lambda) = \sum_{b\in J} M^{ab}(\nu_b,\ \lambda,\ \mu)\eta^b(\nu_b,\ \lambda,\ \mu), \tag{7}$$

其中$\nu_b>0$任意，且

$$M^{ab}(\nu,\ \lambda,\ \mu) = \sigma^{ab}(\lambda,\ \mu)\|F^b(\mu)A(\mu,\ \nu)\|, \tag{8}$$

$$\sigma^{ab}(\lambda,\ \mu) = \delta_{ab} + (\mu-\lambda)[F^a(\lambda),\ Z^b(\mu)], \tag{9}$$

$$\eta^b(\nu,\ \lambda,\ \mu) = \frac{F^b(\mu)A(\mu,\lambda)}{\|F^b(\mu)A(\mu,\nu)\|}. \tag{10}$$

显然

$$\|\eta^b(\nu,\ \nu,\ \mu)\| = 1. \tag{11}$$

由（1.10.9）及引理1，

$$C_{\lambda\nu}^{-1} \leqslant \|\eta^b(\nu,\ \lambda,\ \mu)\| \leqslant C_{\nu\lambda}; \tag{12}$$

$$C_{\nu\lambda}^{-1} M^{ab}(\lambda,\ \lambda,\ \mu) \leqslant M^{ab}(\nu,\ \lambda,\ \mu) \leqslant C_{\lambda\nu} M^{ab}(\lambda,\ \lambda,\ \mu). \tag{13}$$

在(7)中取$\nu_b = \nu > 0$有

$$F^a(\lambda) = \sum_{b \in J} M^{ab}(\nu,\ \lambda,\ \mu)\eta^b(\nu,\ \lambda,\ \mu). \tag{14}$$

由（12）得

$$\frac{1}{\lambda} \geqslant \|F^a(\lambda)\| \geqslant \sum_{b \in J} M^{ab}(\nu,\ \lambda,\ \mu) C_{\lambda\nu}^{-1}, \tag{15}$$

$$\frac{C_{\lambda\nu}}{\lambda} \geqslant \sum_{a \in J} M^{ab}(\nu,\ \lambda,\ \mu). \tag{16}$$

特别地，在（15）中取$\lambda = \nu$得

$$\|F^a(\nu)\| \geqslant \sum_{b \in J} M^{ab}(\nu,\ \nu,\ \mu). \tag{17}$$

于是，对每个$b \in J$，或者

$$\varlimsup_{\mu \to \infty} M^{ab}(\nu,\lambda,\mu) = 0 \quad \text{对一切} \nu > 0,\ \lambda > 0,\ a \in J, \tag{18}$$

或者

$$\varlimsup_{\mu \to \infty} M^{ab}(\nu,\lambda,\mu) > 0 \quad \text{对某个}\ \nu > 0,\ \text{某个}\lambda > 0\text{及某个}\ a \in J. \tag{19}$$

由(13)，上式等价于

$$\varlimsup_{\mu \to \infty} M^{ab}(\nu,\nu,\mu) > 0 \text{对某个}\nu > 0\text{和某个}a \in J. \tag{20}$$

设使（18）成立的$b \in J$全体为G_0，则$J - G_0 \neq \phi$. 因为不然的话，从(14)将导致$F^a(\lambda) = 0$，与$F^a(\lambda)$的前提假设为非零相冲突。因此，存在$b_1 \in J - G_0$，$\nu(b_1) > 0$，$a(b_1) \in J$及子列$\mu_n(1) \to \infty$使

$$M^{a(b_i)b_i}(\nu(b_i),\ \nu(b_i),\ \mu_n(i))$$

$$\longrightarrow M^{a(b_c)b_c}(\nu(b_i))>0. \tag{21}$$

对$i=1$成立。设沿$\mu=\mu_n(1)\longrightarrow\infty$时，(18) 成立的 $b\in J$ 全体为G_1，则$G_0\subset G_1$，$b_1\in J-G_1$。如果$G_1\cup\{b_1\}=J$，则取子列$\mu_n=\mu_n(1)$。否则，存在$b_2\in J-(G_1\cup\{b_1\})$，$\nu(b_2)>0$，$a(b_2)\in J$及$\mu_n(1)$的子列$\mu_n(2)$使(21) 对$i=2$ 成立。由于J有穷，于是存在$\triangle=\{b_1, b_2, \cdots, b_k\}\subset J$及$a(b_1)$，$a(b_2)$，$\cdots$，$a(b_k)\in J$，$\nu(b_1)$，$\nu(b_2)$，$\cdots$，$\nu(b_k)>0$，子列$\mu_n\to\infty$，使得

$$\lim_{n\to\infty}M^{ab}(\nu,\lambda,\mu_n)=0$$

对 $b\in J-\triangle$及一切$a\in J$，$\nu>0$，$\lambda>0$， (22)

$$\lim_{n\to\infty}M^{a(c)c}(\nu(c),\ \nu(c),\ \mu_n)$$

$$=M^{a(c)c}(\nu(c))>0,\ c\in\triangle. \tag{23}$$

由(11) 和 (17)，用对角线方法，可以选取μ_n的子列，仍记为μ_n，使对每个$c\in\triangle$，有

$$\lim_{n\to\infty}M^{ab}(\nu(c),\ \nu(c),\mu_n)=M^{ab}(\nu(c)),\ a,\ b\in J, \tag{24}$$

$$\lim_{n\to\infty}\eta_j^b(\nu(c),\nu(c),\mu_n)=\eta_j^b(\nu(c)),\ b\in J,\ j\in E, \tag{25}$$

而且

$$M^{a(c)c}(\nu(c))>0(c\in\triangle),\ M^{ab}(\nu(c))=0\ \ (c\in\triangle,\ a\in J,\ b\in J-\triangle). \tag{26}$$

在 (14) 中令$\lambda=\nu=\nu(c)$得

$$F^a(\nu(c))=\sum_{b\in J}M^{ab}(\nu(c),\ \nu(c),\mu)\eta^b(\nu(c),\nu(c),\mu),$$

$$c\in\triangle. \tag{27}$$

令$\mu=\mu_n\longrightarrow\infty$得

$$F^a(\nu(c))=\sum_{b\in\triangle}M^{ab}(\nu(c))\eta^b(\nu(c))), \tag{28}$$

从而

$$\|F^a(\nu(c))\|=\sum_{b\in\triangle}M^{ab}(\nu(c))\|\eta^b(\nu(c))\|. \tag{29}$$

在 (17) 中令$\nu=\nu(c)$并取极限得

$$\|F^a(\nu(c))\| \geqslant \sum_{b\in\triangle} M^{ab}(\nu(c)). \tag{30}$$

由（11）及法都引理得$\|\eta^b(\nu(c))\|\leqslant 1$（$c\in\triangle$，$b\in J$），比较（29）（30）得

$$\|\eta^b(\nu(b))\| = 1,\ b\in\triangle \tag{31}$$

这样，当 $b\in\triangle$ 时，$\eta^b(\nu(b),\ \nu(b),\ \mu_n)$ 按坐标收敛于 $\eta^b(\nu(b))$，且范数$\|\eta^b(\nu(b),\ \nu(b),\ \mu_n)\|=1$ 显然收敛于范数 $\|\eta^b(\nu(b))\|=1$，从而在巴拿赫空间l中强收敛，即

$$\|\eta^b(\nu(b),\ \nu(b),\ \mu_n)-\eta^b(\nu(b))\|\to 0,\ b\in\triangle. \tag{32}$$

记

$$\eta^b(\lambda)=\eta^b(\nu(b))A(\nu(b),\ \lambda).$$

则$\eta^b(\lambda)$（$b\in\triangle$）显然是非零行协调族。由引理1，

$$\|\eta^b(\nu(b),\ \lambda,\ \mu_n)-\eta^b(\lambda)\|$$

$$\leqslant C_{\nu(b)\lambda}\|\eta^b(\nu(b),\ \nu(b),\ \mu_n)-\eta^b(\nu(b))\|\to 0, \tag{33}$$

现在，对每个$a\in J$及 $\lambda>0$，可以选取μ_n的子列，仍记为μ_n，使

$$M^{ab}(\nu(b),\ \lambda,\ \mu_n)\to M^{ab}(\lambda),\ b\in\triangle. \tag{34}$$

在（7）中令$\mu=\mu_n\to\infty$并注意（34）和（22）得

$$F^a(\lambda)=\sum_{b\in\triangle} M^{ab}(\lambda)\eta^b(\lambda).$$

对 $b\in J-\triangle$，补定义 $\eta^b(\lambda)=0$，$M^{ab}(\lambda)=0$（$a\in J$）。我们证明了（3）及（i），以及（6）。

由（3），我们有

$$\eta^a(\lambda,\mu)=\frac{F^a(\mu)A(\mu,\lambda)}{\|F^a(\mu)A(\mu,\nu_a)\|}=\sum_{b\in J}H^{ab}(\mu)\eta^b(\lambda), \tag{35}$$

其中 $\nu_a=\nu(a)$，

$$H^{ab}(\mu)=\frac{M^{ab}(\mu)}{\|F^a(\mu)A(\mu,\nu_a)\|}=\frac{M^{ab}(\mu)}{\sum\limits_{b\in J}M^{ab}(\mu)\|\eta^b(\nu_a)\|}. \tag{36}$$

因为

$$\sum_{b\in J}H^{ab}(\mu)\|\eta^b(\nu_a)\|=1. \tag{37}$$

而当$b\in\triangle$时，$\|\eta^b(\nu_a)\|>0$；当$b\in J-\triangle$ 时，$H^{ab}(\mu)=0$，故当μ沿μ_n的某子列趋于无穷时，(5) 中极限存在且有穷。

当$\eta^a(\lambda)\neq 0$ 时，即$a\in\triangle$，由 (33)，$\eta^a(\lambda, \mu_n)=\eta^a(\nu_a, \lambda, \mu_n)$ 强收敛于$\eta^a(\lambda)$。于是当$\eta^a(\lambda)\neq 0$时，由 (35) 得 (4)。当$\eta^a(\lambda)=0$时，取$H^{ab}=0(b\in J)$，(4) 显然成立。引理证完。

§5. 非黏情形

定义1 称行协调族$(\eta(\lambda),\lambda>0)$为黏的，如果$\lim\limits_{\lambda\to\infty}\lambda\|\eta(\lambda)\|=\infty$；否则称为非黏的。如果$Q$过程$\psi(\lambda)$的表现(3.9)和 (4.3)中，每个$\eta^a(\lambda)$都是非黏的，称过程$\psi(\lambda)$为非黏的。

定理1 (i) 将$A_e=H_e\cup B_e$作分法$A_e=a_1\cup a_2\cup\cdots$，诸 a_m 非空且互不相交。记$A=\{a_1,a_2,\cdots\}$。取A的非空子集J，$J=J_H\cup J_B$，其中

$$J_H=\{a\in J;\ a\cap H_e\neq\phi\},\quad J_B=\{a\in J;\ a\cap H_e=\phi\}.$$

(ii) 取$A\times J$矩阵$G=(G^{ab})$满足(3.5)，而且G 的每一列不是零列。记

$$Z^b(\lambda)=\sum_{a\in A}\sum_{c\in a}X^c_{(\lambda)}G^{ab},\quad Z^b=\sum_{a\in A}\sum_{c\in a}X^cG^{ab},\quad b\in J. \tag{1}$$

$$\left.\begin{aligned}Z^*(\lambda)&=Z(\lambda)-\sum_{a\in J}Z^a(\lambda)\\&=\sum_{a\in A}\Big(\sum_{c\in a}X^c(\lambda)\Big)\cdot\Big(1-\sum_{b\in J}G^{ab}\Big),\\Z^*&=Z-\sum_{a\in J}Z^a=\sum_{a\in A}\Big(\sum_{c\in a}X^c\Big)\Big(1-\sum_{b\in J}G^{ab}\Big).\end{aligned}\right\} \tag{2}$$

(iii) 取非零的非黏行协调族$(\bar\eta^a(\lambda),\ \lambda>0)(a\in J)$。记

$$\bar W^{ab}(\lambda)=\lambda[\bar\eta^a(\lambda),\ Z^b]\uparrow\bar W^{ab},\ \lambda\uparrow\infty, \tag{3}$$

$$\bar W^{a*}(\lambda)=\lambda[\bar\eta^a(\lambda),\ Z^*]\uparrow\bar W^{a*},\ \lambda\uparrow\infty, \tag{4}$$

$$\bar\sigma^a=\lambda[\bar\eta^a(\lambda),\ X^0]，(\bar\sigma^a与\lambda无关), \tag{5}$$

$$\bar{\alpha}^a = \lim_{\lambda \to \infty} \lambda \bar{\eta}^a(\lambda) \tag{6}$$

使得

$$\bar{\sigma}^a + \bar{W}^{a*} + \sum_{b \in J} \bar{W}^{ab} \leq 1, \tag{7}$$

$$\bar{\alpha}^a = 0, \text{ 如果} a \in J_H. \tag{8}$$

(iv) 令

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \sum_{a \in J} \sum_{b \in J} Z_i^a(\lambda) K^{ab}(\lambda) \bar{\eta}^b(\lambda), \tag{9}$$

其中$J \times J$矩阵

$$\left.\begin{aligned} &K(\lambda) = (K^{ab}(\lambda)) = (I - \bar{W} + \bar{W}(\lambda))^{-1}, \\ &\bar{W}(\lambda) = (\bar{W}^{ab}(\lambda)) \uparrow \bar{W} = (\bar{W}^{ab}), \ \lambda \uparrow \infty, \end{aligned}\right\} \tag{10}$$

而$I = (\delta_{ab})$是$J \times J$么矩阵。

则$\psi(\lambda)$是非黏的Q过程。$\psi(\lambda)$不中断的充要条件是

$$\bar{W}^{a*} = 0, \ \bar{\sigma}^a + \sum_{b \in J} \bar{W}^{ab} = 1. \tag{11}$$

$\psi(\lambda)$是B型的，当且仅当$J_H = \phi$。$\psi(\lambda)$是F型的，当且仅当

$$\bar{\alpha}^a = 0, \ a \in J_B. \tag{12}$$

每个不是最小的非黏Q过程$\psi(\lambda)$都可以按上面的方式得到。

证 （一）设Q过程$\psi(\lambda)$是非最小的，而且是非黏的。

将(4.3)代入(3.11)和(3.12)屑得

$$\sum_{b \in J} M^{ab}(\lambda)\sigma^b + S^{a*}(\lambda) + \sum_{b \in J} S^{ab}(\lambda) \leq 1, \tag{13}$$

$$\sum_{b \in J} M^{ab}(\lambda)\eta^b(\mu)$$

$$= \sum_{b \in J} (I - S(\lambda) + M(\lambda)W(\mu))^{ab} F^b(\mu), \tag{14}$$

其中$W(\lambda) = (W^{ab}(\lambda))$，$S(\lambda) = (S^{ab}(\lambda))$是$J \times J$矩阵，

$$W^{ab}(\lambda) = \lambda[\eta^a(\lambda), Z^b] \uparrow W^{ab}, \ \lambda \uparrow \infty, \tag{15}$$

$$W^{a*}(\lambda) = \lambda[\eta^a(\lambda), Z^*] \uparrow W^{a*}, \ \lambda \uparrow \infty, \tag{16}$$

$$\sigma^a = \lambda[\eta^a(\lambda), X^0] \text{与}\lambda\text{无关}, \tag{17}$$

$$S^{ab}(\lambda)=\sum_{c\in J}M^{ac}(\lambda)W^{cb}(\lambda), \tag{18}$$

$$S^{a*}(\lambda)=\sum_{c\in J}M^{ac}(\lambda)W^{c*}(\lambda). \tag{19}$$

由于$\psi(\lambda)$非黏，$W=(W^{ab})$及W^{a*}均为有限。

选取子列$\lambda\to\infty$使

$$M^{ab}(\lambda)\to M^{ab},\quad S^{ab}(\lambda)\to S^{ab},\quad S^{a*}(\lambda)\to S^{a*}. \tag{20}$$

显然S^{ab}，S^{a*}都有限。往证M^{ab}有限。

当$\eta^b(\lambda)=0$时，显然$M^{ab}(\lambda)=0\to M^{ab}=0$。设$\Delta=\{b\in J:\ \eta^b(\lambda)\neq 0\}$。由(13)(18)(19)及法都引理得

$$\sum_{c\in J}M^{ac}\sigma^c+S^{a*}+\sum_{b\in J}S^{ab}\leqslant 1, \tag{21}$$

$$\sum_{c\in J}M^{ac}W^{c*}\leqslant S^{a*}, \tag{22}$$

$$\sum_{c\in J}M^{ac}W^{cb}\leqslant S^{ab}. \tag{23}$$

（约定$\infty\cdot 0=0$），从而由(21)—(23)，

$$\sum_{c\in\Delta}M^{ac}\lambda[\eta^c(\lambda),\ 1]=\sum_{c\in J}M^{ac}\lambda[\eta^c(\lambda),1]$$

$$=\sum_{c\in J}M^{ac}\Big(\sigma^c+W^{c*}(\lambda)+\sum_{b\in J}W^{cb}(\lambda)\Big)$$

$$\leqslant\sum_{c\in J}M^{ac}\sigma^c+\sum_{c\in J}M^{ac}W^{c*}+\sum_{b\in J}\Big(\sum_{c\in J}M^{ac}W^{cb}\Big)$$

$$\leqslant 1,$$

故$b\in\Delta$时，$M^{ab}<\infty$。这样，$J\times J$矩阵$M=(M^{ab})$有限。于是(22)(23)中等号成立。

在(14)中取极限得

$$\sum_{b\in J}M^{ab}\eta^b(\mu)=\sum_{b\in J}(I-MW+MW(\mu))^{ab}F^b(\mu). \tag{24}$$

记

$$\bar{\eta}^a(\lambda)=\sum_{c\in J}M^{ac}\eta^c(\lambda). \tag{25}$$

则(24)成为

$$\bar{\eta}^a(\mu)=\sum_{b\in J}(I-\overline{W}+\overline{W}(\mu))^{ab}F^b(\mu). \tag{26}$$

注意 $\bar{\eta}^a(\mu)$是非零的行协调族. 实际上, 如$\bar{\eta}^a(\mu)=\mathbf{0}$, 则$\overline{W}^{ab}(\mu)=0$, $\overline{W}^{ab}=0$, 从而由(26)得$F^a(\mu)=\mathbf{0}$, 此与$F^a(\mu)$的非零性相冲突. 这样,$\mu[\bar{\eta}^a(\mu),\mathbf{1}]>0$.

矩阵$\overline{W}-\overline{W}(\mu)$非负, 且其行和

$$\begin{aligned}&\sum_{c\in J}(\overline{W}^{ac}-\overline{W}^{ac}(\mu))\\&=\sum_{c\in J}\overline{W}^{ac}-\mu[\bar{\eta}^a(\mu),\ \mathbf{1}-X^c-Z^*]\\&=\bar{\sigma}^a+\overline{W}^{a*}+\sum_{c\in J}\overline{W}^{ac}-\mu[\bar{\eta}^a(\mu),\mathbf{1}]\\&\leqslant 1-\mu[\bar{\eta}^a(\mu),\ \mathbf{1}]<1.\end{aligned}$$

于是逆矩阵(10)存在, 非负, 而且

$$K(\lambda)=\sum_{n=0}^{\infty}(\overline{W}-\overline{W}(\lambda))^n. \tag{27}$$

由(26)得

$$F^a(\mu)=\sum_{b\in J}K^{ab}(\mu)\bar{\eta}^b(\mu), \tag{28}$$

从而得(9).

注意到

$$\lim_{\lambda\to\infty}K(\lambda)=\sum_{n=0}^{\infty}(\overline{W}-\overline{W})^n=I, \tag{29}$$

由(28)知, Q条件(3.13)即(8).

(二) 设$\psi(\lambda)$按定理1中的(i)—(iv)确定.

因为

$$\overline{W}(\lambda)=K(\lambda)^{-1}-I+\overline{W},$$

由(28)经简单计算得

$$\lambda[F^a(\lambda),\mathbf{1}]=1-\sum_{b\in J}K^{ab}(\lambda)\Big(1-\bar{\sigma}^b-\overline{W}^{b*}$$

$$-\sum_{c\in J}W^{bc}\Big)-\sum_{b\in J}K^{ab}(\lambda)(W^{b*}-W^{b*}(\lambda))\leqslant 1, \qquad (30)$$

上式不等式中等号成立的充要条件是(7)中等号成立而且$W^{b*}(\lambda)=W^{b*}(b\in J)$。考虑到(1.11.40)有

$(\mu-\lambda)[\bar{\eta}^b(\mu),\ Z^*(\lambda)]=W^{b*}(\mu)-W^{b*}(\lambda)$，故 $\mu\neq\lambda$ 时，$[\bar{\eta}^b(\mu),\ Z^*(\lambda)]=0$，从而$[\bar{\eta}^b(\mu),\ Z^*]=0$，$W^{b*}(\mu)=0$。这样$W^{b*}(\lambda)=W^{b*}$ 等价于$W^{b*}=0$。

容易计算

$$\begin{aligned}&K(\lambda)(W(\lambda)-W(\mu))K(\mu)\\&=K(\lambda)(K(\lambda)^{-1}-K(\mu)^{-1})K(\mu)=K(\mu)-K(\lambda),\end{aligned} \qquad (31)$$

利用上式易验证对$\psi(\lambda)$的预解方程(3.12)成立。已经指出(8)即范条件(3.13)。所以$\psi(\lambda)$是非黏的Q过程。

（三）设$\psi(\lambda)$是B型的，即B条件满足。往证$J_H=\phi$。如果Q保守，结论平凡地成立。如果Q非保守，由定理1.12.1，在(2.10)中有$F^a(\lambda)=0$ $(a\in H_e)$，从而在(3.10)中有$F^a(\lambda)=0$ $(a\in J_H)$。但由(28)，每个$F^a(\mu)\neq 0$ $(a\in J)$。于是必定有$J_H=\phi$。反之，如果$J_H=\phi$，则每个$a\in J=J_B$有$(\lambda I-Q)Z^b(\lambda)=0$，从而$\psi(\lambda)$ 的B条件满足，即$\psi(\lambda)$是B型的。

$\psi(\lambda)$的F条件等价于

$$\begin{aligned}0&=\sum_{a\in J}\sum_{b\in J}Z^a(\lambda)K^{ab}(\lambda)\bar{\eta}^b(\lambda)(\lambda I-Q)\\&=\sum_{a\in J}Z^a(\lambda)\sum_{b\in J}K^{ab}(\lambda)\bar{\alpha}^b.\end{aligned}$$

因$Z^a(\lambda)(a\in J)$线性独立，$K(\lambda)^{-1}$存在。上式等价于$\bar{\alpha}^b=0$，$b\in J$。由(8)，又等价于(12)。证完。

§6．一般构造

引理1 设$a\in J$，且(3.11)和(3.12) 成立。如果对某个$\lambda>0$有

$$S^{aa}(\lambda)\equiv\lambda[F^a(\lambda),\ Z^*]=1, \qquad (1)$$

则(1)对一切$\lambda>0$成立，而且存在非零行协调族$\eta^a(\lambda)$，$\lambda>0$使

$$F^a(\lambda)=\frac{\eta^a(\lambda)}{W^{aa}(\lambda)},\quad \frac{F^a(\mu)A(\mu,\lambda)}{\|F^a(\mu)A(\mu,\nu_a)\|}=\eta^a(\lambda),\tag{2}$$

而$\nu_a>0$使$\|\eta^a(\nu_a)\|=1$。这里$W^{aa}(\lambda)$由(5.15)确定。

证 由(1)及(3.11)，对$b\in J$，$b\neq a$,则有$\lambda[F^a(\lambda),Z^b]=0$，从而$\lambda[F^a(\lambda),Z^b(\mu)]=0$。故(3.12)成为

$$F^a(\lambda)A(\lambda,\mu)=F^a(\mu)+(\mu-\lambda)[F^a(\lambda),Z^a(\mu)]F^a(\mu).$$

因$F^a(\mu)\neq\mathbf{0}$,故 $1+(\mu-\lambda)[F^a(\lambda),Z^b(\mu)]>0$。仿定理2.2.1的证明，知存在非零行协调族$\eta^a(\lambda)$,$\lambda>0$及常数$c^a$使

$$F^a(\lambda)=\frac{\eta^a(\lambda)}{c^a+W^{aa}(\lambda)},\quad \lambda>0.$$

由(1)知$c^a=0$,从而(2)的第一式以及(1)对一切$\lambda>0$成立。适当地将$\eta^a(\lambda)$规范化使$\|\eta^a(\nu_a)\|=1$，则(2)的第二式成立。证完。

定理2 (i) 按照定理5.1中的(i)(ii)选取集合J及$Z^*(\lambda)$，$Z^a(\lambda),a\in J$及$Z^*,Z^a,a\in J$。

(ii) 取J的子集合L(L可以是空集),令

$$L^a=\begin{cases}1, & \text{如}\ a\in L,\\ 0, & \text{如}\ a\in J-L.\end{cases}\tag{3}$$

(iii) 取非零行协调族$(\bar{\eta}^a(\lambda),\lambda>0)$,$a\in J$,使成立

$$\lim_{\lambda\to\infty}\lambda[\bar{\eta}^a(\lambda),1-Z^a]<\infty,\ a\in J,\tag{4}$$

或等价地，按(5.3)—(5.6)的记号

$$\bar{W}^{a*}<\infty,\ \bar{W}^{ab}<\infty,\ a,b\in J,\ a\neq b.\tag{5}$$

(iv) 取非负$J\times J$矩阵$\bar{S}=(\bar{S}^{ab})$使成立

$$\bar{S}^{aa}=0,\ \bar{S}^{a*}\geqslant\bar{W}^{a*},\ \bar{S}^{ab}\geqslant\bar{W}^{ab},\ a,b\in J,\ a\neq b;\tag{6}$$

$$\bar{\sigma}^a+\bar{W}^{a*}+\sum_{b\in J}\bar{S}^{ab}\leqslant L^a,\quad a\in J.\tag{7}$$

(v) 还应使下面二式成立：

$$(L^b+\bar{W}^{bb})^{-1}\,\bar{\alpha}^b=\mathbf{0},\ \text{如}\ b\in J_H,\tag{8}$$

$$N^{ab}(L^b+\bar{W}^{bb})^{-1}\,\bar{\alpha}^b=\mathbf{0},\ \text{如}\ a\in J_H,\ b\in J_B.\tag{9}$$

这里$J\times J$矩阵$N(\lambda)\downarrow N=(N^{ab}),\lambda\uparrow\infty$，

$$N(\lambda)=(I-R(\lambda))^{-1}=\sum_{n=0}^{\infty}R^n(\lambda),$$

$$R^{aa}(\lambda)=0,\ R^{ab}(\lambda)=\frac{\bar{S}^{ab}-\bar{W}^{ab}(\lambda)}{L^a+\bar{W}^{aa}(\lambda)}\quad(a\neq b),\tag{10}$$

$$N=\sum_{n=0}^{\infty}R^n,\quad R=(R^{ab}),\tag{12}$$

$$R^{aa}=0,R^{ab}=\frac{\bar{S}^{ab}-\bar{W}^{ab}}{L^a+\bar{W}^{aa}}\quad(a\neq b),\tag{13}$$

(vi) 令

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+\sum_{a,b\in J}Z_i^a(\lambda)K^{ab}(\lambda)\bar{\eta}_j^b(\lambda),\tag{14}$$

其中$J\times J$矩阵

$$K(\lambda)=(K^{ab}(\lambda))=(\bar{I}-\bar{S}+\bar{W}(\lambda))^{-1}=N(\lambda)D(\lambda),\tag{15}$$

$\bar{I}$是以$L^a,a\in J$为对角线元素的对角型矩阵．$D(\lambda)$是对角线为

$$D^{aa}(\lambda)=\frac{1}{L^a+\bar{W}^{aa}(\lambda)},a\in J\tag{16}$$

的对角型矩阵．

则$\psi(\lambda)$是Q过程．每个Q过程$\psi(\lambda)$都可以按上述方式得到．过程不中断当且仅当

$$\bar{S}^{a*}=0,\ \bar{\sigma}^a+\sum_{b\in J}\bar{S}^{ab}=L^a,\ a\in J.\tag{17}$$

注1．对定理中的过程$\psi(\lambda)$，如果$a\in J-L$，则

$$\lambda[\bar{\eta}^a(\lambda),1]=\bar{W}^{aa}(\lambda)>0,\tag{18}$$

$$F^a(\lambda)=\frac{\bar{\eta}^a(\lambda)}{\bar{W}^{aa}(\lambda)}.\tag{19}$$

实际上，由(6)(7)得(18)．又由

$$F^a(\lambda)=\sum_{b\in J}K^{ab}(\lambda)\bar{\eta}^b(\lambda),$$

得

$$\overline{\eta}^a(\lambda)=\sum_{b\in J}(I-\overline{S}+\overline{W}(\lambda))^{ab}F^b(\lambda). \tag{20}$$

由(6)(7)，上式成为 $\overline{\eta}^a(\lambda)=\overline{W}^{aa}(\lambda)F^a(\lambda)$，从而得(19)。

定理2的证明

（一）设$\psi(\lambda)$是Q过程。则(5.13)(5.14)仍成立。由引理1，可记

$$L=\{a,\ a\in J, S^{a*}(\lambda)<1\}. \tag{21}$$

令

$$\delta^a(\lambda)=\begin{cases}1-S^{aa}(\lambda), & \text{如}a\in L,\\ \|F^a(\lambda)A(\lambda,\nu_a)\|, & \text{如}a\in J-L,\end{cases} \tag{22}$$

则$\delta^a(\lambda)>0$。将$\delta^a(\lambda)$除(5.13)(5.14)得

$$\sum_{b\in J}\overline{M}^{ab}(\lambda)\sigma^b+\overline{S}^{a*}(\lambda)+\sum_{b\in J}\overline{S}^{ab}(\lambda)\leqslant L^a, \tag{23}$$

$$\sum_{b\in J}\overline{M}^{ab}(\lambda)\eta^b(\mu)=\sum_{b\in J}(\overline{I}-\overline{S}(\lambda)+\overline{M}(\lambda)W(\mu))^{ab}F^b(\mu), \tag{24}$$

其中

$$\left.\begin{aligned}&\overline{M}^{ab}(\lambda)=\frac{M^{ab}(\lambda)}{\delta^a(\lambda)},\ \overline{S}^{aa}(\lambda)=0,\ \overline{S}^{a*}_{(\lambda)}=\frac{S^{a*}(\lambda)}{\delta^a(\lambda)},\\ &\overline{S}^{ab}(\lambda)=\frac{S^{ab}(\lambda)}{\delta^a(\lambda)}(b\neq a)\end{aligned}\right\} \tag{25}$$

满足

$$\left.\begin{aligned}&\overline{S}^{ab}(\lambda)=\sum_{c\in J}\overline{M}^{ac}(\lambda)W^{cb}(\lambda),\ b\neq a,\\ &\overline{S}^{a*}(\lambda)=\sum_{c\in J}\overline{M}^{ac}(\lambda)W^{c*}(\lambda).\end{aligned}\right\} \tag{26}$$

在引理4.2中的序列μ_n中，选取子列$\lambda\to\infty$使对一切$a,b\in J$有

$$\overline{M}^{ab}(\lambda)\to M^{ab},\quad \overline{S}^{ab}(\lambda)\to\overline{S}^{ab},\quad \overline{S}^{a*}(\lambda)\to\overline{S}^{a*}. \tag{27}$$

由(23)(26)及法都引理得

$$\sum_{b\in J} \bar{M}^{ab}\sigma^b + \bar{S}^{a*} + \sum_{b\in J} \bar{S}^{ab} \leqslant L^a, \tag{28}$$

$$\left.\begin{aligned} &\bar{S}^{aa} = 0,\ \bar{S}^{ab} \geqslant \sum_{c\in J} \bar{M}^{ac}W^{cb}\ (b \neq a), \\ &\bar{S}^{a*} \geqslant \sum_{c\in J} \bar{M}^{ac}W^{c*}\ . \end{aligned}\right\} \tag{29}$$

(约定$\infty \cdot 0 = 0 \cdot \infty = 0$)

显然　$\bar{S}^{a*}(a\in J)$及$J\times J$矩阵$\bar{S} = (\bar{S}^{ab})$有穷。往证$J\times J$矩阵$\bar{M} = (\bar{M}^{ab})$有穷。

当$a\in J-L$时，由(4.6)，$\bar{M}^{ab} = H^{ab} < \infty (b\in J)$。下面设$a\in L$。令

$$\wedge = \left\{c;\ \ c\in J, \eta^c(\lambda) \neq 0, \sigma^c + W^{c*} + \sum_{b\neq c} W^{cb} = 0\right\}.$$

如果$\eta^c(\mu) = 0$，由约定$M^{ac}(\lambda) = 0 (a\in J)$，从而有$\bar{M}^{ac} = 0\ (a\in J)$。如果$\eta^c(\mu) \neq 0$，且$c\in J-\wedge$，则由(5.25)(5.26)有

$$\sum_{c\in J} \bar{M}^{ac}\left(\sigma^c + W^{c*} + \sum_{b\neq c} W^{cb}\right)$$

$$\leqslant \sum_{c\in J} \bar{M}^{ac}\sigma^c + \bar{S}^{a*} + \sum_{b\in J} \bar{S}^{ab} \leqslant L^a,$$

故　$\bar{M}^{ac} < \infty$，对$c\in J-\wedge, a\in J$。

注意　$c\in\wedge$时，

$$\mu[\eta^c(\mu), 1 - Z^a] = \sigma^c + W^{c*}\ (\mu) + \sum_{b\neq a} W^{cb}(\mu) = 0,$$

故　$\mu[\eta^c(\mu), 1] = W^{ca}(\mu) > 0$，从而

$$\mu[\eta^c(\mu) - W^{ca}(\mu)F^a(\mu), Z^a] = W^{ca}(\mu)\delta^a(\mu) > 0. \tag{30}$$

因为

$$\sum_{b\in J}\sum_{c\in J} \bar{M}^{ac}(\lambda)W^{cb}(\mu)F^b(\mu)$$

$$= \sum_{c\in J-\wedge} \bar{M}^{ac}\ (\lambda) \sum_{b\in J} W^{cb}(\mu)F^b\ (\mu) + \sum_{c\in\wedge} \bar{M}^{ac}(\lambda)W^{ca}(\mu)F^a(\mu),$$

故(24)可改写为

$$\sum_{c\in\Lambda}\bar{M}^{ac}(\lambda)(\eta^c(\mu)-W^{ca}(\mu)F^a(\mu))$$

$$=F^a(\mu)-\sum_{b\neq a}\bar{S}^{ab}(\lambda)F^b(\mu)$$

$$+\sum_{c\in J-\Lambda}\bar{M}^{ac}(\lambda)\Big(\sum_{b\in J}W^{cb}(\mu)F^b(\mu)-\eta^c(\mu)\Big),$$

上式两边乘μZ^a得

$$\sum_{c\in\Lambda}\bar{M}^{ac}(\lambda)W^{ca}(\mu)\delta^a(\mu)=S^{aa}(\mu)-\sum_{b\neq a}\bar{S}^{ab}(\lambda)S^{ba}(\mu)$$

$$+\sum_{c\in J-\Lambda}\bar{M}^{ac}(\lambda)\sum_{b\in J}(W^{cb}(\mu)S^{ba}(\mu)-W^{ca}(\mu)).$$

当$\lambda\to\infty$时，右方极限有穷，从而左方极限也有穷。注意(30)，我们得 $\bar{M}^{ac}<\infty(c\in\Lambda)$。这样，完成了$(\bar{M}^{ab})$有穷的证明。

在(24)中令$\lambda\to\infty$得

$$\sum_{b\in J}\bar{M}^{ab}\eta^b(\mu)=\sum_{b\in J}(\bar{I}-\bar{S}+\bar{M}W(\mu))^{ab}F^b(\mu). \qquad (31)$$

倘若记

$$\bar{\eta}^a(\mu)=\sum_{b\in J}\bar{M}^{ab}\eta^b(\mu), \qquad (32)$$

则(28)(29)(31)成为(6)(7)和

$$\bar{\eta}^a(\mu)=\sum_{b\in J}(\bar{I}-\bar{S}+\bar{W}(\mu))^{ab}F^b(\mu). \qquad (33)$$

往证逆矩阵

$$\bar{K}(\mu)=(\bar{I}-\bar{S}+\bar{W}(\mu))^{-1} \qquad (34)$$

存在而且非负。

实际上，$\bar{I}-\bar{S}+\bar{W}(\mu)$的对角线元素非负，非对角线元素非正。为证逆矩阵存在，只需证行和大于零。当$a\in J-L$时，由(6)(7)知，行和等于$\bar{W}^{aa}(\mu)=\mu[\bar{\eta}^a(\mu),1]>0$。当$a\in L$时，必定$\bar{\eta}^a(\mu)\neq 0$。因为如果$\bar{\eta}^a(\mu)=0$，则从(33)得

$$F^a(\mu)=\sum_{b\in J}\bar{S}^{ab}F^b(\mu),$$

即 (3.7)(3.8) 对 $h^{ab}=\bar{S}^{ab}$ 成立，由引理3.2，这不可能。所以当$a\in L$时有$\bar{\eta}^a(\mu)\neq 0$，从而由(6)(7)，行和

$$1-\sum_{b\in J}\bar{S}^{ab}+\sum_{b\in J}\bar{W}^{ab}\geqslant 1-(\bar{\eta}^a+\bar{S}^{a*}+\sum_{b\in J}\bar{S}^{ab})+\bar{\sigma}^a$$

$$+W^{*}+\sum_{b\quad J}{}^{-\,ab}(\mu)\geqslant\mu[\bar{\eta}^a(\mu),1]>0.$$

这样，(15) 成立，而且从 (33) 得

$$F^a(\lambda)=\sum_{b\in J}\bar{K}^{ab}(\lambda)\eta^b(\lambda), \tag{35}$$

得证 (14)。

(二) 由 (15) 及 (10)—(13)，Q条件等价于

$$\sum_{b\in J}N^{ab}(L^b+\bar{W}^{bb})^{-1}\bar{\sigma}^b=0,\ a\in J_H. \tag{36}$$

因 $N^{aa}\geqslant 1$ $(a\in J_H)$，故上式等价于(8)(9)。

(三) 设$\psi(\lambda)$按定理中的(i)—(vi)确定。

注意$\bar{W}(\lambda)=K(\lambda)^{-1}-I-\bar{S}$，仿定理5.1关于范条件的证明可知$\lambda[F^a(\lambda),1]\leqslant 1$，并且等号成立当且仅当 (17) 成立。

对于 (34) 中的$\bar{K}(\lambda)$，(5.31) 仍然成立，因而推出$\psi(\lambda)$的预解方程成立。已指出 (8)(9) 等价于Q条件。所以$\psi(\lambda)$是Q过程。定理证完。

定理3 设$\psi(\lambda)$是定理2中的Q过程。则$\psi(\lambda)$为B型的充要条件是$J_H=\phi$；$\psi(\lambda)$为F型的充要条件是$\bar{\sigma}^a=0$ $(a\in J)$。

证 同定理5.1。

定理4 设$\psi(\lambda)$是定理2中的Q过程。则$\psi(\lambda)$是非黏的充要条件是

$$\lim_{\lambda\to\infty}\lambda\|\bar{\eta}^a(\lambda)\|<\infty,\ a\in J. \tag{37}$$

证 设(37)成立，按定义5.1，$\psi(\lambda)$是非黏的。反之，设$\psi(\lambda)$

是非黏的，即$\psi(\lambda)$有表现（3.9）和（4.3），并且

$$\lim_{\lambda\to\infty}\lambda\|\eta^a(\lambda)\|<\infty,\quad a\in J. \tag{38}$$

$\psi(\lambda)$有表现（5.9）和（6.14），因$Z^a(\lambda)$，$a\in J$是线性独立的，故

$$\sum_{b\in J}K^{ab}(\lambda)\eta^b(\lambda)=\sum_{b\in J}\bar{K}^{ab}(\lambda)\bar{\eta}^b(\lambda),\ a\in J.$$

从而

$$\bar{\eta}^a(\lambda)=\sum_{b\in J}\left[\bar{K}(\lambda)^{-1}K(\lambda)\right]^{ab}\eta^b(\lambda)$$

右乘$A(\lambda,\mu)$得

$$\bar{\eta}^a(\mu)=\sum_{b\in J}\left[\bar{K}(\lambda)^{-1}K(\lambda)\right]^{ab}\eta^b(\mu), \tag{39}$$

$$\mu\|\bar{\eta}^a(\mu)\|=\sum_{b\in J}\left[\bar{K}(\lambda)^{-1}K(\lambda)\right]^{ab}\mu\|\eta^b(\mu)\|.$$

由上式及（38）得(37)。证完

注1 在Q过程$\psi(\lambda)$的表现（14）中，如果$\bar{\eta}^a(\lambda)$，$a\in J$和$\tilde{\eta}^a(\lambda)$，$a\in J$对应于同一个$\psi(\lambda)$，由定理4的证明(39)看出，存在$J\times J$矩阵$R=(r_{ab})$，使R的逆矩阵存在，而且

$$\bar{\eta}^a(\lambda)=\sum_{b\in J}r_{ab}\tilde{\eta}^b(\lambda),\quad \lambda>0.$$

§7. 等价构造

定理1 (i) 同定理6.2的(i)。

(ii) 取非零行协调族$(\eta^a(\lambda),\lambda>0)$，$a\in J$，使

$$\lim_{\lambda\to\infty}\lambda[\eta^a(\lambda),1-Z^a]<\infty,\quad a\in J, \tag{1}$$

或等价地

$$W^{a*}<\infty,\quad W^{ab}<\infty\ (b\neq a),\quad a\in J. \tag{2}$$

(iii) 取$J\times J$矩阵$T=(T^{ab})$使

$$O^a \leqslant -W^{a*} + \sum_{b \in J} T^{ab}, \tag{3}$$

$$W^{ab} \leqslant -T^{ab} \quad (b \neq a) \tag{4}$$

$$(T^{bb}+W^{bb})^{-1}\bar{\alpha}^b = 0, \quad b \in J_H \tag{5}$$

$$F^{ab}(T^{bb}+W^{bb})^{-1}\bar{\sigma}^b = 0, \quad a \in J_H, \quad b \in J_B. \tag{6}$$

这里$J \times J$矩阵$F(\lambda) = (F^{ab}(\lambda)) \downarrow F = (F^{ab})$，$\lambda \uparrow \infty$，而$J \times J$矩阵

$$F(\lambda) = (I - V(\lambda))^{-1} = \sum_{n=0}^{\infty} V(\lambda)^n \downarrow F = \sum_{n=0}^{\infty} V^n, \quad \lambda \uparrow \infty,$$

而
$$V(\lambda) = (V^{ab}(\lambda)) \downarrow V = (V^{ab}), \quad \lambda \uparrow \infty;$$

$$\left.\begin{aligned} &V^{aa}(\lambda) = 0, V^{ab}(\lambda) = \frac{(-T^{ab}) - W^{ab}(\lambda)}{T^{aa} + W^{aa}(\lambda)} (b \neq a), \\ &V^{aa} = 0, V^{ab} = \frac{(-T^{ab}) - W^{ab}}{T^{aa} + W^{aa}} \quad (b \neq a). \end{aligned}\right\} \tag{7}$$

(V) 令

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \sum_{a \in J} \sum_{b \in J} Z_i^a(\lambda) K^{ab}(\lambda) \eta_j^b(\lambda), \tag{8}$$

其中

$$K(\lambda) = (T + W(\lambda))^{-1} = (I - V(\lambda))^{-1} E(\lambda), \tag{9}$$

而

$$E(\lambda) = \mathrm{diag}\{(T^{aa} + W^{aa}(\lambda))^{-1}\}. \tag{10}$$

则$\psi(\lambda)$是非最小的Q过程。任何非最小的Q过程都可以按上述方式而得到。

注 当Q保守时，定理1中的$\psi(\lambda)$就是Williams [1]中的构造。

定理2 设$\psi(\lambda)$是定理6.2形式的Q过程。令

$$T = I - \bar{S}, \eta^a(\lambda) = \bar{\eta}^a(\lambda), a \in J, \tag{11}$$

则$\psi(\lambda)$是定理1形式的Q过程。反之，设$\psi(\lambda)$是定理1形式的Q过程。令

$$L = \{a; \ a \in J, \ T^{aa} > 0\}. \tag{12}$$

$$\left.\begin{aligned} &\bar{S}^{a*} = 0, \quad \bar{S}^{ab} = 0 (b \in J), \text{ 如果 } a \in J - L, \\ &\bar{S}^{a*} = \frac{W^{a*}}{T^{aa}} \quad \bar{S}^{aa} = 0, \quad \bar{S}^{ab} = \frac{-T^{ab}}{T^{aa}} (b \neq a), \text{ 如果 } a \in L, \end{aligned}\right\} \tag{13}$$

$$
\bar{\eta}^{a}(\lambda)=\begin{cases}\eta^{a}(\lambda), & \text{如果}a\in J-L,\\ \dfrac{\eta^{a}(\lambda)}{T^{aa}} & \text{如果}a\in L.\end{cases} \tag{14}
$$

则$\psi(\lambda)$是定理6.2形式的Q过程。

定理1和定理2的证明

设$\psi(\lambda)$具有定理1中的形式，按（12）—（14）确定L，$\bar{S}^{a*}$，$\bar{S}=(\bar{S}^{ab})$，$\bar{\eta}^{a}(\lambda)(a\in J)$，则$\psi(\lambda)$具有定理6.2的形式，因而$\psi(\lambda)$是$Q$过程。

设$\psi(\lambda)$是Q过程，依定理6.2，它必具有定理6.2中的形式，按（11）确定$T=(T^{ab})$及$\eta^{a}(\lambda)$，$a\in J$，则$\psi(\lambda)$具有定理1中的形式.证完。

§8. 非双有限构造的注

注1 假定$A_e=H_e\cup B_e$是无限集。作分法$A_e=a_1\cup a_2\cup\cdots\cdots$，集合$A=\{a_1,a_2,\cdots\cdots\}$可以是无限集。只要$A$的非空子集$J$是有限的，或者$J$是无限的，但在定理5.1中按（5.27）定义$K(\lambda)$，在定理6.2中$\bar{K}(\lambda)$按下式定义：

$$
\bar{K}(\lambda)=\left\{\sum_{n=0}^{\infty}R^{n}(\lambda)\right\}D(\lambda),
$$

$R(\lambda)=(R^{ab}(\lambda))$如(5.10)，对角型矩阵$D(\lambda)$如(5.16)在定理7.1中$K(\lambda)$按下式定义：

$$
K(\lambda)=\left\{\sum^{\infty}V(\lambda)^{n}\right\}E(\lambda),
$$

$V(\lambda)=(V^{ab}(\lambda))$如（7.7），对角型矩阵$E(\lambda)$如(7.10)。则按定理5.1，定理6.2，定理7.1得到的$\psi(\lambda)$都是非最小的Q过程。

注2 假定H_e无限，B_e有限，作A_e的分法后，$A=\{a_1,a_2,\cdots\cdots\}$仍可以是无限的.只要取$A$的非空子集$J$使得$J_H=\phi$(此时，$J$必是有限集)，则按定理5.1，定理6.2和定理7.1构造的$\psi(\lambda)$，是满足向后方程组的非最小Q过程，而且此时定理6.2或定理7.1构造的Q过程已穷尽了满足向后方程组的全部非最小的Q过程。

第四篇　可列马尔科夫过程的轨道结构

第八章　W变换和强极限

§1. 引　　言

王梓坤教授在研究生灭过程构造论时，引进了过程的g_n、f_n变换和强极限概念，并成功地用概率方法解决了生灭过程的构造问题，见王梓坤[3，7]，王梓坤与杨向群 [1,2]。其基本思想是将一般的有比较复杂的轨道的过程变换为有比较简单的轨道的过程。然后证明一般的过程是简单过程的强极限。侯振挺教授[2]将g_n变换推广到一般的可列马氏过程。W 变换，是g_n、f_n变换的进一步推广，因而王梓坤[3]，王梓坤与杨向群[1,2] 及侯振挺[2]中关与g_n,f_n 的结果都可以作为我们现在的结果的特殊情形而得到。本章内容取自杨向群[7,8]。

§2. W变换的定义

先考虑对函数的W变换。设 $X=\{x(t),t<\sigma\}$ 是定义域为$[0,\sigma)(\sigma\leqslant\infty)$取值于$\bar{E}=E\cup\{\infty\}$的函数。称$\{\tau,\beta\}$为$X$的偶，如果

$$0=\tau_0\leqslant\beta_0\leqslant\tau_1\leqslant\beta_1\leqslant\cdots\leqslant\sigma. \tag{1}$$

对于偶$\{\tau,\beta\}$，令

$$\sigma_n=\begin{cases}0, & \text{如果} n=0,\\ \sum\limits_{k=1}^{n}(\tau_k-\beta_{k-1}), & \text{如果} n>0.\end{cases} \tag{2}$$

$$\sigma_n = \sum_{k=0}^{n} (\beta_k - \tau_k), \quad n \geqslant 0. \tag{3}$$

$$\bar{\sigma} = \lim_{n\to\infty} \sigma_n, \quad \bar{\delta} = \lim_{n\to\infty} \delta_n. \tag{4}$$

对$t \in [0, \bar{\sigma})$，令

$$a_t = \beta_n + (t - \sigma_n), \quad 如果 t \in [\sigma_n, \sigma_{n+1}). \tag{5}$$

$$\bar{x}(t) = x(a_t), \quad t \in [0, \bar{\sigma}). \tag{6}$$

定义1　将函数$X = \{x(t), t < \sigma\}$变为函数$\bar{X} = \{\bar{x}(t), t < \bar{\sigma}\}$的变换，称为对$X$的$W$变换，记为$W_{\tau,\beta}$。于是$\bar{X} = W_{\tau,\beta}(X)$。

直观地说，抛弃函数X对应于$[\tau_i, \beta_i)$的那些段，保留对应于$[\beta_i, \tau_{i+1})$的那些段并向左移动，使$[\beta_0, \tau_1)$移至$[0, \tau_1 - \beta_0)$，其余诸$[\beta_i, \tau_{i+1})(i = 1, 2, \cdots)$按原次序联结而不相交，所得的函数即$\bar{X}$

借助于对函数W变换，可以定义对过程X的W变换。

定义2　设$X = \{x(t), t < \sigma\} \in \mathscr{X}$。设对每个$\omega \in \Omega$，存在偶$\{\tau(\omega), \beta(\omega)\}$，因而可以确定$W_{\tau(\omega),\beta(\omega)}(X(\omega)) = \bar{X}(\omega) = \{\bar{x}(t, \omega), t < \bar{\sigma}(\omega)\}$。称$\bar{X}$为对过程$X$的$W$变换，记为$\bar{X} = W_{\tau,\beta}(X)$。

§3. 强极限定理

一、X为函数的情形

定义1　设A为$[0, \infty)$中的勒贝格集，称$[\lambda, \eta)$为A的构成区间，如果满足：

(i)　$(\lambda, \eta) \subset A$；

(ii)　最大性：如果$[\bar{\lambda}, \bar{\eta}) \supset [\lambda, \eta)$，且$(\bar{\lambda}, \bar{\eta}) \subset A$，则$[\bar{\lambda}, \bar{\eta}) = [\lambda, \eta)$。

记A的构成区间所成的集合为$\mathscr{U}(A)$。令

$$C_1(A) = \bigcup_{[\lambda,\eta) \in \mathscr{U}(A)} (\lambda, \eta), \tag{1}$$

$$C_2(A) = \bigcup_{[\lambda,\eta) \in \mathscr{U}(A)} [\lambda, \eta), \tag{2}$$

$$\overline{C}_2^+(A)=\{t\mid \text{存在严格下降的}t_n\in C_2(A)\text{使}t_n\downarrow t\}$$
$$=\{t\mid \text{存在不增的}t_n\in C_2(A)\text{使}t_n\downarrow t\}. \tag{3}$$

并称$\overline{C}_2^+(A)$为A的右闭包。

定义2 称变换$\gamma: u\to\gamma(u)$为偶$\{\tau,\beta\}$或偶序列$\{\tau^n,\beta^n\}(n\geqslant 1)$所确定的变换，如果

$$\gamma(u)=L\{A\cap[0,u)\}, \tag{4}$$

其中L为勒贝格测度，而

$$A=\begin{cases}\bigcup\limits_{k=0}^{\infty}[\beta_k,\tau_{k+1}), & \text{对于偶}\{\tau,\beta\},\\ \bigcup\limits_{n=1}^{\infty}\bigcup\limits_{k=0}^{\infty}[\beta_k^n,\tau_{k+1}^n), & \text{对于偶序列}\{\tau^n,\beta^n\}.\end{cases} \tag{5}$$

定义3 称函数$X=\{x(t),t<\sigma\}$为右连续的，如果

$$\lim_{s\downarrow t}x(s)=x(t)\in\overline{E}, \text{ 对一切}t<\sigma. \tag{6}$$

定理1 设$X=\{x(t),\ t<\sigma\}$为右连续函数，存在偶序列$\{\tau^n,\beta^n\}(n\geqslant 1)$ 使

$$A^n=\bigcup_{k=0}^{\infty}[\beta_k^n,\tau_{k+1}^n)\uparrow A. \tag{7}$$

记γ^n和γ分别为对指定n的偶$\{\tau^n,\beta^n\}$及偶序列$\{\tau^n,\beta^n\}(n\geqslant 1)$所确定的变换。令

$$\tau_k^{n,m}=\gamma^n(\tau_k^m),\quad \beta_k^{n,m}=\gamma^n(\beta_k^m),\quad m<n, \tag{8}$$

$$W_{\beta^n,\tau^n}(X)=X^n=\{x^n(t),\ t<\sigma^n\}. \tag{9}$$

则 (i) $\{\tau^{n,m},\beta^{n,m}\}(m<n)$是$X^n$的偶，

而且 $$X^m=W_{\tau^{n,m},\beta^{n,m}}(X^n),\quad m<n. \tag{10}$$

(ii) 当$n\to\infty$时，存在极限

$$\sigma^n\uparrow\overline{\sigma}=L(A), \tag{11}$$

$$\lim_{n\to\infty}x^n(t)=x(a_t), \text{ 对一切}t\in[0,\overline{\sigma}). \tag{12}$$

其中a_t是方程

$$\left.\begin{array}{l}L\{C_2(A)\cap[0,u)\}=t,\\ u\in\overline{C}_2^+(A).\end{array}\right\} \tag{13}$$

的唯一解。

(iii) $\bar{X}=\{x(\alpha_t),\ t<\bar{\sigma}\}$右连续。

这个定理的直观意义如下。假设施行W_{τ^m,β^m}变换时X变成X^m，而且对$m<n$，施行W_{τ^m,β^m}变换时被保留的段，在施行W_{τ^n,β^n}变换时仍然被保留。那么(10)表明：由X变换成X^m可以分两步得到。第一步，将X对应于$\bigcup_{i=0}^{\infty}[\tau_i^n,\beta_i^n)$的段抛弃，将对应于$\bigcup_{i=0}^{\infty}[\beta_i^n,\tau_{i+1}^n)$的段保留并按原次序向左平移联结而不相交，便得$X^n$。在这一过程中，对应于$[\beta_k^m,\tau_k^m)$的段被保留下来，向左移后$\beta_k^m$变成$\beta_k^{n,m}$，$\tau_k^m$变成$\tau_k^{n,m}$。第二步，将$X^n$对应于$\bigcup_{i=0}^{\infty}[\tau_i^{n,m},\beta_i^{n,m})$的段抛弃，将对应于$\bigcup_{i=0}^{\infty}[\beta_i^{n,m},\tau_{i+1}^{n,m})$的段保留并按原次序向左平移联结而不相交，便得到$X^m$。结论(iii)表明：将$X$对应于$A=\bigcup_{n=1}^{\infty}\bigcup_{k=0}^{\infty}[\beta_k^n,\tau_{k+1}^n)$的段都保留，而将其余诸段抛弃，并将保留的段按原次序向左平移不相交联结而得的函数正是$\bar{X}$。在这一平移过程中，α_t变成t。$\bar{X}$也是右连续函数。结论(ii)表明，函数X^n的极限函数是$\bar{X}$。

二、X为过程的情形

定义4 设$X=\{x(t),\ t<\sigma\}$，$X^n=\{x^n(t),\ t<\sigma^n\}(n\geqslant 1)$均$\in\mathscr{X}$。称$X$为$X^n$的强极限，记为$X=\lim\limits_{n\to\infty}X^n$，如果对几乎一切$\omega$有

$$\left.\begin{aligned}&\sigma^n(\omega)\uparrow\sigma(\omega),\\&\lim_{n\to\infty}x^n(t,\omega)=x(t,\omega),\quad 对一切t<\sigma(\omega).\end{aligned}\right\}\tag{14}$$

根据定义，显然有

定理2 设$X,X^n\in\mathscr{X}$，$\lim\limits_{n\to\infty}X^n=X$，则对一切$i$，$j\in E$，$t\geqslant 0$，

$$\lim_{n\to\infty}P_{ij}^n(t)=P_{ij}(t).\tag{15}$$

其中$\{P_{ij}^n(t)\}$，$\{P_{ij}(t)\}$分别为X^n和X的转移概率。

定义5 设$X\in\mathscr{X}$，$\beta_0\leqslant\tau_1\leqslant\beta_1$都是$X$的马时，满足$P(\beta_0<\sigma)>0$，并在$(\beta_0<\tau_1)$上有

$$\beta_0+\theta_{\beta_0}\tau_1=\tau_1,\quad \beta_0+\theta_{\beta_0}\beta_1=\beta_1.\tag{16}$$

其中θ为推移算子。设τ_n，β_n已确定，如果$\beta_n=\sigma$，则定义$\tau_{n+1}=\beta_{n+1}=\sigma$，否则定义

$$\tau_{n+1}=\beta_n+\theta_{\beta_n}\tau_1,\quad \beta_{n+1}=\beta_n+\theta_{\beta_n}\beta_1. \tag{17}$$

称这样的偶$\{\tau,\beta\}$为标准偶。

回忆Q矩阵有限的过程类$\mathscr{X}_s$及D型过程类$\mathscr{X}_D$。

定理3 设$X\in\mathscr{X}_s$，标准偶$\{\tau,\beta\}$满足下面的条件(a)、(b)：

$$\left.\begin{aligned}(a)\ &P\{t+\theta_t\tau_1=\tau_1\text{对一切}t\in[\beta_0,\tau_1)\}=1,\\ &P\{t+\theta_t\beta_1=\beta_1\text{对一切}t\in[\beta_0,\tau_1)\}=1.\end{aligned}\right\} \tag{18}$$

(b) 对按（2.5）确定的a_t有

$$P\{x(a_t)=\infty\}=0,\quad t\geqslant 0. \tag{19}$$

则下述结论成立：

(i) 条件(b)成立的一个充分条件是下面的条件(c)成立。即

$$\left.\begin{aligned}(c)\ &P\{x(t)\in E\text{对一切}t\in[\beta_0,\ \tau_1)\}=1,\\ &P\{x(\beta_1)=\infty\}=0.\end{aligned}\right\} \tag{20}$$

(ii) 如条件(a)、(b)成立，则$\bar{X}=W_{\tau\beta}(X)\in\mathscr{X}_s$。

(iii) 如条件(a)、(c)成立，则$\bar{X}=W_{\tau,\beta}(X)\in\mathscr{X}_D$。

定理4 设$X\in\mathscr{X}_s$，对每个$n\geqslant 1$，存在标准偶$\{\tau^n,\beta^n\}$且满足条件$(a)(c)$，而且

$$P\{x(\inf\beta_0^n)=\infty\}=0. \tag{21}$$

设对几乎一切$\omega\in\Omega$有

$$A^n(\omega)=\bigcup_{k=0}^{\infty}[\beta_k^n(\omega),\ \tau_{k+1}^n(\omega))\uparrow A(\omega). \tag{22}$$

则 (i) $X^n=W_{\tau^n,\beta^n}(X)\in\mathscr{X}_D$，并且满足（10），其中的$\tau_k^{n,m}$，$\beta_k^{n,m}(m<n,\ k\geqslant 0)$为

$$\tau_k^{n,m}=L\{A^n\cap[0,\ \tau_k^m)\},\quad \beta_k^{n,m}=L\{A^n\cap[0,\ \beta_k^m)\}. \tag{23}$$

(ii) $X^n(n\geqslant 1)$的强极限存在：

$$\lim_{n\to\infty}X^n=\bar{X}. \tag{24}$$

其中$\bar{X}=\{x(a_t),t<\bar{\sigma}\}\in\mathscr{X}_s$，$\bar{\sigma}(\omega)=L\{A(\omega)\}$，$a_t(\omega)$是方程

$$\left.\begin{array}{l} L\{C_2(A(\omega))\cap[0,\ u)\}=t, \\ u\in\bar{C}_2^+(A(\omega)). \end{array}\right\} \tag{25}$$

的唯一解。

§4. 定理3.1的证明

设$\{\tau,\ \beta\}$为函数$X=\{x(t),\ t<\sigma\}$的偶。令

$$A=\bigcup_{k=0}^{\infty}[\beta_k,\ \tau_{k+1}),\ B=\bigcup_{k=0}^{\infty}[\tau_k,\ \beta_k). \tag{1}$$

$$\gamma(u)=L\{A\cap[0,\ u)\},\ \rho(u)=L\{B\cap[0,\ u)\},\quad u<\sigma. \tag{2}$$

显然有

引理1 (i) $\gamma(u)+\rho(u)=u$。

(ii) $\gamma(u)$, $\rho(u)$是非降连续函数。

(iii) 如果$u\in A$, $u<t<\sigma$, 则$\gamma(u)<\gamma(t)$。

如果$u\in B$, $u<t<\sigma$, 则$\rho(u)<\rho(t)$。

(iv) 如果$u\in[\beta_k,\ \tau_{k+1})$, 则$\rho(u)=\rho(\beta_k)=\bar{\sigma}_k$,

如果$u\in[\tau_k,\ \beta_k)$, 则$\gamma(u)=\gamma(\tau_k)=\sigma_k$。

(v) 对$t\in[0,\bar{\sigma})$, $\bar{\sigma}=L(A)=\gamma(\sigma)$, (2.5) 中的$\alpha_t$是方程

$$\left.\begin{array}{l} \gamma(u)=t, \\ u\in A. \end{array}\right\}. \tag{3}$$

的唯一解。

引理2 当$u<\sigma$时，下列条件彼此等价。

(i) $\alpha_t<u$;

(ii) $\gamma(u)>t$;

(iii) $\rho(u)<u-t$。

证 由引理1 (i)知本引理的 (ii)和 (iii)等价。

设$\alpha_t<u$。由引理 1 (v) 知$\alpha_t\in A,\gamma(\alpha_t)=t$, 由引理 1(iii), $\gamma(u)>\gamma(\alpha_t)=t$。反之，设$u\leqslant\alpha_t$。由引理 1 (ii), $\gamma(u)\leqslant\gamma(\alpha_t)=t$。故本引理的 (i)和(ii) 等价。证毕。

引理3 设函数X有二偶$\{\tau, \beta\}$及$\{\tau', \beta'\}$。下面带有"$'$"或"$\sim$"的记号，其意义是明显的。假设

$$A \subset A' \quad (\text{或} B \supset B')。 \tag{4}$$

记$\bar{X} = W_{\tau,\beta}(X)$，$\bar{X}' = W_{\tau',\beta'}(X)$。则

(i) $\bar{\sigma} \leqslant \bar{\sigma}'$且对$t \in [0, \bar{\sigma})$有$\alpha_t' \leqslant \alpha_t$。

(ii) 令

$$\tilde{\tau}_k = L\{A' \cap [0, \tau_k)\}, \quad \tilde{\beta}_k = L\{A' \cap [0, \beta_k)\}。 \tag{5}$$

则$\{\tilde{\tau}, \tilde{\beta}\}$是$\bar{X}'$的偶，而且

$$\bar{X} = W_{\tilde{\tau},\tilde{\beta}}(\bar{X}')。 \tag{6}$$

注 此引理的直观意义是：如果对X施行变换$W_{\tau,\beta}$时比施行变换$W_{\tau',\beta'}$时抛弃的多，则$\bar{X} = W_{\tau,\beta}(X)$可以分两步抛弃而得到。首先，对$X$施以变换$W_{\tau',\beta'}$，即抛弃$X$对应于$B'$的段，保留对应于$A'$的段并向左移动联结而得$\bar{X}'$。在这一移动过程中，$\tau_k$，$\beta_k$变成$\tilde{\tau}_k$，$\tilde{\beta}_k$。然后对$\bar{X}'$施以变换$W_{\tilde{\tau},\tilde{\beta}}$，即抛弃$\bar{X}'$对应于$\tilde{B} = \bigcup_{k=0}^{\infty}[\tilde{\tau}_k, \tilde{\beta}_k)$的段，保留对应于$\tilde{A} = \sum_{k=0}^{\infty}[\tilde{\beta}_k, \tilde{\tau}_{k+1})$的段并向左平移联结而得$\bar{X}$。

证 由(4)得$\bar{\sigma} = L(A) \leqslant L(A') = \bar{\sigma}'$。倘若$\alpha_t < \alpha_t'$。由引理1结论(iii)(v)有$\alpha_t \in A \subset A'$，$\alpha_t \in A'$，从而$t = L\{A \cap [0, \alpha_t)\} \leqslant L\{A' \cap [0, \alpha_t)\} < L\{A' \cap [0, \alpha_t')\} = t$。矛盾。故$\alpha_t' \leqslant \alpha_t$。

显然$\{\tilde{\tau}, \tilde{\beta}\}$是$\bar{X}'$的偶。设$W_{\tilde{\tau},\tilde{\beta}}(\bar{X}') = \tilde{X} = \{\tilde{x}(t), t < \tilde{\sigma}\}$。则

$$\tilde{\sigma}_n = \sum_{k=1}^{n}(\tilde{\tau}_k - \tilde{\beta}_{k-1}) = \sum_{k=1}^{n} L\{A' \cap [\beta_{k-1}, \tau_k)\}$$

$$= L\left\{A' \cap \left(\bigcup_{k=1}^{n}[\beta_{k-1}, \tau_k)\right)\right\}.$$

由(4)，$\tilde{\sigma}_n = L\left\{\bigcup_{k=1}^{n}[\beta_{k-1}, \tau_k)\right\} = \sigma_n$，从而$\tilde{\sigma} = \bar{\sigma}$。为证$\tilde{X} = \bar{X}$，

尚需证明当$t \in [\widetilde{\sigma}_n, \widetilde{\sigma}_{n+1}) = [\sigma_n, \sigma_{n+1})$时有$\widetilde{x}(t) = \bar{x}(t)$，即

$$\bar{x}'(\widetilde{\beta}_n + t - \widetilde{\sigma}_n) = x(\beta_n + t - \sigma_n). \tag{7}$$

令$u = \widetilde{\beta}_n + t - \widetilde{\sigma}_n (< \bar{\sigma}')$。为证（7），只需证明$\beta_n + t - \sigma_n = \alpha_u'$即可。由引理1结论(v)，即要证明

$$\left.\begin{aligned} &L\{A' \cap [0, \beta_n + t - \sigma_n)\} = \widetilde{\beta}_n + t - \widetilde{\sigma}_n, \\ &\beta_n + t - \sigma_n \in A'. \end{aligned}\right\} \tag{8}$$

因为$0 \leqslant t - \sigma_n < \sigma_{n+1} - \sigma_n = \tau_{n+1} - \beta_n$，故$\beta_n + t - \sigma_n \in [\beta_n, \tau_{n+1}) \subset A \subset A'$。其次，

$$\begin{aligned} &L\{A' \cap [0, \beta_n + t - \sigma_n)\} \\ &= L\{A' \cap [0, \beta_n)\} + L\{A' \cap [\beta_n, \beta_n + t - \sigma_n)\} \\ &= \widetilde{\beta}_n + L\{[\beta_n, \beta_n + t - \sigma_n)\} = \widetilde{\beta}_n + t - \sigma_n = \widetilde{\beta}_n + t - \widetilde{\sigma}_n \end{aligned}$$

得证（8），证毕。

引理4 设函数$X = \{x(t), t < \sigma\}$满足定理3.1中的条件，

$$A = \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{\infty} [\beta_k^n, \tau_{k+1}^n), \tag{9}$$

$\gamma(u) = L\{A \cap [0, u)\}$，$C_1(A)$，$C_2(A)$，$\bar{C}_2^+(A)$由(3.1)-(3.3)确定。则

(i) 定理3.1结论(i)成立。

(ii) $$C_1(A) \subset A \subset C_2(A) \subset \bar{C}_2^+(A) \subset [0, \sigma), \tag{10}$$

$$L\{C_1(A)\} = L\{A\} = L\{C_2(A)\} = L\{\bar{C}_2^+(A)\} = \bar{\sigma}. \tag{11}$$

(iii) (a) $\gamma(u)$在$[0, \sigma)$上不降；

(b) 如$[a, b) \cap C_2(A) = \phi$，则对$u \in [a, b)$，有$\gamma(u) = \gamma(a)$。

(c) 如果$u \in \bar{C}_2^+(A)$，且$u < s < \sigma$，则$\gamma(u) < \gamma(s)$。特别$\gamma(u)$在$\bar{C}_2^+(A)$中严格增加。

(d) 对变换γ有$\gamma[0, \sigma) = [0, \bar{\sigma})$，$\bar{\sigma} = L(A)$。

(e) γ作为$\bar{C}_2^+(A)$到$[0, \bar{\sigma}) = \gamma[\bar{C}_2^+(A)]$上的变换是一

对一的，因而有逆变换γ^{-1}，并且γ与γ^{-1}是保测的。特别，如果$[\lambda,\ \eta)\in\mathscr{U}(A)$，$t\in[\gamma(\lambda),\ \gamma(\eta))$，则

$$\gamma^{-1}(t)=\lambda+t-\gamma(\lambda). \tag{12}$$

(iv) 当$n\to\infty$时，存在极限

$$\sigma^n=L(A^n)\uparrow\bar{\sigma}=L(A), \tag{13}$$

$$\alpha_t^n\downarrow_t\equiv\gamma^{-1}(t),\quad t\in[0,\bar{\sigma}). \tag{14}$$

证 由引理3的结论(ii)得本引理的(i)。(10)式明显．易知$\mathscr{U}(A)$是可列集，故$C_1(A)$，A，$C_2(A)$最多相差一可列集，因而得证(11)前面两个等号。

本引理的结论(iii)(a)(b)(d)明显。为证(iii)(c)，首先注意：如$t\in C_2(A)$，$t<s<\sigma$，则有$\gamma(t)<\gamma(s)$，故对$u\in\bar{C}_2^+(A)$，$u<s<\sigma$，存在$t_n\in C_2(A)$，$t_n\downarrow u$，从而当n充分大时有$u\leqslant t_n<s<\sigma$，由(iii)(a)及刚才所述，$\gamma(u)\leqslant\gamma(t_n)<\gamma(s)$。得证(iii)($c$)。

往证(iii)(d)。由(iii)(c)，γ作为$\bar{C}_2^+(A)$到$\gamma(\bar{C}_2^+(A))$上的变换是一对一的，因而γ^{-1}存在。为证保测性，令$\mathscr{L}$为满足$F\subset\bar{C}_2^+(A)$及$L\{\gamma(F)\}=L\{F\}$的勒贝格可测集F全体。令$\mathscr{C}=\{[a,\ b)\mid[a,\ b)\subset\bar{C}_2^+(A)\}$。易知$\mathscr{L}$是$\lambda$系，$\mathscr{C}$是$\pi$系，而且$\mathscr{C}\subset\mathscr{L}$，从而$\mathscr{L}$包含$\bar{C}_2^+(A)$中一切Borel集，见Дынкин[2,引理1.1]。因此$\mathscr{L}$包含$\bar{C}_2^+(A)$中一切勒贝格可测集。

由引理3结论(i)得(13)(14)中极限存在。对$t\in[0,\ \bar{\sigma})$，当n充分大时，$t\in[0,\ \sigma^n)$。由引理1结论(v)，

$$\left.\begin{aligned}&L\{A^n\cap[0,\ \alpha_t^n)\}=t,\\&\alpha_t^n\in A^n.\end{aligned}\right\} \tag{15}$$

因$A^n\subset A\subset C_2(A)$。令$n\to\infty$得极限$\alpha_t=\lim\limits_{n\to\infty}\alpha_t^n$满足

$$\left.\begin{aligned}&\gamma(\alpha_t)=L\{A\cap[0,\ \alpha_t)\}=t,\\&\alpha_t\in\bar{C}_2^+(A).\end{aligned}\right\} \tag{16}$$

注意(10)(11)得α_t是方程(3.13)的解。由本引理的(iii)(c)，α_t是唯一解，而$\gamma^{-1}(t)$也是解，故$\alpha_t=\gamma^{-1}(t)$，$\gamma(\bar{C}_2^+(A))=[0,$

$\bar{\sigma}$)，$L\{\bar{C}_2^+(A)\}=L\{\gamma(\bar{C}_2^+(A))\}=\bar{\sigma}$，证毕。

定理3.1的证明。

由X的右连续性及(13)(14)得(3.11)(3.12)，从而由引理4得证定理3.1的结论(i)和(ii)。往证定理3.1的结论(iii)。设$t\in[0,\ \bar{\sigma})$，且$s\downarrow t$。由引理4(iii)(c)(e)，$\gamma^{-1}(s)\downarrow$，设极限为u。由(3.13)知

$$\left.\begin{array}{l}L\{C_2(A)\cap[0,\ \gamma^{-1}(s))\}=s,\\ \gamma^{-1}(s)\in\bar{C}_2^{\dagger}(A).\end{array}\right\}$$

令$s\downarrow t$得u满足(3.13)，从而$u=\gamma^{-1}(t)$。即$\gamma^{-1}(s)\downarrow\gamma^{-1}(t)$。于是

$$\lim_{s\downarrow t}\bar{x}(s)=\lim_{s\downarrow t}x(\alpha_s)=\lim_{s\downarrow t}x(\gamma^{-1}(s))=x(\gamma^{-1}(t))=x(\alpha_t)$$

$=\bar{x}(t)$，证毕。

§5. 一些引理

设$X=\{x(t),\ t<\sigma\}\in\mathscr{X}_*$。如果$\beta$是$X$的马时，$\mathscr{F}_\beta$为$\beta$前域，$\mathscr{F}_\beta'$为$\beta$后域。令

$$x_\beta'(t)=x(\beta+t),\quad t<\sigma_\beta'=\sigma-\beta. \tag{1}$$

称$X_\beta'=\{x_\beta'(t),\ t<\sigma_\beta'\}$或简记$X'$为$\beta$后过程。记$\mathscr{L}_t^0(\beta)$或简记$\mathscr{L}_t^0$为$\{x_\beta'(u),\ u\leqslant t\}$所产生的$\Omega_\beta=(\beta<\sigma)$上的完备波雷尔域。$\mathscr{L}_\infty^0(\beta)=\mathscr{F}_\beta'$。

显然，如$\tau\leqslant\beta$为X的两个马时，则有

$$\mathscr{F}_\tau\cap\Omega_\beta\subset\mathscr{F}_\beta,\quad \mathscr{F}_\tau'\supset\mathscr{F}_\beta'. \tag{2}$$

引理1 设β为X的马时，则

$$\theta_\beta\mathscr{F}_t^0=\mathscr{L}_t^0(\beta)\subset\mathscr{F}_\beta'\cap\mathscr{F}_{\beta+t}. \tag{3}$$

证 设$A\in\mathscr{F}_t^0$。由Дынкин[2，引理1.5]，存在$t_k\in[0,\ t]$及$E^\infty=E\times E\times E\times\cdots\cdots$中的波雷尔集$\Gamma$使

$$A=\{[x(t_1),\ x(t_2),\ \cdots\cdots]\in\Gamma\}. \tag{4}$$

故由Дынкин[1，第122页]。

$$\theta_\beta A=\{[x'_\beta(t_1),\ x'_\beta(t_2),\cdots\cdots]\in\Gamma\}\in\mathscr{L}_t^0(\beta). \tag{5}$$

故$\theta_\beta\mathscr{F}_t^0\subset\mathscr{L}_t^0(\beta)$。类似可证反包含成立。故$\theta_\beta\mathscr{F}_t^0=\mathscr{L}_t^0(\beta)$。

显然$\mathscr{L}_t^0(\beta)\subset\mathscr{F}'_\beta$。要证$\{x'(s)=i\}\in\mathscr{F}_{\beta+t}(s\leqslant t, i\in E)$，即要证对任意$u\geqslant 0$，$s\leqslant t$，

$$\{x(\beta+s)=i,\ \beta+t<u<\sigma\}\in\mathscr{F}_u^0.$$

当$u\leqslant t$时，左方为空集。故不妨设$u>t$。

令 $\beta_n=\frac{k+1}{2^n}(u-t)$，如果$\frac{k}{2^n}(u-t)\leqslant\beta<\frac{k+1}{2^n}(u-t)$

由 (6.6.1)，$P\{x(\beta_n+s)=\infty\}=0$。由$X$的右连续性以及$(\beta+t<u)\subset(\beta_n+t\leqslant u)$，

$$\begin{aligned}&\{x(\beta+s)=i,\ \beta+t<u<\sigma\}\\&=\{\lim_{n\to\infty}x(\beta_n+s)=i,\ \beta<u-t,\ u<\sigma\}\in\mathscr{F}_u^0,\end{aligned}$$ 证毕

引理2 设τ，β是X的马时，则$\theta_\beta\tau$是X'_β的马时。

证 $$\begin{aligned}\{\theta_\beta\tau<u\}&=\bigcup_{t\geqslant 0}\{\theta_t\tau<u\}\cap(\beta=t)\\&=\bigcup_{t\geqslant 0}\theta_t(\tau<u)\cap(\beta=t)=\theta_\beta(\tau<u).\end{aligned}$$

由 $(\tau<u)\in\mathscr{F}_u^0$，由引理1，$(\theta_\beta\tau<u)\in\mathscr{L}_u^0$，从而 $(\theta_\beta\tau<u<\sigma'_\beta)\in\mathscr{L}_u^0$。

引理3 设$\{\tau,\ \beta\}$是X的标准偶，则τ_n，β_n是X的马时。对指定n，令

$$\tau'_0=\beta'_0=0, \tau'_k=\tau_{n+k}-\beta_n, \beta'_k=\beta_{n+k}-\beta_n,\ k\geqslant 1. \tag{6}$$

则$\{\tau',\ \beta'\}$是X'_{β_n}的标准偶。

证 已知β_0，τ_1，β_1是X的马时，设τ_n，β_n是X的马时，由引理1和2，

$$\left.\begin{aligned}&(\theta_{\beta_n}\tau_1<t<\sigma'_{\beta_n})\in\mathscr{L}_t^0(\beta_n)\subset\mathscr{F}_{\beta_n+t},\\&(\theta_{\beta_n}\beta_1<t<\sigma'_{\beta_n})\in\mathscr{L}_t^0(\beta_n)\subset\mathscr{F}_{\beta_n+t}.\end{aligned}\right\} \tag{7}$$

依定理 6.6.3，$\tau_{n+1}=\beta_n+\theta_{\beta_n}\tau_1$和 $\beta_{+1}=\beta_n+\theta_{\beta_n}\beta_1$是$X$的马时。

由引理2知 $\tau'_1=\tau_{n+1}-\beta_n=\theta_{\beta_n}\tau_1$，$\beta'_1=\beta_{n+1}-\beta_n=\theta_{\beta_n}\beta_1$是$X'_{\beta_n}$的马时。又

$$\beta_1' + \theta_{\beta_1'}\tau_1' = \beta_{n+1} - \beta_n + \theta_{\beta_{n+1}-\beta_n}\theta_{\beta_n}\tau_1$$

$$= \beta_{n+1} + \theta_{\beta_{n+1}}\tau_1 - \beta_n = \beta_{n+2} - \beta_n = \tau_2'$$

仿此可以证明(3.17)对$\{\tau', \beta'\}$成立，即$\{\tau', \beta'\}$是X'_{β_n}的标准偶。

引理4 设$\{\tau, \beta\}$是X的标准偶，则按(2.2)(2.3)确定的σ_n和δ_n分别为$\mathscr{F}_{\tau_n}$和$\mathscr{F}_{\beta_n}$可测。

证 要证对任意$u \geqslant 0$，$t \geqslant 0$有

$$(\sigma_n < u,\ \tau_n < t < \sigma) \in \mathscr{F}_t^0,$$

$$(\delta_n < u,\ \beta_n < t < \sigma) \in \mathscr{F}_t^0.$$

实际上，设R为$[0, \infty)$中有理数集，

$$(\sigma_n < u,\ \tau_n < t < \sigma) = \left\{\sum_{k=1}^{n}(\tau_k - \beta_{k-1}) < u,\ \tau_n < t < \sigma\right\}$$

$$= \bigcup_{\substack{r_k \in R \\ \sum_{k=1}^{n} r_k < u}} (\tau_k - \beta_k < r_k,\ \tau_k < t)(\tau_n < t < \sigma),$$

$$(\delta_n < u,\ \beta_n < t < \sigma) = \left\{\sum_{k=0}^{n}(\beta_k - \tau_k) < u,\ \beta_n < t < \sigma\right\}$$

$$= \bigcup_{\substack{r_k \in R \\ \sum_{k=1}^{n} r_k < u}} (\beta_k - \tau_k < r_k,\ \beta_k < t)(\beta_n < t < \sigma).$$

依侯、郭[1，引理1.4.2]，上述集合$\in \mathscr{F}_t^0$，证毕。

引理5 设$\{\tau, \beta\}$是X的标准偶。按(2.5)确定α_t。对任意固定的$t \geqslant 0$，α_t是X的马时。

证 因$t \leqslant \alpha_t < \sigma$，故不妨设$u > t$。由引理2，

$$(\alpha_t < u < \sigma) = \bigcup_{k=1}^{\infty}(\beta_{k-1} \leqslant u < \tau_k,\ \alpha_t < u < \sigma)$$

$$+ \bigcup^{\infty}(\tau_k \leqslant u < \beta_k,\ \alpha_t < u < \sigma)$$

$$=\bigcup_{k=1}^{\infty}(\beta_{k-1}\leqslant u<\tau_k,\ \rho(u)<u-t,\ u<\sigma)$$

$$+\bigcup_{k=1}^{\infty}(\tau_k\leqslant u<\beta_k,\ \gamma(u)>t,\ u<\sigma)$$

$$=\bigcup_{k=1}^{\infty}(\delta_{k-1}<u-t,\ \beta_{k-1}\leqslant u<\sigma)(u<\tau_k)$$

$$+\bigcup_{k=1}^{\infty}(\sigma_k>t,\ \tau_k\leqslant u<\sigma)(u<\beta_k)。$$

依引理4，上式$\in\mathscr{F}_u^0$，证毕。

引理6 设$\{\tau,\ \beta\}$是X的标准偶。α_t按(2.5)确定。如果定理3.3中条件(c)满足，则定理3.3中条件(b)成立。更精确些，

$$P\{x(t)\in E\text{对一切}t\in A\}=1, \tag{8}$$

其中A如(4.1)。特别地有

$$P\{x(\alpha_t)\in E\text{对一切}t\in[0,\ \bar{\sigma})\}=1, \tag{9}$$

$$P\{x(\beta_k)=\infty\text{对某个}k\}=0。 \tag{10}$$

如果定理3.3中条件(a)(c)满足，则对几乎一切$\omega\in\Omega_{\beta_k}=(\beta_k<\sigma)$及一切$u\in[\beta_k(\omega),\ \tau_{k+1}(\omega)]$有

$$u+(\theta_u\tau_1)(\omega)=\tau_{k+1}(\omega),$$

$$u+(\theta_u\beta_1)(\omega)=\beta_{k+1}(\omega)。 \tag{11}$$

证 (一) 先假设$P\{\beta_0=0\}=1$。

令 $A_k=\{x(u)\in E\text{对一切}u\in[\beta_k,\ \tau_{k+1})\}\cap\Omega_{\beta_k}$。

由(3.17)及$\beta_0=0$，不难看出$A_k=\theta_{\beta_k}A_0$。由定理3.3条件(c)及强马氏性，

$$P\{A_1\}=P\{\theta_{\beta_1}A_0\}=\sum_{k\in E}P\{x(\beta_1)=k\}P\{A_0\,|\,x(0)$$

$$=k\}=\sum_{k\in E}P\{x(\beta_1)=k\}=P\{\Omega_{\beta_1}\}。 \tag{12}$$

类似可证 $P\{A_k\}=P\{\Omega_{\beta_k}\}$。得证(8)。

令 $\triangle_k=\{u+\theta_u\tau_1=\tau_{k+1}\text{对一切}u\in[\beta_k,\ \tau_{k+1})\}\cap\Omega_{\beta_k}$。

同样有$\triangle_k=\theta_{\beta_k}\triangle_0$。由定理3.3的条件(a)(c)及强马氏性，仿(12)可证$P(\triangle_k)=P(\Omega_{\beta_k})$。得证(11)的第一式。第二式类似证明。

(二) 假定$P\{\beta_0=0\}\leqslant 1$。

此时考虑β_0后过程$X'=X'_{\beta_0}$。由引理2，$\{\tau',\ \beta'\}$是X'的标准偶，其中

$$\tau'_0=\beta'_0=0,\ \tau'_l=\tau_l-\beta_0, \beta'_l=\beta_l-\beta_0,\ l\geqslant 1. \qquad (13)$$

由于定理3.3的条件(a)(c)对X和$\{\tau,\beta\}$满足，因而对X'和$\{\tau',\ \beta'\}$也满足。按(一)，本引理的结论对X'和$\{\tau',\ \beta'\}$成立。而由(13)，

$\{x'(t)\in E$对一切$t\in A'\}=\{x(t)\in E$对一切$t\in A\}$，$\Omega_{\beta'_k}=\Omega_{\beta_k}$。$u+\theta_u\tau'_1=\tau'_{k+1}$对一切$u\in[\beta'_k,\ \tau'_{k+1})$当且只当$u+\theta_u\tau_1=\tau_{k+1}$对一切$u\in[\beta_k,\ \tau_{k+1})$。因此本引理的结论对$X$和$\{\tau,\ \beta\}$成立，证毕。

引理7 设$\{\tau,\ \beta\}$是X的标准偶，$P\{\beta_0=0\}=1$。定理3.3中的条件(a)、(c)满足。则对任意固定的$s,t\geqslant 0$，有

$$\alpha_{s+t}=\alpha_s+\theta_{\alpha_s}(\alpha_t). \qquad (15)$$

证 令$X'=X'_{\alpha_s}$。由引理2,

$$\beta'_0=0,\ \tau'_1=\theta_{\alpha_s}\tau_1,\ \beta'_1=\theta_{\alpha_s}\beta_1. \qquad (15)$$

是X'的马时，因而可以确定X'的标准偶$\{\tau',\ \beta'\}$以及对应的α'_t。因$P\{\beta_0=0\}=1$，易见$\alpha'_t=\theta_{\alpha_s}(\alpha_t)$。

由引理1(V)，$\alpha_s\in A=\bigcup_{k=0}^{\infty}[\beta_k,\ \tau_{k+1})$。为明确计，设$\alpha_s\in[\beta_k,\ \tau_{k+1})$。由引理6，$\alpha_s+\theta_{\alpha_s}\tau_1=\tau_{k+1}$，$\alpha_s+\theta_{\alpha_s}\beta_1=\beta_{k-1}$。故由(15)，$\tau'_1=\tau_{k+1}-\alpha_s$，$\beta'_1=\beta_{k+1}-\alpha_s$。象引理3的证明一样可得

$$\tau'_l=\tau_{k+l}-\alpha_s,\ \beta'_l=\beta_{k+l}-\alpha_s\ . \qquad (16)$$

由引理1(V)，$\alpha'_t=\theta_{\alpha_s}(\alpha_t)\in A'=\bigcup_{k=0}^{\infty}[\beta'_k,\ \tau'_{k+1})$。为

明确计，设$\alpha_i' \in [\beta_l', \tau_{l+1}')$。按(16) 即$\beta_{k+l}-\alpha_s \leqslant \theta_{\alpha_s}(\alpha_t) < \tau_{k+l}-\alpha_s$，从而 $\beta_{k+l} \leqslant \alpha_s+\theta_{\alpha_s}(\alpha_t) < \tau_{k+l}$，即 $\alpha_s+\theta_{\alpha_s}(\alpha_t) \in A$。

其次，由引理1(V)，

$$t=L\{A' \cap [0,\ \alpha_t')\}=L\left\{\bigcup_{l=0}^{\infty}[\beta_l',\tau_{l+1}')\cap[0,\ \alpha_t')\right\}$$

$$=L\left\{\bigcup_{l=0}^{\infty}[\beta_{k+l}-\alpha_s,\tau_{k+l}-\alpha_s)\cap[0,\ \theta_{\alpha_s}(\alpha_t'))\right\}$$

$$=L\left\{\bigcup_{l=0}^{\infty}[\beta_{k+l},\ \tau_{k+l})\cap[\alpha_s,\ \alpha_s+\theta_{\alpha_s}(\alpha_t))\right\}$$

$$=L\{A\cap[\alpha_s,\ \alpha_s+\theta_{\alpha_s}(\alpha_t))\}.$$

因而 $L\{A\cap[0,\ \alpha_s+\theta_{\alpha_s}(\alpha_t))\}=L\{A\cap[0,\alpha_s)\}+t=s+t$。

于是 $\alpha_s+\theta_{\alpha_s}(\alpha_t)$ 是方程

$$\left.\begin{array}{l}\gamma(u)=s+t,\\ u\in A,\end{array}\right\}$$

的解，而α_{s+t}是上面方程的唯一解。所以(14) 成立，证毕。

§6. 强极限定理的证明

定理3.3的证明。

(一) 假设$P\{\beta_0=0\}=1$。

设$t_1<t_2<\cdots<t_{n+1}, i_1, i_2, \cdots, i_{n+1}\in E$。由引理5.5和5.1 以及 (5.2)，

$$\Lambda_1=\bigcap_{k=1}^{n-1}\{x(\alpha_{t_k})=i_k\}\cap(\alpha_{t_n}<\sigma)\in\mathscr{F}_{\alpha_{t_n}}.$$

$$\Lambda_2=\{x(\alpha_{t_{n+1}})=i_{n+1}\}\in\mathscr{F}'_{\alpha_{t_n}}.$$

由引理7及强马氏性，

$$P\{x(\alpha_{t_{n+1}})=i_{n+1}\mid x(\alpha_{t_k})=i_k,\ 1\leqslant k\leqslant n\}$$

$$= P\{x(\alpha_{t_{n+1}}) = i_{n+1} | \wedge_1, \alpha_{t_n} < \sigma, x(\alpha_{t_n}) = i_n\}$$

$$= P\{x[\alpha_{t_n} + \theta_{\alpha_{t_n}}(\alpha_{t_{n+1}} - t_n)] = i_{n+1} | x(\alpha_{t_n}) = i_n\}$$

$$= P\{x(\alpha_{t_{n+1} - t_n}) = i_{n+1} | x(0) = i_n\}.$$

注意(3.19)，得证$\bar{X} = W_{\tau,\beta}(X)$的齐次马氏性。

（二）假设$P\{\beta_0 = 0\} \leqslant 1$。

考虑 $X' = X'_{\beta_0}$，以及按 (5.6) 中 $n = 0$ 确定的X'的标准偶$\{\tau', \beta'\}$和对应的α'_t。则(5.13)成立，并且

$$\alpha'_t = \alpha_t - \beta_0，从而 x'(\alpha'_t) = x(\alpha_t)。 \tag{1}$$

由定理3.3的条件$(a)(b)$对X和 $\{\tau, \beta\}$满足，从而对 X' 和 $\{\tau', \beta'\}$ 也满足。于是由(一)，$W_{\tau',\beta'}(X')$具有齐次马氏性。但$W_{\tau,\beta}(X) = W_{\tau',\beta'}(X')$，故$\bar{X} = W_{\tau,\beta}(X)$具有齐次马氏性。

（三）因$X \in \mathscr{X}_1$，所以$\bar{X} \in \mathscr{X}_1$。由引理6知定理3.3的条件$(c)$蕴含条件$(b)$，并且有定理3.3的结论(iii)，证毕。

定理3.4的证明。

除了定理3.4中$\bar{X}$是过程尚需证明外，其余结论都从定理3.1和3.3推出。

往证 $P\{\bar{x}(t) = \infty\} = 0 (t \geqslant 0)$。

由引理6，对几乎一切ω，$x(t, \omega) \in E$对一切 $t \in A''(\omega)$，从而

$$x(t, \omega) \in E，对一切 t \in A(\omega)。 \tag{2}$$

故由引理 4.4(iii)(e)， 当 $t \in \gamma[C_1(A(\omega))]$ 时， $\gamma^{-1}(t) \in C_1[A(\omega)] \subset A(\omega)$，故由(2)，$\bar{x}(t, \omega) = x(\gamma^{-1}(t), \omega) \in E$。因此如果令$\bar{S}_i(\omega) = \{t | \bar{x}(t, \omega) = i\} (i \in \bar{E})$，则有

$$\gamma[C_1(A(\omega))] \subset \{t | \bar{x}(t, \omega) \in E\} = \bigcup_{i \in E} \overline{S_i}(\omega)$$

$$\subset \bigcup_{i \in E} \overline{S_i}(\omega) \cup \overline{S_\infty}(\omega) = [0, \sigma(\omega))。$$

由引理4.4(ii)，$\sigma(\omega) = L\{\gamma[C_1(A(\omega))]\}$，故$L\{\overline{S_\infty}(\omega)\} = 0$。由Fubini定理，知存在$L$零集$T$使当 $t \bar{\in} T$ 时，$P\{\bar{x}(t) = \infty\} = 0$。

由(3.22)，$\beta_0^n \downarrow \inf_n \beta_0^n = \alpha_0$。故(3.21)成为$P\{\bar{x}(0)=\infty\}=0$。于是$0 \in T$。

今设$t_0 \in T$，显然$t_0>0$。因$L(T)=0$，故可选$0<t<t_0$使t_0-t，$t \bar{\in} T$。注意对每个$\omega \in \Omega$，在$\bar{E}$中$\lim_{n\to\infty} x^n(t)=\lim_{n\to\infty} x(\alpha_t^n)=x(\alpha_\cdot)$。故对有限集$D \subset E$，令$\bar{D}=\bar{E}-D$，则

$$P\{\bar{x}(t_0)\in\bar{D}\}=P\{\lim_{n\to\infty} x(\alpha_{t_0}^n)\in\bar{D}\}$$

$$=\lim_{n\to\infty} P\{x(\alpha_{t_0}^n)\in\bar{D}\}.$$

记$X'=X'_{\beta_0^n}$，$\alpha'_t=\alpha_t^n-\beta_0^n$。则由(1)，引理7及$X'$的强马氏性，

$$P\{x(\alpha_{t_0}^n)\in\bar{D}\}=P\{x'(\alpha'_{t_0})\in\bar{D}\}$$

$$=P\{x'[\alpha'_t+\theta_{\alpha'_t}(\alpha'_{t_0-t})]\in\bar{D}\}$$

$$=E\{P_{x'(\alpha'_t)}[x'(\alpha'_{t_0-t})\in\bar{D}]\}$$

$$=E\{P_{x(\alpha_t^n)}[x(\alpha_{t_0-t}^n)\in\bar{D}]\}$$

于是$P\{\bar{x}(t_0)\in\bar{D}\}=\lim_{n\to\infty} E\{P_{x(\alpha_t^n)}[x(\alpha_{t_0-t}^n)\in\bar{D}]\}$。

因$t\in T$，对几乎一切ω，$x(\alpha_t,\omega)\in E$，故n充分大时$x(\alpha_t^n,\omega)=x(\alpha_t,\omega)$。因而

$$P\{\bar{x}(t_0)\in\bar{D}\}=E\{\lim_{n\to\infty} P_{x(\alpha_t)}[x(\alpha_{t_0-t}^n)\in\bar{D}]\}$$

$$=E\{P_{x(\alpha_t)}[x(\alpha_{t_0-t})\in\bar{D}]\}$$

$$=\sum_{k\in E} P\{x(\alpha_t)=k\}P_k\{x(\alpha_{t_0-t})\in\bar{D}\}$$

令$D\uparrow E$得

$$P\{\bar{x}(t_0)=\infty\}=\sum_{k\in E} P\{x(\alpha_t)=k\}P_k\{x(\alpha_{t_0-t})=\infty\}$$

$$=\sum_{k\in E} P\{x(\alpha_t)=k\}\cdot 0=0.$$

既然 $P\{\bar{x}(t)=\infty\}=P\{x^n(t)=\infty\}=0 \ (t\geqslant 0)$，由引理5.3知，$\{x^n(t_k)=i_k,\ 1\leqslant k\leqslant l\}$当$n\to\infty$时依分布收敛于$\{\bar{x}(t_k)=i_k,\ 1\leqslant k\leqslant l\}$。从而由$X$的齐次马氏性得$\bar{X}$的齐次马氏

性，证毕。

§7. 几种特殊的强极限定理

定理1 设$X\in\mathscr{X}_D$。指定一个状态$i\in E$。设β_0为X的马时，满足$P\{\beta_0<\sigma\}>0$。令

$$\tau_1=\inf\{t\mid\beta_0\leqslant t<\sigma,\ x(t)=i\},$$
$$\beta_n=\inf\{t\mid\tau_n\leqslant t<\sigma, x(t)\neq i\},$$
$$\tau_{n+1}=\inf\{t\mid\beta_n\leqslant t<\sigma,\ x(t)=i\}.$$

约定$\inf\phi=\sigma$，ϕ为空集。则$\tilde{X}=W_{\tau,\beta}(X)\in\mathscr{X}_D$，并且$\tilde{X}$不取值$i$。

证 易知$\{\tau,\beta\}$是标准偶，并且定理3.3的条件(a)满足。因$X\in\mathscr{X}_D$，条件(b)也当然满足。引用定理3.3便得定理1。

定理2 设$X\in\mathscr{X}_D$，E的子集M和N互不相交。令

$$\beta_0=\inf\{t\mid 0\leqslant t<\sigma,\ x(t)\in N\},$$
$$\tau_1=\inf\{t\mid\beta_0\leqslant t<\sigma,\ x(t)\in M\},$$
$$\beta_n=\inf\{t\mid\tau_n\leqslant t<\sigma,\ x(t)\in N\},$$
$$\tau_{n+1}=\inf\{t\mid\beta_n\leqslant t<\sigma,\ x(t)\in M\}.$$

则$\tilde{X}=W_{\tau,\beta}(X)\in\mathscr{X}_D$，并且$\tilde{X}$的值$\in E-M$。

证 同定理1。

定理3 设$X\in\mathscr{X}_S$，有穷集$D_n\uparrow E$。令

$$\beta_0^n=\inf\{t\mid 0\leqslant t<\sigma,\ x(t)\in D_n\},$$
$$\tau_1^n=\inf\{t\mid\beta_0^n\leqslant t<\sigma,\ x(t)\overline{\in}D_n\},$$
$$\beta_k^n=\inf\{t\mid\tau_k^n\leqslant t<\sigma,\ x(t)\in D_n\},$$
$$\tau_{k+1}^n=\inf\{t\mid\beta_k^n\leqslant t<\sigma,\ x(t)\overline{\in}D_n\}.$$

则$X^n=W_{\tau^n,\beta^n}(X)$是最小过程，其状态空间为$D_n$，而且

$$\lim_{n\to\infty}X^n=X. \tag{1}$$

证 引用定理3.4，易知$\{\tau^n,\ \beta^n\}$是X的标准偶，并且对X和$\{\tau^n,\ \beta^n\}$，定理3.3中条件$(a)(c)$满足。又因$P\{x(0)=\infty\}=0$，故$\inf_n\beta_0^n=0$，从而(3.21)成立。因$D_n\uparrow$，故(3.22)成立。

$$\left.\begin{aligned}&\text{注意}x(t)\in D_n,\ \text{如果}\ t\in A^n=\bigcup_{k=0}^{\infty}[\beta_k^n,\ \tau_{k+1}^n),\\&x(t)\overline{\in}D_n,\ \text{如果}\ t\in B_n=\bigcup_{k=0}^{\infty}[\tau_k^n,\ \beta_k^n).\end{aligned}\right\}\quad(2)$$

由定理3.4(i) 得 $X^n=W_{\tau^n,\beta^n}(X)\in\mathscr{X}_D$。由 X^n 的定义，X^n 在 $[0,\ \sigma^n)$中没有飞跃点，故X^n是最小过程。

$$\text{往证}\quad\{t\,|\,x(t)\in E\}=A=\bigcup_{n=1}^{\infty}A^n.\qquad(3)$$

设$x(t)=i\in E$，则存在 n 使$i\in D_n$，于是 $x(t)\in D_n$。由(2)，$t\in A^n\subset A$。所以$\{t\,|\,x(t)\in E\}\subset A$。由(2)，反包含关系也成立。

由(3)，$[0,\ \sigma)$与A 只相差L零集 $S_\infty=\{t\,|\,x(t)=\infty\}$。故有

$$\gamma(u)=L\{A\cap[0,\ u)\}=L\{[0,\ \sigma)\cap[0,\ u)\}=u.$$

所以$\alpha_t=\gamma^{-1}(t)=t$，$\bar{\sigma}=\sigma$，从而(3.14)成为(1)。

定理4 设$X\in\mathscr{X}_e$是非最小过程，有穷集$D_n\uparrow E$。令$\beta_0^n=0$，

$$\left.\begin{aligned}&\tau_1^n\text{为}X\text{的第一个飞跃点},\\&\beta_k^n=\inf\{t\,|\,\tau_k^n\leqslant t<\sigma,\ x(t)\in D_n\},\\&\tau_{k+1}^n\text{为}X\text{在}\beta_k^n\text{后的第一个飞跃点}.\end{aligned}\right\}\quad(4)$$

则$X^n=W_{\tau^n,\beta^n}(X)\in\mathscr{X}_1$(一阶过程类)，(3.10) 成立，并且(1)仍成立。

证 仿定理3引用定理3.4，我们只需说明现在的情形下，(3)仍成立即可。为此注意

$$\left.\begin{aligned}&x(t)\in E,\ \text{如果}t\in A^n,\\&x(t)\overline{\in}D_n,\ \text{如果}t\in B^n.\end{aligned}\right\}\quad(5)$$

并仿定理3证明可知(3)成立。

注 定理4中将X变成X^n 的变换正是g_n变换，定理4正是侯振挺[2]中的基本结果，即侯、郭[1]中的第一构造定理。

第九章　飞跃区间和流入分解

§1. 引　言

本章中我们研究过程的流入。我们引进飞跃区间即∪区间的概念，并研究了飞跃区间与飞跃点集之间的联系，找到了柯氏方程组成立的用飞跃区间表达的充要条件。我们发现，用飞跃区间来研究过程的流入是有效的。特别，我们得到过程的流入分解定理。利用此分解，对各种流入都出现的情况，我们可以对各别的流入单独研究。利用飞跃区间的结果，我们得到过程的两类强极限定理，即变换$_Mg_n$，$_Mf_n$以及对应于它们的强极限定理。作为这些定理的简单推论，就是侯振挺[2]中的基本结果即侯振挺等[1]中的第一构造定理，以及王梓坤[3]中的定理5.3。

§2. 飞跃区间的定义

设 $X=\{x(t),\ t<\sigma\}\in\mathscr{X}$。由于(6.6.1)，不失一般性，今后我们将假定，对每个$\omega\in\Omega$，

$$x(r,\ \omega)\in E,\ 对一切r\in R\cap[0,\ \sigma(\omega)). \tag{1}$$

这里以及今后R表示[0，∞)中有理数的集合。

定义1　称$t\in(0,\ \sigma(\omega))$为$X(\omega)$的连续点，如果$\lim\limits_{s\uparrow t}x(s,\ \omega)=x(t,\ \omega)\in E$。连续点集记为 $G(\omega)$。称 $D(\omega)=[0,\ \sigma(\omega)]-G(\omega)$为$X(\omega)$的间断点集。

回忆定义6.8.1，$X(\omega)$ 的跳跃点集记为 $T(\omega)$，飞跃点集记

为$\Gamma(\omega)$，约定 $0\in T(\omega)\cap\Gamma(\omega)$，但称 $t=0$为$X(\omega)$ 的第零个跳跃点和第零个飞跃点。$X(\omega)$ 有第一个间断点 $\tau_1(\omega)>0$，以及第一个飞跃点$\tau(\omega)\geqslant\tau_1(\omega)>0$。

显然，如果$\sigma(\omega)<\infty$且$\sigma(\omega)\in T(\omega)$，则$x(\sigma(\omega)-0,\ \omega)\in H$（非保守状态集）。易见$D(\omega)$和$\Gamma(\omega)$是闭集。

对每个$s\in(0,\ \sigma(\omega))$，可以定义

$$\left.\begin{aligned}&\mu_s(\omega)=\max\{D(\omega)\cap[0,\ s]\},\\&\nu_s(\omega)=\min\{D(\omega)\cap[s,\sigma(\omega)]\},\\&\lambda_s(\omega)=\max\{\Gamma(\omega)\cap[0,\ s]\},\\&\eta_s(\omega)=\min\{\Gamma(\omega)\cap[s,\ \sigma(\omega)]\}.\end{aligned}\right\}\tag{1}$$

对$s=0$，定义

$$\nu_0(\omega)=\tau_1(\omega),\ \ \eta_0(\omega)=\tau(\omega),\ \ \mu_0(\omega)=0,\ \lambda_0(\omega)=0. \tag{2}$$

对$s\geqslant0$，分别称$\mu_s(\omega)$，$\lambda_s(\omega)$，$\nu_s(\omega)$，$\eta_s(\omega)$ 为s 前的最后一个间断点，s 前的最后一个飞跃点，s 后的第一个间断点，s 后的第一个飞跃点。设 $s>0$。如果 $s\in D(\omega)$ 或$s\in\Gamma(\omega)$，则显然有$\mu_s(\omega)=\nu_s(\omega)=s$或$(\lambda_s(\omega)=\eta_s(\omega)=s)$。如果$s\overline{\in}D(\omega)$或$s\overline{\in}\Gamma(\omega)$，则有$\mu_s(\omega)<s<\nu_s(\omega)$或$\lambda_s(\omega)<s<\eta_s(\omega)$。

定义2 称$[\lambda,\ \eta)$为$X(\omega)$ 的一个飞跃区间即$\cup$区间，如果$\lambda,\ \eta\in\Gamma(\omega)$，且$(\lambda,\eta)\cap\Gamma(\omega)=\phi$(空集)。$X(\omega)$的$\cup$区间全体记为$\mathscr{U}(\omega)$。

很明显，$\mathscr{U}(\omega)$中的$\cup$区间按其在$[0,\ \sigma]$中的位置有“先”和“后”的次序。

由王梓坤[1，定理3.2.4系]知，如$[\lambda,\eta)$为$\cup$区间，则(λ,η)中的不连续点是跳跃点，而且

$$x(t,\ \omega)\in E,\ \text{对一切}t\in(\lambda,\ \eta). \tag{3}$$

定义3 设$[\lambda,\ \eta)\in\mathscr{U}(\omega)$，$M\subset\bar{E}$，$N\subset\bar{E}$。如果$x(\lambda,\omega)\in M$，称$[\lambda,\ \eta)$为${}_M\cup$区间。如果 $x(\eta-0,\ \omega)\in N$，称$[\lambda,\ \eta)$ 为$\cup_N$区间。类似可以定义${}_M\cup_N$区间，${}_i\cup$区间等等。记 $X(\omega)$ 的${}_M\cup$区间全体为${}_M\mathscr{U}(\omega)$。类似记号$\mathscr{U}_N(\omega)$，${}_i\mathscr{U}(\omega)$等，其意义明确。

由定理6.7.2，$X(\omega)$没有$\cup_{E-H}$区间，换言之，如果$X(\omega)$有$\cup_E$区间，它必是$\cup_H$区间。

由王梓坤[1，定理3.2.3]，对任意固定的$t>0$，对几乎一切$\omega\in(t<\sigma)$有

$$\lambda_t(\omega)\leqslant\mu_t(\omega)<t<\nu_t(\omega)\leqslant\eta_t(\omega), \tag{4}$$

从而易知对几乎一切$\omega\in\Omega$，

$$\lambda_r(\omega)<r<\eta_r(\omega),\ \text{对一切}r\in R\cap[0,\ \sigma(\omega)). \tag{5}$$

$$\{[\lambda_r(\omega),\ \eta_r(\omega))\mid r\in R\cap[0,\ \sigma(\omega))\}=\mathscr{U}(\omega). \tag{6}$$

于是$\mathscr{U}(\omega)$是可列集。

记 $\mathscr{U}^\tau(\omega)=\{[\lambda,\ \eta)\mid[\lambda,\ \eta)\in\mathscr{U}(\omega),\ \lambda\geqslant\tau(\omega)\}$. (7)

类似记号${}_M\mathscr{U}^\tau(\omega)$，$\mathscr{U}_N^\tau(\omega)$等，其意义明确。记

$$C_2(\omega)=\bigcup_{[\lambda,\eta)\in\mathscr{U}(\omega)}[\lambda,\ \eta), \tag{8}$$

$$C_1(\omega)=\bigcup_{[\lambda,\eta)\in\mathscr{U}(\omega)}(\lambda,\ \eta). \tag{9}$$

类似记号${}_MC_2(\omega)$，$C_{1N}^\tau(\omega)$等等，其意义是明确的。

§3．飞跃点和飞跃区间

定义1 设 $t\in\Gamma(\omega)$。称 t 为 $X(\omega)$ 的右孤立飞跃点，如果存在 $\varepsilon>0$ 使 $(t,\ t+\varepsilon)\cap\Gamma(\omega)=\phi$。全体右孤立飞跃点集记为$\Gamma^r(\omega)$。类似定义左孤立飞跃点集$\Gamma^l(\omega)$。

对$M\subset E$，令

$$\Gamma_M^r(\omega)=\{t\mid t\in\Gamma^r(\omega),\ x(t,\ \omega)\in M\}, \tag{1}$$

$$\overline{\Gamma_M^r}^+(\omega)=\{t\mid\text{存在不增的}t_n\in\Gamma_M^r(\omega)\text{使}t_n\downarrow t\}. \tag{2}$$

并称$\overline{\Gamma_M^r}^+(\omega)$为$\Gamma_M^r(\omega)$的右闭包。类似地，令

$$\Gamma_M^l(\omega)=\{t\mid t\in\Gamma^l(\omega),\ x(t-0,\ \omega)\in M\}, \tag{3}$$

及左闭包

$$\overline{\Gamma_M^l}^-(\omega)=\{t\mid\text{存在不降的}t_n\in\Gamma_M^l(\omega)\text{使}t_n\uparrow t\}. \tag{4}$$

显然$0\in\Gamma^l(\omega)\cap\Gamma^r(\omega)$。如果$\sigma(\omega)<\infty$，则$\sigma(\omega)\in\Gamma^r(\omega)$。

定理1 对几乎一切$\omega\in\Omega$，

$$\Gamma^l(\omega)=\{\eta\,|\,[\lambda,\ \eta)\in\mathscr{U}(\omega)\}\cup\{0\}. \tag{5}$$

$$\Gamma^r(\omega)=\{\lambda\,|\,[\lambda,\ \eta)\in\mathscr{U}(\omega)\}\cup\{\sigma(\omega)\,|\,\sigma(\omega)<\infty\}. \tag{6}$$

证 只证(6)。右方集合含于$\Gamma^r(\omega)$明显。设$t\in\Gamma^r(\omega)$并且$t\neq\sigma(\omega)$。依定义存在$\varepsilon>0$使$(t,\ t+\varepsilon)\cap\Gamma(\omega)=\phi$，故任取$r\in(t,\ t+\varepsilon)\cap R$，$t=\lambda_r(\omega)<r<\eta_r(\omega)$。由(2.6)知$t\in$(6)右方集，证毕。

定理2 对几乎一切ω，

$$\Gamma(\omega)=\overline{\Gamma}^{r+}(\omega)=\overline{\Gamma}^{l-}(\omega). \tag{7}$$

证 $\overline{\Gamma}^{r+}(\omega)\subset\Gamma(\omega)$明显。设$t\in\Gamma(\omega)$。如果$t=0$或$t=\sigma(\omega)$，显然$t\in\Gamma^r(\omega)\subset\overline{\Gamma}^{r+}(\omega)$。如果$t\in(0,\ \sigma(\omega))$并且$t\overline{\in}\Gamma^r(\omega)$。任取严格下降的$r_n\downarrow t$，$r_n\in R\cap[0,\ \sigma(\omega))$。依$\lambda_{r_n}(\omega)$的定义有$t\leqslant\lambda_{r_n}(\omega)<r_n$。由$t\overline{\in}\Gamma^r(\omega)$，故$t<\lambda_{r_n}(\omega)<r_n$。于是存在$r_n$的子列$r_n'$使$\lambda_{r_n'}(\omega)$严格下降。当$r_n'\downarrow t$时，$\lambda_{r_n'}(\omega)\downarrow t$。由定理1，$r_{r_n'}(\omega)\in\Gamma^r(\omega)$。于是$t\in\overline{\Gamma}^{r+}(\omega)$。所以$\Gamma(\omega)=\overline{\Gamma}^{r+}(\omega)$。类似证明$\Gamma(\omega)=\overline{\Gamma}^{l-}(\omega)$，证毕。

定理3 对几乎一切$\omega\in\Omega$，

$$S_E(\omega)=C_1(\omega)\cup\{\lambda\,|\,\lambda\in\Gamma^r(\omega),\ x(\lambda,\ \omega)\in E\}\subset C_2(\omega). \tag{8}$$

$$[0,\ \sigma(\omega))-C_1(\omega)\subset S_\infty(\omega)\subset\Gamma(\omega). \tag{9}$$

$$\mathrm{L}\{C_1(\omega)\}=\mathrm{L}\{C_2(\omega)\}=\sigma(\omega). \tag{10}$$

其中

$$S_M(\omega)=\{t\,|\,t\in[0,\sigma(\omega)),\ x(t,\ \omega)\in M\},\quad M\subset\overline{E}. \tag{11}$$

证 (8)中包含关系明显。$S_\infty(\omega)\subset\Gamma(\omega)$也明显。设$t\in S_E(\omega)$，则必存在$i\in E$，使$t$属于$X(\omega)$的某个$i$区间$[a,\ b)$中。任取$r\in[a,\ b)\cap R$，则$t\in[a,\ b)\subset[\lambda_r(\omega),\ \eta_r(\omega))$，此时

或者 $t\in(\lambda_r(\omega),\ \eta_r(\omega))\subset C_1(\omega)$，或者 $t=\lambda_r(\omega)\in\Gamma^r(\omega)$，而 $x(t,\ \omega)=i\in E$。因此 $S_E(\omega)\subset C_1(\omega)\cup\{\lambda|\lambda\in\Gamma^r(\omega),\ x(\lambda,\ \omega)\in E\}$。反包含明显。得证(8)，从而得证(9)。因 $[0,\ \sigma(\omega))=S_E(\omega)\cup S_\infty(\omega)$，且 $L\{S_\infty(\omega)\}=0$。由(8)(9)得(10)，证毕。

系 $L\{\Gamma(\omega)\}=0$。

证 因 $\Gamma(\omega)\subset[0,\sigma(\omega))-C_1(\omega)$。由(10)得 $L\{\Gamma(\omega)\}=0$。

令 $A(\omega)=S_E(\omega)-\Gamma(\omega)\doteq[0,\ \sigma(\omega))-\Gamma(\omega)$。 (12)

$\mathscr{U}[A(\omega)]$表示$A(\omega)$中构成区间全体。

定理4 对几乎一切$\omega\in\Omega$，

$$\mathscr{U}[A(\omega)]=\mathscr{U}(\omega)。 \tag{13}$$

证 设 $[\lambda,\ \eta)\in\mathscr{U}[A(\omega)]$，依定义 $(\lambda,\ \eta)\subset A(\omega)$。再依(12)，$(\lambda,\ \eta)\cap\Gamma(\omega)=\phi$。而且任取$t\in(\lambda,\ \eta)$有$x(t,\ \omega)\in E$，$(\lambda,\ \eta)\subset(\lambda_t(\omega),\ \eta_t(\omega))\subset A(\omega)$。由$[\lambda,\ \eta)$的最大性，$[\lambda,\ \eta)=[\lambda_t(\omega),\ \eta_t(\omega))\in\mathscr{U}(\omega)$。

设$[\lambda,\ \eta)\in\mathscr{U}(\omega)$。依定义 $(\lambda,\ \eta)\cap\Gamma(\omega)=\phi$。注意(2.3)有$(\lambda,\ \eta)\subset A(\omega)$。如果有$[\lambda',\eta')\supset[\lambda,\eta)$，且$(\lambda',\eta')\subset A(\omega)$。则 $\lambda'\leqslant\lambda<\eta\leqslant\eta'$，且对任意 $s\in(\lambda',\ \eta')$ 有 $s\overline{\in}\Gamma(\omega)$。因而对任意$t\in(\lambda,\ \eta)$，有$\lambda_t(\omega)\leqslant\lambda'$，$\eta'\leqslant\eta_t(\omega)$。而 $\lambda,\ \eta\in\Gamma(\omega)$，故又有$\lambda=\lambda_t(\omega)$，$\eta=\eta_t(\omega)$。这样，$\lambda'=\lambda=\lambda_t(\omega)$，$\eta'=\eta=_t(\omega)$。得证$[\lambda,\ \eta)$的最大性。从而$[\lambda,\ \eta)\in\mathscr{U}[A(\omega)]$。证毕。

定理5 设$M\subset E$，$N\subset E$，H为非保守状态集。则下列诸集都是$\mathscr{F}^0_\infty$可测集。

$F_1=\{\omega|\mathscr{U}(\omega)$有最后一个$\cup$区间，它是$\cup_N$区间$\}$

$F_2=\{\omega|\mathscr{U}(\omega)=\mathscr{U}_N(\omega)\}$，

$F_3=\{\omega|\mathscr{U}(\omega)$中有最后一个$\cup$区间，除最后一个外，$\mathscr{U}(\omega)$中其余区间都是$\cup_N$区间$\}$，

$F_4=\{\omega|\mathscr{U}^\tau(\omega)={}_M\mathscr{U}^\tau(\omega)\}$，

$F_5=\{\omega|\mathscr{U}(\omega)$中至多有一个$\cup_H$区间，如果有，它是最后一个$\cup$区间$\}$，

$F_6=\{\omega|\mathscr{U}(\omega)$中至少有一个$\cup_H$区间，而且这样的$\cup_H$区间不

是最后一个∪区间}。

证 显然，对$r\geqslant 0$，$k\in \bar{E}$，

$$\{x(\eta_r-0)=k\}\in \mathscr{F}^0_\infty. \tag{14}$$

次证

$$\{x(\lambda_r)=k\}\in \mathscr{F}^0_r. \tag{15}$$

实际上，当$k\neq\infty$时，由（2.1），

$$\{x(\lambda_r)=k\}=\{x(\lambda_r)=k,\ \lambda_r<r\}$$

$$=\bigcup_{i=1}^{\infty}\bigcap_{n=i}^{\infty}\bigcup_{m=0}^{2^i-2}\left\{x\left(\frac{m+1}{2^n}r\right)=k,\right.$$

$$\left[\frac{m+1}{2^n}r,\ r\right)\cap T(\omega)\text{为有穷集},$$

$$\left.\left[\frac{m-1}{2^n}r,\ \frac{m+1}{2^n}r\right)\cap T(\omega)\text{为无穷集}\right\}\in\mathscr{F}^0_r,$$

$$\{x(\lambda_r)=\infty\}=(r<\sigma)-\bigcup_{k\in E}\{(\lambda_r)=k\}\in\mathscr{F}^0_r.$$

于是

$$F_1=\bigcap_{r\in R}\{r<\sigma,\ \eta_r=\sigma,\ x(\eta_r-0)\in N\}\in\mathscr{F}^0_\infty,$$

$$F_2=\bigcap_{r\in R}\{(\sigma\leqslant r)\cup(r<\sigma,\ x(\eta_r-0)\in N)\}\in\mathscr{F}^0_\infty,$$

$$F_3=\bigcap_{r\in R}\{(\sigma\leqslant r)\cup(r<\sigma,\ \eta_r=\sigma)\cup(r<\sigma,\ \eta_r<\sigma,$$

$$x(\eta_r-0)\in N)\}\in\mathscr{F}^0_\infty,$$

$$F_4=\bigcap_{r\in R}\{(\sigma\leqslant r)\cup(r<\tau)\cup(\tau\leqslant r<\sigma,\ x(\lambda_r)\in M)\}$$

$$\in\mathscr{F}^0_\infty,$$

$$F_6=\bigcup_{r\in R}\{\eta_r<\sigma,\ x(\eta_r-0)\in H\}\in\mathscr{F}^0_\infty,$$

$$F_5=\Omega-F_6\in\mathscr{F}^0_\infty,$$

证毕。

§4. 飞跃区间和柯氏方程组

设$M\subset \bar{E}$，$N\subset \tilde{E}$，$S\subset E$。令

$$\xi_{MS}=\inf\{t\mid \tau\leqslant t<\sigma,\ x(\lambda_t)\in M,\ x(t)\in S\}, \tag{1}$$

$$\rho_{SN}=\sup\{t\mid 0\leqslant t<\sigma,\ x(t)\in S,\ t(\eta_t-0)\in N\}, \tag{2}$$

$$\delta_S=\inf\{t\mid \tau\leqslant t<\sigma,\ t\in \Gamma,\ x(t)\in S\}. \tag{3}$$

这里τ是X 的第一个飞跃点，Γ 是飞跃点集。约定对空集ϕ，$\inf\phi=\sigma$，$\sup\phi=0$。

引理1 设S为有穷集，则(1)、(3) 中的inf可用min 代替。

证 设$t_n\downarrow\xi_{MS}$，$\tau\leqslant t_n<\sigma$，$x(\lambda_{t_n})\in M$，$x(t_n)\in S$。因S有限，由X的右连续性，

$$x(\xi_{MS})=\lim_{n\to 1}x(t_n)\in S.$$

因此ξ_{MS}属于某$i(\in S)$区间$[a,\ b)$中，当n充分大时，$t_n\in[a,\ b)$。因而$x(\xi_{MS})=x(t_n)$，$\lambda_{\xi_{MS}}=\lambda_{t_n}$。于是$x(\lambda_{\xi_{MS}})=x(\lambda_{t_n})\in M$。得证(1) 中可用min代替inf，其余的证明类似，证毕。

引理2

$$\begin{aligned}\xi_{MS}&=\inf\{\xi_{Mj}\mid j\in S\}=\inf\{\xi_{kS}\mid k\in M\}\\&=\inf\{\xi_{kj}\mid k\in M,\ j\in S\},\end{aligned} \tag{4}$$

$$\begin{aligned}\rho_{SN}&=\sup\{\rho_{Sj}\mid j\in N\}=\sup\{\rho_{kN}\mid k\in S\}\\&=\sup\{\rho_{kj}\mid k\in S,\ j\in N\},\end{aligned} \tag{5}$$

$$\delta_S=\inf\{\delta_k\mid k\in S\}. \tag{6}$$

证 令

$$A_{MS}=\{t\mid \tau\leqslant t<\sigma,\ x(\lambda_t)\in M,\ x(t)\in S\}. \tag{7}$$

显然 $A_{Mj}\subset A_{MS}(j\in S)$，故 $\xi_{Mj}\geqslant\xi_{MS}(j\in S)$。另一方面，对任意$\varepsilon>0$，存在$t\in A_{MS}=\bigcup_{j\in S}A_{Mj}$使$t<\xi_{MS}+\varepsilon$。故存在 $j\in S$ 使 $t\in A_{Mj}$，从而$\xi_{Mj}\leqslant t<\xi_{MS}+\varepsilon$。得证$\xi_{MS}=\inf\{\xi_{Mj}\mid j\in S\}$。其余类似证明，证毕。

引理3 ξ_{MS}，δ_S是X的马时，ρ_{SN}是随机变量。

证 由引理2，只需对 $k\in\bar{E}$，$j\in E$证明ξ_{kj}，δ_j是马时，ρ_{kj}是随机变量即可。

实际上，注意（3.15），对任意$u\geqslant 0$，

$$(\xi_{kj}<u<\sigma)=\bigcup_{\substack{r<u\\ r\in R}}\{\tau\leqslant r<\sigma,\ x(\lambda_r)=k,$$

$$x(r)=j,\ u<\sigma\}\in\mathscr{F}_u^0.$$

其次，

$$(\delta_j<u<\delta)=\bigcup_{\substack{r<u\\ r\in R}}\Big\{(\tau\leqslant r<\sigma,\ x(r)=j)$$

$$\cap\Big[\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty}\bigcup_{v=1}^{2^m-1}\Big(X\text{在}\Big[\frac{v-1}{2^m}r,\ \frac{v}{2^m}r\Big)\text{中有无穷多个跳跃点，在}$$

$$\Big[\frac{v}{2^m}r,\ r\Big)\text{中取常值}j\ \Big)\Big]\Big\}\in\mathscr{F}_u^0,$$

$$(\rho_{kj}>u)=\bigcup_{\substack{r\in R\\ r>u}}\{0\leqslant\sigma<0,\ x(r)=k,\ x(\eta_r-0)=j\}$$

$$\in\mathscr{F}_\infty^0,$$

证毕。

定义1 设$X\in\mathscr{X}_S$。称X为纯自M流入，如果 $P_i\{\mathscr{U}^\tau={}_M\mathscr{U}^\tau\}=1(i\in E)$；称$X$纯流出到$N$，如果$P_i\{\mathscr{U}=\mathscr{U}_N\}=1(i\in E)$；称$X$拟流出到$N$，如果$P_i(\Omega_0)=1(i\in E)$，这里

$$\begin{aligned}\Omega_0&=(F_2-F_1)\cup F_3\\&=\{\omega|\mathscr{U}(\omega)\text{中没有最后一个}\cup\text{区间，且}\\&\quad\mathscr{U}(\omega)=\mathscr{U}_N(\omega)\}\cup\{\omega|\mathscr{U}(\omega)\text{中有最后一个}\cup\text{区间，}\\&\quad\text{除最后一个}\cup\text{区间外，其余均是}\cup_N\text{区间}\},\end{aligned}$$

F_1，F_2，F_3按定理3.5确定。

定理4 设$X\in\mathscr{X}_S$，则下列结论彼此等价。

(i) 过程X纯自∞流入，即$P_i\{\mathscr{U}^\tau={}_\infty\mathscr{U}^\tau\}=1(i\in E)$。

(ii) $P_i\{\xi_{EE}<\sigma\}=0$ $(i\in E)$，ξ_{EE}按(4.1) 确定。

(iii) 过程X满足向前方程组。

(iv) 对任意$t>0$，$P_i\{\tau<t<\sigma,\ x(\lambda_t)\in E\}=0$ ($i\in E$)。

如果上面结论之一成立，则

(v) $P_i\{\xi_{\infty E}=\tau\}=1$ ($i\in E$)。更精确些，对任意$i\in E$，P_i 几乎一切$\omega\in(\tau\vee\sigma)$有 $\tau(\omega)\in\overline{\Gamma}^{\tau+}_{\infty}(\omega)$。

证 显然(i) $\Rightarrow$(iv)。反之，由 (iv) 推出$P_i\{x(\lambda_r)\in E$对一切$r\in R\cap(\tau,\ \sigma)\}=0$ ($i\in E$)。由此并注意 (2.6) 得 $P_i\{{}_E\mathscr{U}^{\tau}=\mathscr{U}^{\tau}\}=0(i\in E)$，此即条件(i)。

(i)$\Rightarrow$(iii)。设$t>0$，因$P_i\{x(t)=\infty\}=0$，故对P_i几乎一切$\omega\in(t<\sigma)$，$t\in S_E(\omega)$。由定理3.3，$t\in C_2(\omega)$。因而存在$[\lambda,\eta)\in\mathscr{U}(\omega)$使$t\in[\lambda,\ \eta)$。如果$[\lambda,\ \eta)=[0,\ \tau(\omega))$，则显然$t$前有最后一个间断点$\mu_t(\omega)\in T(\omega)$。否则，由结论(i)，$[\lambda,\ \eta)$ 是${}_\infty\cup$区间，$x(\lambda)=\infty$，故$t\in(\lambda,\ \eta)$，从而$\lambda<\mu_t(\omega)<\eta$，$\mu_t(\omega)\in T(\omega)$。依定理6.8.3，$X$满足向前方程组。

(iii)$\Rightarrow$(ii)。依定理6.8.3，若结论(iii)成立，则$p_k(\Omega_1)=1$ ($k\in E$)，这儿

$$\Omega_1=\{\omega|\text{对一切}r\in R\cap[0,\ \sigma(\omega)),\ x(r,\ \omega)\in E,\ \mu_r(\omega)\in T(\omega)\}.$$

如果$\omega\in(\xi_{ij}<\sigma)(i,\ j\in E)$，则$x(\lambda_{\xi_{ij}},\ \omega)=i$，故存在$r\in R\cap[0,\sigma(\omega))$使$\lambda_{\xi_{ij}(\omega)}$与$r$在同一个$i$区间中，从而$\mu_r(\omega)=\lambda_{\xi_{ij}(\omega)}(\omega)\overline{\in}T(\omega)$。因此 $\omega\overline{\in}\Omega_1$，即 $(\xi_{ij}<\sigma)\subset\Omega-\Omega_1$。于是由引理2，$(\xi_{EE}<\sigma)=\bigcup\limits_{i,j\in E}(\xi_{ij}<\sigma)\subset\Omega-\Omega_1$。得证结论(ii)。

(ii) $\Rightarrow$(i)。如果$\omega\in(\mathscr{U}^{\tau}\neq{}_\infty\mathscr{U}^{\tau})$，则存在${}_E\cup$区间$[\lambda,\ \eta)\in\mathscr{U}^{\tau}(\omega)$。因而存在某$i\in E$使$\xi_{Ei}(\omega)<\sigma(\omega)$。于是

$$(\mathscr{U}^{\tau}\neq{}_\infty\mathscr{U}^{\tau})\subset\bigcup_{i\in E}(\xi_{Ei}<\sigma)=(\xi_{EE}<\sigma)\text{。得证结论(i)。}$$

(i) $\Rightarrow$(v)。由结论(i)，对P_i几乎一切ω，$\Gamma^{\tau}_E(\omega)-\{0\}=\phi$。而且由定理3.2，如果$\tau(\omega)<\sigma(\omega)$，则或者$\tau(\omega)$是某$\cup$区间的左

端点，此时$x(\tau(\omega),\ \omega)=\infty$，$\tau(\omega)\in\Gamma_{\infty}^{l}(\omega)$；或者存在严格下降的$\lambda_n\downarrow\tau(\omega)$，$\lambda_n$是$\cup$区间的左端点。由结论（i），$x(\lambda_n,\ \omega)=\infty$，因此 $\tau(\omega)\in\Gamma_{\infty}^{\bar{l}+}(\omega)$，证毕。

定理5 设$X\in\mathscr{X}s$，则下列结论彼此等价。

（i）过程X满足向后方程组。

（ii）过程拟流出到∞。

（iii）$P_i\{\omega|\mathscr{Y}(\omega)$中至多有一个$\cup_H$区间，如果有，它是最后一个$\cup$区间$\}=1$，$(i\in E)$。

（iv）$P_i\{\tau_1\in\Gamma(\omega),\ \tau_1<\sigma\}=0$，$(i\in H)$，其中$\tau_1$为第一个跳跃点，$H$为非保守状态集。

（v）对任意$t\geqslant0$，$P_i\{t<\sigma,\ x(\eta_t-0)\in H,\ \eta_t<\sigma\}=0(i\in E)$。

证 （ii）与（iii）是同一意思。

（i）⇒（iii）。由定理 6.8.3，$P_i(\Omega_2)=1$。这里 $\Omega_2=\{\omega|$对一切$r\in R\cap[0,\ \sigma(\omega))$，$\nu_r(\omega)\in T(\omega)\}$。由定理6.7.2，$P_i(\Omega_3)=1$。这里$\Omega_3=\{\omega|$对一切$r\in R\cap[0,\ \sigma(\omega))$，如果$x(\eta_r-0,\ \omega)\in H$并$\eta_r(\omega)<\sigma(\omega)$，则$x(\eta_r,\ \omega)=\infty\}$。试证$\Omega_2\cap\Omega_3\subset F_5$。实际上，设$\omega\in\Omega_2\cap\Omega_3$。如果$X(\omega)$有一个$\cup_H$区间$[\lambda,\ \eta)$，即$x(\eta-0,\ \omega)\in H$，故存在 $r\in R\cap[0,\ \sigma(\omega))$ 使 $\eta(\omega)=\nu_r(\omega)=\eta_r(\omega)$。如果$\eta(\omega)<\sigma(\omega)$，因$\omega\in\Omega_2$有$\eta(\omega)=\nu_r(\omega)\in T(\omega)$，因 $\omega\in\Omega_3$ 有$\eta=\eta_r(\omega)\in\Gamma(\omega)$。矛盾。于是必定$\eta(\omega)=\sigma(\omega)$，即$[\lambda,\ \eta)$是最后一个$\cup$区间。得证$\Omega_2\cap\Omega_3\subset F_5$，从而得结论(iii)。

(iii)⇒(iv)明显。由结论(iv)及强马氏性易得结论(v)。

(v)⇒(i)。由结论(v) 得 $P_i\{\Omega_4\}=1(i\in E)$，这里 $\Omega_4=\{\omega|$对任意$r\in R\cap[0,\ \sigma(\omega))$，如果$x(\eta_r-0,\ \omega)\in H$，则必定 $\eta_r(\omega)=\sigma(\omega)\}$。

固定$t\geqslant0$，因$P_i\{x(t)=\infty\}=0$。由定理3.3，对P_i几乎一切$\omega\in(t<\sigma)\cap\Omega_4$，$t<S_E(\omega)\subset C_2(\omega)$。因而有 $[\lambda,\ \eta)\in\mathscr{Y}(\omega)$ 使 $t\in[\lambda,\ \eta)$。如果$x(\eta-0,\ \omega)=\infty$，则有 $\lambda\leqslant t<\nu_t(\omega)<\eta$，因而$\nu_t(\omega)\in T(\omega)$。否则$x(\eta-0,\ \omega)\in H$。倘若$\nu_t(\omega)<\eta$，显然 $\nu_t(\omega)\in T(\omega)$。倘若$\nu_t(\omega)=\eta$，则有某$r\in R\cap[0,\ \sigma(\omega))$使 $\eta=\nu_t(\omega)=$

$\nu_t(\omega)=\eta_t(\omega)$。因$\omega\in\Omega_4$，故$\eta=\sigma(\omega)$.这样 $\nu_t(\omega)=\sigma(\omega)\in T(\omega)$。因此恒有$\nu_t(\omega)\in T(\omega)$。由定理6.8.3，$X$满足向后方程组，证毕。

由定理4和5立即得到下面的两个定理。

定理6 设$X\in\mathscr{X}s$，则下列结论彼此等价。

(i) X同时满足向后向前方程组。

(ii) X纯自∞流入，拟流出到∞。

(iii) $P_i\{\xi_{EE}<\sigma\}=0$ $(i\in E)$，$P_i\{\tau_1\in\Gamma,\tau_1<\sigma\}=0(i\in H)$。

(iv) $P_i\{\mathscr{U}^{\tau}={}_{\infty}\mathscr{U}^{\tau}\}=1$，$P_i\{\mathscr{U}$中至多有一个$\cup_H$区间，如果有，它是最后一个$\cup$区间$\}=1$，$(i\in E)$。

(v) $P_i\{\tau<t<\sigma,\ x(\lambda_t)\in E\}=0(t>0,\ i\in E)$，

$P_i\{t<\sigma,\ x(\eta_t-0)\in H,\ \eta_t<\sigma\}=0(t\geqslant0,\ i\in E)$。

定理7 设$X\in\mathscr{X}s$，则下列结论彼此等价。

(i) X既不满足向后也不满足向前方程组。

(ii) 存在$i\in E$，$k\in H$使$P_i\{\xi_{EE}<\sigma\}>0$，

$P_k\{\tau_1\in\Gamma,\ \tau_1<\sigma\}>0$。

(iii) 存在i，$k\in E$使$P_i\{\mathscr{U}^{\tau}={}_E\mathscr{U}^{\tau}\}>0$，$P_k\{\mathscr{U}$中有$\cup_H$区间，但不是最后一个$\cup$区间$\}>0$。

(iv) 存在$t_1>0$，$t_2\geqslant0$，i，$k\in E$使$P_i\{\tau<t_1<\sigma,\ x(\lambda_{t_1})\in E\}>0$，$P_k\{t_2<\sigma,\ X(\eta_{t_2}-0)\in H,\ \eta_{t_2}<\sigma\}>0$。

定理8 设$X\in\mathscr{X}s$，$M\subset\bar{E}$。则下列结论彼此等价。

(i) 过程X纯自M流入，即$P_i\{\mathscr{U}^{\tau}={}_M\mathscr{U}^{\tau}\}=1$ $(i\in E)$。

(ii) $P_i\{\xi_{\bar{M}E}<\sigma\}=0$ $(i\in E,\ \bar{M}=\bar{E}-M)$。

(iii) 对任意$t>0$，$P_i\{\tau<t<\sigma,\ x(\lambda_t)\in\bar{M}\}=0(i\in E)$。或等价地，对任意$i\in E$，$t>0$，对$P_i$几乎一切$\omega\in(\tau<t<\sigma)$，有$x[\lambda_t(\omega),\ \omega]\in M$。

如果上面结论之一成立，则

(iv) $P_i\{\xi_{ME}=\tau\}=1$ $(i\in E)$。更精确些，对任意 $i\in E$，对P_i几乎一切 $\omega\in(\tau<\sigma)$，$\tau(\omega)\in\vec{\Gamma}_M^{+}(\omega)$。

证 (i)$\Rightarrow$(iii)。仿定理4中证明其中的(i)与(iv)等价一样进行。

(i)⇒(ii)。设$\omega\in(\xi_{\overline{M}E}<\sigma)=\bigcup_{k\in E}(\xi_{\overline{M}k}<\sigma)$。则存在某$k\in E$，使$\xi_{\overline{M}E}(\omega)<\sigma(\omega)$。故$[\lambda_{\xi_{\overline{M}k}}(\omega),\ \eta_{\xi_{\overline{M}k}}(\omega))\in\mathscr{U}^{\tau}(\omega)$，而且由$x[\lambda_{\xi_{\overline{M}i}}(\omega),\ \omega]\in\overline{M}$知是$_{\overline{M}}\cup$区间，故$\omega\in(\mathscr{U}^{\tau}\neq{}_{M}\mathscr{U}^{\tau})$。

(ii) ⇒(i)。由X的齐次性及结论(ii)得，对一切$r\in R$，$P_i\{\theta_r(\xi_{\overline{M}E}<\sigma)\}=0$，$\theta$为推移算子。令$\Omega_0=\bigcap_{r\in R}\{\Omega-\theta_r(\xi_{\overline{M}E}<\sigma)\}$，则$P_i\{\Omega_0\}=1$。当$\omega\in\Omega_0$时，对一切$r\in[0,\ \sigma(\omega))$，在$[\max(r,\ \tau(\omega)),\ \sigma(\omega))$中没有$_{\overline{M}}\cup$区间，从而$[\tau(\omega),\ \sigma(\omega))$中的$\cup$区间全是$_{M}\cup$区间，即$\Omega_0\subset(\mathscr{U}^{\tau}={}_{M}\mathscr{U}^{\tau})$。

设结论(i)—(iii) 之一成立，因而对P_i几乎一切ω，$\Gamma_{\overline{M}}^{\tau}(\omega)-\{0\}$是空集，因而$\tau(\omega)\in\overline{\Gamma}_{M}^{\tau+}(\omega)$。由定理 3.2，$\tau(\omega)\in\Gamma(\omega)=\overline{\Gamma}^{\tau+}(\omega)=\overline{\Gamma}_{M}^{\tau+}(\omega)$，证毕。

§5. $_{M}g_{n}$ 变换及其强极限定理

设$X=\{x(t),\ t<\sigma\}\in\mathscr{X}_{s}$，$M\subset\overline{E}$，有穷集$D_n\uparrow E$。令

$$\left.\begin{array}{l}
{}_{M}\beta_0^n=0,\\
{}_{M}\tau_1^n\text{为}X\text{的第一个飞跃点},\\
{}_{M}\beta_k^n=\min\{t\mid{}_{M}\tau_k^n\leqslant t<\sigma,X(\lambda_t)\in M,\ X(t)\in D_n\},\\
{}_{M}\tau_{k+1}^n\text{为}{}_{M}\beta_k^n\text{后的第一个飞跃点}.
\end{array}\right\}\qquad(1)$$

λ_t是t前的最后一个飞跃点。由引理4.1，(1) 中的min是存在的，只要其后的集合不是空集，否则取最小值为σ。

由引理4.3，${}_{M}\tau_k^n$，${}_{M}\beta_k^n$ 是X的马时，易知对每个 n，$\{{}_{M}\tau^n,\ {}_{M}\beta^n\}$是$X$的标准偶，且满足定理8.3.3中的条件 ($a$)、($c$)，因而依定理8.3.3，${}_{M}X^n=W_{M^{\tau^n}}$，${}_{M}t^n(X)\in\mathscr{X}_D$。实际上 ${}_{M}X^n\in\mathscr{X}_1$，即${}_{M}X^n$是一阶过程。

定义1 按上面方式将 $X\in\mathscr{X}_s$变为${}_{M}X^n\in\mathscr{X}_1$ 的变换 $W_{M}\tau^n$，${}_{M}\beta^n$，我们称为${}_{M}g_n$变换。当$M=\overline{E}$时，简记 ${}_{\overline{E}}g_n$为g_n。

这样，

$${}_MX^n={}_Mg_n(X)。\tag{2}$$

显然，当$n\uparrow\infty$时，

$$\left.\begin{aligned}{}_MA^n&=\bigcup_{k=0}^{\infty}[{}_M\beta_k^n,\ {}_M\tau_{k+1}^n)\uparrow{}_MA,\\ {}_MA_1^n&=\bigcup_{k=0}^{\infty}({}_M\beta_k^n,\ {}_M\tau_{k+1}^n)\uparrow{}_MA_1。\end{aligned}\right\}\tag{3}$$

定理1 对几乎一切$\omega\in\Omega$有

$$\mathscr{U}[{}_MA_1^\tau(\omega)]={}_M\mathscr{U}^\tau(\omega),\tag{4}$$

$$C_2[{}_MA^\tau(\omega)]=C_2[{}_MA_1^\tau(\omega)]={}_MC_2^\tau(\omega)。\tag{5}$$

其中${}_MA_1^\tau={}_MA_1-(0,\ \tau)$，${}_MA^\tau={}_MA-[0,\ \tau)$，$\tau$ 为第一个飞跃点。$\mathscr{U}(A)$，$C_2(A)$按定义8.3.1 理解，${}_M\mathscr{U}(\omega)$，${}_MC_2^\tau(\omega)$ 按定义2.3及(2.8) 理解。

证 首先注意，根据定义，对任意$t\in{}_MB^n=\bigcup\limits_{k=1}^{\infty}[{}_M\tau_k^n,{}_M\beta_k^n)$，必定有

$$x(t)\bar{\in}D_n\text{或者}x(t)\in D_n,\ x(\lambda_t)\bar{\in}M。\tag{6}$$

而对于$\tau\leqslant t\in{}_MA^n$，必定有

$$x(\lambda_t)\in M。\tag{7}$$

往证(4)，从而由(4)得(5)。

(一) 设 $[\lambda,\ \eta)\in\mathscr{U}[{}_MA_1^\tau(\omega)]$。因 $(\lambda,\ \eta)\subset{}_MA_1^\tau(\omega)$，而${}_MA_1^\tau(\omega)\cap\Gamma(\omega)=\phi$，故$(\lambda,\eta)\cap\Gamma(\omega)=\phi$，因此任取$t\in(\lambda,\eta)$，有$(\lambda_t,\ \eta_t)\supset(\lambda,\ \eta)$。如果$\lambda_t<\lambda$，则可任取$s\in(\lambda_t,\ \lambda)$。因$x(s,\ \omega)\in E$，故存在$n$使$x(s,\ \omega)\in D_n$。而$\lambda_s=\lambda_t$，因$t\in(\lambda,\ \eta)\subset{}_MA_1^\tau(\omega)\subset{}_MA^\tau(\omega)$，故由(6) (7)，$x(\lambda_s,\ \omega)=x(\lambda_t,\ \omega)\in M$，且$\eta_s=\eta_t\geqslant\eta$。由 (6)，$s\bar{\in}{}_MB^n(\omega)$，$s\in{}_MA^n(\omega)$。即存在$l\geqslant1$使$s\in[{}_M\beta_l^n(\omega),\ {}_M\tau_{l+1}^n(\omega))$。因 $\eta_s={}_M\tau_{l+1}^n(\omega)\geqslant\eta$， 这样 $(s,\ \eta)\subset({}_M\beta_l^n(\omega),\ {}_M\tau_{l+1}^n(\omega))\subset{}_MA_1^n(\omega)$。这与$[\lambda,\ \eta)$是${}_MA_1^\tau(\omega)$的成构区间冲突。于是$\lambda_t=\lambda$。同样可证$\eta_t=\eta$。从而$\lambda$，$\eta\in\Gamma(\omega)$。

综合上述得$[\lambda,\ \eta)\in {}_M\mathscr{U}^{\tau}(\omega)$。

（二）设$[\lambda,\ \eta)\in {}_M\mathscr{U}^{\tau}(\omega)$，即$\lambda\geqslant\tau(\omega)$，$\lambda$，$\eta\in\Gamma(\omega)$，$(\lambda,\eta)\cap\Gamma(\omega)=\phi$，$x(\lambda,\ \omega)\in M$。

任取$s\in(\lambda,\eta)$，则$\lambda=\lambda_s$，$\eta=\eta_s$，且$x(s,\ \omega)\in E$。故存在n使$x(s,\ \omega)\in D_n$，且$x(\lambda_s,\ \omega)=x(\lambda,\ \omega)\in M$。由(6)（7)得$s\in {}_MA_n$，故存在$k\geqslant1$使$s\in[{}_M\beta_k^{\tau}(\omega),\ {}_M\tau_{k+1}^{\tau}(\omega))$，从而$\eta=\eta_s={}_M\tau_{k+1}^{\tau}$。于是$(s,\ \eta)\subset({}_M\beta_k^{\tau}(\omega),\ {}_M\tau_{k+1}^{\tau}(\omega))\subset {}_MA_1^{\tau}(\omega)$。由$s$的任意性，$(\lambda,\eta)\subset {}_MA_1^{\tau}(\omega)$。

其次，如果$[\lambda',\ \eta')\supset[\lambda,\ \eta)$，且$(\lambda',\ \eta')\subset {}_MA_1^{\tau}(\omega)$，可任取$t\in(\lambda,\ \eta)$。因$\lambda$，$\eta\in\Gamma(\omega)$，故$\lambda=\lambda_t$，$\eta=\eta_t$。另一方面，因$(\lambda',\ \eta')\subset {}_MA_1^{\tau}(\omega)$且${}_MA_1^{\tau}(\omega)\cap\Gamma(\omega)=\phi$，故$\lambda_t\leqslant\lambda'$，$\eta'\leqslant\eta_t$。于是$\lambda'=\lambda$，$\eta'=\eta$。即$[\lambda',\ \eta')=[\lambda,\ \eta)$。得证$[\lambda,\ \eta)$的最大性。

综合上述，得$[\lambda,\ \eta)\in\mathscr{U}[{}_MA_1^{\tau}(\omega)]$，证毕。

注 $\mathscr{U}[{}_MA^{\tau}(\omega)]={}_M\mathscr{U}^{\tau}(\omega)$未必成立。例如，如果$X\in\mathscr{X}_1$，则${}_EA(\omega)=[0,\ \sigma(\omega))$，故$\mathscr{U}[{}_EA^{\tau}(\omega)]=[\tau(\omega),\ \sigma(\omega))$。但${}_E\mathscr{U}^{\tau}(\omega)=\{[\lambda_n,\ \lambda_{n+1})|n=1,\ 2,\ \cdots\}$，$\lambda_n$为$X(\omega)$的第$n$个飞跃点。

由于(3)，依定理8.3.1和8.3.4，

$$ {}_MX^n={}_Mg_n(X)\in\mathscr{X}_1, \tag{8}$$

$$ {}_Mg_m({}_MX^n)={}_MX^m,\quad m<n. \tag{9}$$

且${}_MX^n$的强极限存在，而且就是${}_MX\in\mathscr{X}_1$，即

$$\lim_{n\to\infty}{}_MX^n={}_MX\in\mathscr{X}_1, \tag{10}$$

其中 $${}_MX=\{x({}_M\gamma^{-1}(t)),\ t<{}_M\sigma\},\quad {}_M\sigma=L\{{}_MA\}. \tag{11}$$

${}_M\gamma$为偶序列$\{{}_M\tau^n,\ {}_M\beta^n\}$ $(n\geqslant1)$所确定的变换，即

$$ {}_M\gamma(u)=L\{{}_MA\cap[0,\ u)\},\quad u\in[0,\ \sigma). \tag{12}$$

由定理1及引理8.4.4结论(ii)，

$$ {}_M\gamma(u)=L\{C_2[{}_MA(\omega)]\cap[0,u)\}=L\{{}_MC_2(\omega)\cap[0,u)\}, \tag{13}$$

$$ {}_M\sigma=L\{C_2[{}_MA(\omega)]\}=L\{{}_MC_2(\omega)\}. \tag{14}$$

由引理8.4.4，逆变换${}_M\gamma^{-1}$将$[0,\ {}_M\sigma)$映象到$\bar{C}_2^{+}({}_MA)={}_M\bar{C}_2^{+}(\omega)$

上。这样，我们得到下面定理的前一部分。

定理2 设$X\in\mathscr{X}_s$，则(8)—(10)成立，而且$_MX$是纯自M流入的过程。如果X满足向后方程组，则X^n，$_MX$都满足向后方程组，且与X有相同的Q矩阵。

注 $_MX$的直观意义如下。将X对应于$[0,\tau)$及$_MU$区间的段全部保留，而将其余的段抛弃，并将保留的段按原次序向左平移不相交联结而得过程$_MX\in\mathscr{X}_s$。

证 因$_MX$的U区间中，除去第一个U区间$[0,\tau(\omega))$外，其余全是$_MU$区间，故$_MX$纯自M流入。引用定理4.5(iii)便得本定理中关于方程组的结论。由于X，X^n，$_MX$有相同的第一个飞跃点$\tau(\omega)$，故其Q阵相同，证毕。

系1 设$X\in\mathscr{X}_s$，并且$P\{_M\mathscr{U}^\tau=\mathscr{U}^\tau\}=1$，

则
$$\lim_{n\to\infty}{_MX^n}=X. \tag{15}$$

证 这是很显然的，因为在系1的假设下，$_MX=X$。

特别，当$M=\bar{E}$时，系1的假设恒满足，因此我们有

系2 设$X\in\mathscr{X}_s$，

则
$$\lim_{n\to\infty}X^n=X, \tag{16}$$

其中
$$X^n=g_n(X)\in\mathscr{X}_1, \tag{17}$$

$$X^m=g_m(X^n),\quad m<n. \tag{18}$$

如果X满足向后方程组，则X^n也满足向后方程组，且X与X^n有相同的Q矩阵。

系2正是定理8.7.4，即侯振挺[1]的基本结果，即第一构造定理。

§6. 过程的流入分解

定理1 设$X\in\mathscr{X}_s$，有穷集$M\subset E$。如果
$$P\{x(\tau)\in M\mid\tau<\sigma\}=1, \tag{1}$$
其中τ是第一个飞跃点。则$X\in\mathscr{X}_1$，即X是一阶过程。特别当X纯

自M流入时，(1) 成立，因而X是一阶过程。

证 因x纯自M流入时，X的一切U区间（除第一个外）都是${}^{M}U$区间，故$\Gamma' \subset \Gamma'_M$，$\bar{\Gamma}'^{+} \subset \bar{\Gamma}'^{+}_M$。

设$\tau<\sigma$。由定理3.2，$\tau \in \Gamma \subset \bar{\Gamma}'^{+}_M$。如果$\tau \in \Gamma'_M$,则$x(\tau) \in M$，否则存在严格下降的$\lambda_n \in \Gamma'_M$使$\lambda_n \downarrow \tau$。但$x(\lambda_n) \in M$，由$X$的右连续性和$M$的有穷性，必然$x(\tau)=\lim\limits_{n\to\infty} x(\lambda_n) \in M$。因此恒有$x(\tau) \in M$。得证(1)。

由 (1) 及X的强马氏性可得X的第一，第二，第三…个飞跃点τ^1，τ^2，τ^3，…，并且如果$\tau^n<\sigma$，则$x(\tau^n) \in M$。倘若对$t<\sigma(\omega)$，$X(\omega)$在$[0,t]$中有无穷多个飞跃点，则$X(\omega)$在$[0,t]$中无穷多次取M中的值。因M有穷，存在$i \in M$使$X(\omega)$在$[0,t]$中无穷多次取i值，因而$X(\omega)$在$[0,t]$中有无穷多个i区间。这与定理6.7.1矛盾。于是X在$[0,t]$($t<\sigma$) 中只有有穷多个飞跃点。所以X是一阶过程，证毕。

定理2 设$X \in \mathscr{X}_1$。对任意$i \in \bar{E}$，令

$$ {}_iX=\{x({}_i\gamma^{-1}(t)),\ t<{}_i\sigma\}. \tag{2}$$

其中${}_iC_2$按(2.8)确定，而

$$ {}_i\sigma=L\{{}_iC_2\},{}_i\gamma(u)=L\{{}_iC_2 \cap [0,u)\}. \tag{3}$$

${}_i\gamma^{-1}$是${}_i\gamma$的逆变换，它将$[0,{}_i\sigma)$映象到${}_i\bar{C}_2^+$上。那么，

(i) 对$i \in E$,${}_iX \in \mathscr{X}_1$是纯自i流入的过程。如果X满足向后方程组，则${}_iX$也满足向后方程组，并且X与${}_iX$有相同的Q矩阵。

(ii) ${}_\infty X \in \mathscr{X}_1$是纯自$\infty$流入的过程，满足向前方程组。${}_\infty X$与$X$有相同的$Q$矩阵。

证 在定理5.2中令$M=\{i\}$ ($i \in \bar{E}$)，注意定理1，并引用定理4.4和定理4.5即可。

注 定理2的直观意义见定理5.2注。

§7. ${}^{M}\mathbf{f}_n$ 变换及其强极限定理

设$X \in \mathscr{X}_1$,Γ为飞跃点集，$M \subset E$，有限集$D_n \uparrow E$。令

$$\left.\begin{array}{l}{}_M\bar{\beta}_0^n=0,\\ {}_M\tau_1^n\text{为}X\text{的第一个飞跃点},\\ {}_M\bar{\beta}_k^n=\min\{t\ {}_M\tau_k^n\leqslant t<\sigma,t\in\Gamma,x(t)\in D_n\cap M\},\\ {}_M\tau_{k+1}^n\text{为}{}_M\bar{\beta}_k^n\text{后的第一个飞跃点。}\end{array}\right\}\qquad(1)$$

易证${}_M\tau_k^n$，${}_M\bar{\beta}_k^n$是X的马时，对每个n，$\{{}_M\tau^n,\ {}_M\bar{\beta}^n\}$是X的标准偶，并且定理8.3.3的条件(a)(c)满足，因而依定理8.3.3，${}_MX^n=W_{{}_M\bar{\tau}^n,{}_M\bar{\beta}^n}(X)\in\mathscr{X}_D$。实际上${}_MX^n\in\mathscr{X}_1$是一阶过程。

定义1 按上面方式将$X\in\mathscr{X}$,变成${}_MX^n$的变换$W_{{}_M\bar{\tau}^n,{}_M\bar{\beta}^n}$，我们称为${}_Mf_n$变换。当$M=E$时，简记${}_Ef_n$为$f_n$。

这样，${}_MX^n={}_Mf(X)$。 (2)

显然，当$n\uparrow\infty$时，

$$\left.\begin{array}{l}{}_M\bar{A}^n=\bigcup\limits_{k=0}^{\infty}[{}_M\bar{\beta}_k^n,\ {}_M\tau_{k+1}^n)\uparrow{}_M\bar{A},\\ {}_M\bar{A}_1^n=\bigcup\limits_{k=0}^{\infty}({}_M\bar{\beta}_k^n,{}_M\tau_{k+1}^n)\uparrow{}_M\bar{A}_1.\end{array}\right\}\qquad(3)$$

仿照定理5.1可证得

定理1 对几乎一切$\omega\in\Omega$，

$$\mathscr{U}[{}_M\bar{A}_1^\tau(\omega)]={}_M\mathscr{U}\tau(\omega),\qquad(4)$$

$$C_2[{}_M\bar{A}^\tau(\omega)]=C_2[{}_M\bar{A}_1^\tau(\omega)]={}_MC_1^\tau(\omega).\qquad(5)$$

其中${}_M\bar{A}_1^\tau={}_M\bar{A}_1-(0,\tau)$，${}_M\bar{A}^\tau={}_M\bar{A}-[0,\tau)$，诸$\mathscr{U}$及$C$量仍按定义8.3.1，定义2.3及(2.8)理解。

由于(3)，依定理8.3.1和8.3.4，

$${}_MX^n={}_Mf_n(X)\in\mathscr{X}_1,\qquad(6)$$

$${}_Mf_m({}_MX^n)={}_MX^n,\ m<n.\qquad(7)$$

${}_MX^n$的强极限过程存在，由(4)(5)，强极限过程就是定理5.2中的过程${}_MX\in\mathscr{X}_*$。引用定理4.5(iii)就得到下面的定理。

定理2 设$X\in\mathscr{X}_*$，则(6)(7)成立，且

$$\lim_{n\to\infty}{}_MX^n={}_MX\in\mathscr{X}_*.\qquad(8)$$

如果X满足向后方程组，则${}_MX^n$和${}_MX$也满足向后方程组，并且X,

${}_MX^{n}$，${}_MX$有相同的Q矩阵。

系1 设$X\in\mathscr{X}_s$。如果$P\{{}_M\mathscr{U}^{\tau}=\mathscr{U}^{\tau}\}=1$，则

$$\lim_{n\to\infty}{}_MX^{n}=X。\tag{9}$$

系2 设$X\in\mathscr{X}_s$，如果$\{{}_E\mathscr{U}^{\tau}=\mathscr{U}^{\tau}\}=1$，即过程$X$不可能自无穷流入，则

$$\lim_{n\to\infty}X^{n}=X。\tag{10}$$

其中$X^{n}=f_n(X)\in\mathscr{X}_1$。如果$X$满足向后方程组，则$X^{n}$也满足，且与$X$有相同的$Q$矩阵。

注 当X是生灭过程时，系2正是王梓坤[3，定理5.3]中当$S=\infty$时考虑的情形。

第十章　过程的延拓

§1. 引　　言

过程的构造问题本质上就是过程的延拓问题．如果最小Q过程中断，构造一切Q过程的问题，就相当于构造最小过程的一切延拓过程，并使延拓过程与最小过程有相同的Q矩阵．当Q保守时，Doob[1]最早引进最小过程的延拓，即所谓杜勃过程．钟开莱在Chung[1]中对杜勃过程的构造作了较严密的论证．杨向群[1]以及Kunita[1]在Q保守时，分别对过程加上一些限制后，考虑了比杜勃过程更广泛的延拓．

本章中我们对Q过程或h阶瞬返过程不加限制．例如，不要求Q保守，不要求存在"中心"等等，在§2中我们甚至不要求Q矩阵有限．

我们主要考虑非黏延拓．由于D型延拓可能改变过程的Q矩阵，因此我们引进D^*型延拓，以保持Q矩阵不变．当然还可以考虑其它类型的延拓．例如，对某些最小过程，可以引进所谓V型延拓，使得延拓后的过程$X=\{x(t),t<\sigma\}$，对任意$t<\sigma(\omega)$，$X(\omega)$在$[0,t]$中只有有穷多个∪区间，而且除第一个∪区间外，其余∪区间都是∞∪区间．还可以引进D型、V型的混合延拓，本章中我们只作简单而初步的讨论．

§2. D型延拓

设$\bar{P}(t)=\{\bar{p}_{ij}(t)\}(i,j\in E,t\geqslant 0)$满足(1.2.$A$—$C$)，并且

$$\sum_j \bar{p}_{ij}(t)<1,\text{对某}i\in E. \tag{1}$$

设给定分布$\pi=\{\pi_i,i\in E\}$满足

$$0<\sum_j \pi_j\leqslant 1. \tag{2}$$

引理1 存在概率空间$(\Omega,\mathscr{F},P)$，在其上可以定义一列过程$X^n=\{x^n(t),\ t<\sigma^n\}(n\geqslant 0)$，它们具有下列性质：

(i) $X^n\in\mathscr{X}$，$(n\geqslant 0)$，且有相同的转移概率矩阵$\bar{P}(t)$。

(ii) $\{\sigma^n=0\}\cup\{\sigma^n=\infty\}\subset\{\sigma^{n+1}=0\}$[1)]，$n\geqslant 0$。

(iii) $P\{x^{n+1}(0)=j|0<\sigma^n<\infty\}=\pi_j$，

$$P\{\sigma^{n+1}=0|0<\sigma^n<\infty\}=1-\sum_j \pi_j.$$

(iv) 在条件$(0<\sigma^n<\infty)$或$\{x^{n+1}(0)=i\}$之下，$X^m(m\leqslant n)$与$X^m(m>n)$条件独立。即设$0\leqslant t_{m1}<t_{m2}<\cdots<t_{ml_m}$，$j_{m1}$，$j_{m2}$，…，$j_{ml_m}\in E$，令

$$A_m=\{x^m(t_{mk})=j_{mk},1\leqslant k\leqslant l_m\}, \tag{3}$$

则对任意$l\geqslant 1$，$n\geqslant 0$有

$$P\left\{\bigcap_{a=0}^{n+l}A_a|\Delta\right\}=P\left\{\bigcap_{a=0}^{n}A_a|\Delta\right\}P\left\{\bigcap_{a=n+1}^{n+l}A_a|\Delta\right\}. \tag{4}$$

其中$\Delta=\{0<\sigma^n<\infty\}$或者$\Delta=\{x^{n+1}(0)=i\}$。

证 利用作独立乘积空间的技巧，不难证明，存在概率空间$(\Omega,\ \mathscr{F},\ P)$，在其上可以定义一列过程$X^0=\{x^0(t),\ t<\sigma^0\}$，$X_i^n=\{x_i^n(t),\ t<\sigma_i^n\}$，$(n\geqslant 1,\ i\in E)$以及取值于$\bar{E}$的随机变量族$f^n(n\geqslant 0)$，它们具有下列性质1°—3°：

1°. X^0，X_i^n均$\in\mathscr{X}$，$(n\geqslant 1,\ i\in E)$，且转移概率都是$\bar{P}(t)$。

1) 一般说来，过程的初始分布全质量为1.但在本引理中，对$X^n(n\geqslant 1)$不作此要求，即允许$P\{0<\sigma^n\}\leqslant 1$ $(n\geqslant 1)$。

2°. $P\{x^0(0)\in E\}=P\{\sigma^0>0\}=1$,

$P\{x_i^n(0)=i\}=1$, $(n\geqslant 1,\ i\in E)$,

$P\{f^n=i\}=\pi_i$, $(i\in E)$, $P\{f^n=\infty\}=1-\sum\limits_i \pi_i$.

3°. 诸X^0, $X_i^n(n\geqslant 1,\ i\in E)$, $f^n(n\geqslant 0)$相互独立。

设$C(t)$为集合$\{t\,|\,0<t<\infty\}$的示性函数。对$\omega\in\{f^0=i,\ C(\sigma^0)=1\}$, $(i\in E)$, 令$x^1(t,\omega)=x_i^1(t,\omega)$, $t<\sigma^1(\omega)=\sigma_i^1(\omega)$; 否则令$\sigma^1(\omega)=0$。对于$\omega\in\{f^1=i, C(\sigma^1)=1\}(i\in E)$, 令$x^2(t,\omega)=x_i^2(t,\ \omega)$, $t<\sigma^2(\omega)=\sigma_i^2(\omega)$; 否则令$\sigma^2(\omega)=0$。如此继续，我们得到一列过程$X^n=\{x^n(t),\ t<\sigma^n\}(n\geqslant 0)$。

易见$X^n\in\mathscr{X}(n\geqslant 0)$且转移概率为$P(t)$，即性质(i)成立。根据$X^n$的构造，也具有性质(ii)。

因为σ^n只依赖于X^0, $X_i^m(m\leqslant n,\ i\in E)$及$f^m(m<n)$，由1°和3°,对$j\in E$,

$$P\{x^{n+1}(0)=j,\ 0<\sigma^n<\infty\}=P\{f^n=j,\ 0<\sigma^n<\infty\}$$
$$=P\{f^n=j\}P\{0<\sigma^n<\infty\}=\pi_jP\{0<\sigma^n<\infty\}.$$

此即性质(iii)。

为证性质(iv)，暂时固定n及$j\in E$. 令$\bar X^0=X_j^{n+1}$, $\bar X_i^m=X_i^{n+m+1}$, $\bar f^m=f^{n+m+1}$. 象刚才按照X^0, $X_i^m(m\geqslant 1,\ i\in E)$, $f^m(m\geqslant 0)$定义$X^m(m\geqslant 0)$一样，按照$\bar X^0$, $\bar X_i^m(m\geqslant 1,\ i\in E)$, $\bar f^m(m\geqslant 0)$, 我们可以确定$\bar X^m=\{\bar x^m(t),\ t<\bar\sigma^m\}(m\geqslant 0)$. 显然$\bar X^m(m\geqslant 0)$只依赖于$X_i^m(m>n,\ i\in E)$及$f^m(m>n)$，并且如令

$$\bar A_{n+m+1}=\{\bar x^m(t_{n+m+1,k})=j_{n+m+1,k},\ 1\leqslant k\leqslant l_{n+m+1}\},$$

$$\bar N=\bigcap_{a=n+1}^{n+l}\bar A_a,\quad M=\bigcap_{a=0}^{n}A_a,\quad N=\bigcap_{a=n+1}^{n+l}A_a,$$

则易见

$$\{x^{n+1}(0)=j\}\cap\bar N=\{x^{n+1}(0)=j\}\cap N,$$

$$P\{\bar N\}=P\{N\,|\,x^{n+1}(0)=j\}.$$

于是由1°, 3°, 性质(iii)以及对$\Delta=\{0<\sigma^n<\infty\}$有$\Delta\cap\{x^{n+1}(0)\in E\}=\Delta$，从而

$$
\begin{aligned}
\mathbf{P}\{MN\Delta\} &= \sum_j P\{M\Delta,\ x^{n+1}(0)=j,\ N\} \\
&= \sum_j P\{M\Delta,\ f^n=j,\tilde{N}\} \\
&= \sum_j P\{M\Delta\}\mathbf{P}\{f^n=j\}\mathbf{P}\{\tilde{N}\} \\
&= \sum_j \mathbf{P}\{M\Delta\}\pi_j\mathbf{P}\{N|x^{n+1}(0)=j\} \\
&= P(M\Delta)\sum_j \mathbf{P}\{x^{n+1}(0) \\
&= j|\Delta\}P\{N|x^{n+1}(0)=j\} \\
&= P\{M\Delta\}P\{N|\Delta\}.
\end{aligned}
$$

由此知性质（iv）对 $\Delta=(0<\sigma^n<\infty)$ 成立。类似可证对 $\Delta=\{x^{n+1}(0)=i\}$也成立，证毕。

定理2 设定义在同一概率空间上的过程列 $X^n=\{x^n(t),\ t<\sigma^n\}(n\geqslant 0)$具有引理1中的性质(i)—(iv)。令

$$\tau^0=0,\ \tau^{n+1}=\sum_{a=0}^{n}\sigma^a,\ \sigma=\sum_{a=0}^{\infty}\sigma^a. \tag{5}$$

对$0\leqslant t<\sigma$，令

$$x(t)=x^n(t-\tau^n),\ 如果\tau^n\leqslant t<\tau^{n+1}. \tag{6}$$

则　(i) $X=\{x(t),\ t<\sigma\}\in\mathscr{X}$.

(ii) X的转移概率$\{p_{ij}(t)\}$由下式给出：

$$p_{ij}(t)=\bar{p}_{ij}(t)+\int_0^t \pi_j(t-s)dK_i(s). \tag{7}$$

其中　$$\pi_j(t)=\sum_i \pi_i\bar{p}_{ij}(t). \tag{8}$$

而$K_i(t)$如下定义：

$$
\left.
\begin{aligned}
&L_i(t)=1-\sum_j \bar{p}_{ij}(t),\ L=\sum_i \pi_i L_i, \\
&L^0(t)=\begin{cases}0, & 如t\leqslant 0\\ 1, & 如t>0,\end{cases}\quad L^{n+1}=L^n*L, \\
&K_i=\sum_{n=0}^{\infty} L_i*L^n,\ *表示卷积。
\end{aligned}
\right\} \tag{9}
$$

(iii) X不中断的充要条件是

$$\sum_j \pi_j = 1. \tag{10}$$

证 因X^n是典范过程，故只要说明X是齐次马氏链，则X也是典范过程，因而结论 (i) 成立。

设$0\leqslant t_1<t_2<\cdots t_l<t_{l+1}$，$j_1$，$j_2$，$\cdots$，$j_{l+1}\in E$，

$$\text{令}\quad \Lambda_k=\bigcap_{a=1}^{k}\{x(t_a)=j_a\},$$

$$\Delta(m_1,\ m_2,\ \cdots,m_k)=\bigcap_{a=1}^{k}\{x^{m_a}(t_a-\tau^{m_a})=j_a,$$

$$\tau^{m_a}\leqslant t_a<\tau^{m_a+1}\}. \tag{12}$$

$$\text{显然}P\{\Lambda_{l+1}\}=\sum_{0\leqslant m_1\leqslant\cdots\leqslant m_{l+1}} P\{\Delta(m_1,m_2\cdots,m_{l+1})\}. \tag{13}$$

下面简记$\Delta(m_1,\ \cdots,\ m_k)=\Delta_k$。

设$m_l=m_{l+1}=m$。则存在$k<l$使$m_1\leqslant\cdots\leqslant m_k<m_{k+1}=\cdots m_l=m_{l+1}=m$，因而

$$P\{\Delta(m_1,\ \cdots,\ m_{l+1})\}$$

$$=\sum_i P\{\Delta_k,\tau^m\leqslant t_{k+1},x^m(0)=i,x^m(t_a-\tau^m)=j_a,$$

$$k+1\leqslant a\leqslant l+1\}$$

$$=\sum_i\int_0^{t_{k+1}} P\{x^m(t_a-\tau^m)=j_a,k+1\leqslant a\leqslant l+1\,|\,\Delta_k,\tau^m=s,$$

$$x^m(0)=i\}d_sP\{\Delta_k,\tau^m\leqslant s,x^m(0)=i\}. \tag{14}$$

由于Δ_k，τ^m只依赖于$X^n(n\leqslant m-1)$，故由引理1(iv)，上式被积表达式等于

$$P\{x^m(t_a-s)=j_a,\ k+1\leqslant a\leqslant l+1\,|\,\Delta_k,\ \tau^m=s,\ x^m(0)=i\}$$

$$=P\{x^m(t_a-s)=j_a,\ k+1\leqslant a\leqslant l+1\,|\,x^m(0)=i\}$$

$$=P\{x^m(t_a-s)=j_a,k+1\leqslant a\leqslant l\,|\,x^m(0)=i\}p_{j_lj_{l+1}}(t_{l+1}-$$

$$-t_l)$$

$$=P\{x^m(t_a-s)=j_a,\ k+1\leqslant a\leqslant l\,|\,\Delta_k,\ \tau^m=s,$$

$$x^m(0)=i\}\ \bar{p}_{j_l j_{l+1}}(t_{l+1}-t_l).$$

代入(14)并逆转刚才的运算得当$m_l=m_{l+1}=m$时。

$$P\{\Delta(m_1,\ \cdots,\ m_{l+1})\}=P\{\Delta_l\}\bar{p}_{j_l j_{l+1}}(t_{l+1}-t_l). \tag{15}$$

今设$m_l=m<m_{l+1}=r$。由引理1(iv)，

$$\begin{aligned}&P\{\Delta(m_1,\cdots,m_{l+1})\}\\&=\sum_i P\{\Delta_l,\ x^r(0)=i,\ x^r(t_{l+1}-\tau^r)=j_{l+1},t_l<\tau^r\leqslant t_{l+1}\}\\&=\sum_i\int_{t_l}^{t_{l+1}}P\{x^r(t_{l+1}-\tau^r)=j_{l+1}\mid\Delta_l,\ \tau^r=s,\\&\quad x^r(0)=i\}d_sP\{\Delta_l,\ \tau^r\leqslant s,\ x^r(0)=i\}.\end{aligned} \tag{16}$$

由引理1性质(iv)，被积表达式等于

$$P\{x^r(t_{l+1}-s)=j_{l+1}\mid x^r(0)=i\}=\bar{p}_{ij_{l+1}}(t_{l+1}-s). \tag{17}$$

又由引理1性质(ii)—(iv)，

$$\begin{aligned}&P\{\Delta_l,\ \tau^r\leqslant s,\ x^r(0)=i\}\\&=P\{\Delta_l,\ \tau^r\leqslant s,\ 0<\sigma^{r-1}<\infty,\ x^r(0)=i\}\\&=P\{\Delta_l,\ \tau^r\leqslant s,0<\sigma^{r-1}<\infty\}\pi_i\end{aligned} \tag{18}$$

如果能够证明：对$s\geqslant t_l$，

$$P\{\Delta_l,\tau^r\leqslant s,0<\sigma^{r-1}<\infty\}=P\{\Delta_l\}(L_{j_l}*L^{r-m-1})(s-t_l) \tag{19}$$

则将(15)—(19)代入(13)中得

$$\begin{aligned}P\{\Delta_{l+1}\}=&\sum_{0\leqslant m_1\leqslant\cdots\leqslant m_l}P\{\Delta_l\}\Big\{\bar{p}_{j_l j_{l+1}}(t_{l+1}-t_l)\\&+\sum_{m_{l+1}-m_l=1}^{\infty}\int_{t_l}^{t_{l+1}}\pi_{j_{l+1}}(t_{l+1}-s)d_s(L_{j_l}*L^{m_{l+1}-m_l-1})\\&\cdot(s-t_l)\Big\}\\=&\sum_{0\leqslant m_1\leqslant\cdots\leqslant m_l}P\{\Delta_l\}\Big\{\bar{p}_{j_l j_{l+1}}(t_{l+1}-t_l)\\&+\int_{t_l}^{t_{l+1}}\pi_{j_{l+1}}(t_{l+1}-s)d_sK_{j_l}(s-t_l)\Big\}\end{aligned}$$

$$=P\{\Delta_l\}p_{j_l j_{l+1}}(t_{l+1}-t_l).$$

由此说明X是以$\{p_{ij}(t)\}$为转移概率的齐次马氏过程。

往证(19)，即要证

$$P\{\Delta_l,\tau^r\leqslant s+t_l,0<\sigma^{r-1}<\infty\}$$
$$=P\{\Delta_l\}(L_{j_l}*L^{r-m-1})(s),s\geqslant 0. \tag{20}$$

设$r=m+1$。上式左方等于

$$\sum_i P\{\Delta_{l-1},\ \tau^m-\sigma^m\leqslant s+t_l,x^m(0)=i,$$
$$x^m(t_l-\tau^m)=j_l,\tau^m\leqslant t_l\}$$
$$=\sum_i\int_0^{t_l}P\{x^m(t_l-\tau^m)=j_l,\ \tau^m+\sigma^m\leqslant s+t_l|\Delta_{l-1},$$
$$\tau^m=u,\ x^m(0)=i\}\cdot d_uP\{\Delta_{l-1},$$
$$x^m(0)=i,\ \tau^m\leqslant u\}.$$

由引理1(iv)，被积表达式等于

$$P\{x^m(t_l-u)=j_l,\ \sigma^m\leqslant s+t_l-u|x^m(0)=i\}$$
$$=P\{x^m(t_l-u)=j_l|x^m(0)=i\}L_{j_l}(s).$$

代回原来的式子并逆转刚才的计算得(20)左方等于$P\{\Delta_l\}L_{j_l}(s)$，即(20)对$r=m+1$正确。

用归纳法证(20)。设$r>m+1$。(20)左方等于

$$\sum_i P\{\Delta_l,\tau^{r-1}+\sigma^{r-1}\leqslant s+t_l,x^{r-1}(0)=i\}$$
$$=\sum_i\int_{t_l}^{s+t_l}P\{\tau^{r-1}+\sigma^{r-1}\leqslant s+t_l|\Delta_l,\tau^{r-1}=u,$$
$$x^{r-1}(0)=i\}\cdot d_uP\{\Delta_l,\tau^{r-1}\leqslant u,x^{r-1}(0)=i\}$$
$$=\sum_i\int_{t_l}^{s+t_l}P\{\sigma^{r-1}\leqslant s+t_l-u|x^{r-1}(0)$$
$$=i\}d_uP\{\Delta_l,\ \tau^{r-1}\leqslant u,\ x^{r-1}(0)=i\}.$$

但是 $\quad P\{\sigma^{r-1}\leqslant s+t_l-u|x^{r-1}(0)=i\}=L_i(s+t_l-u)$,

又由引理1(ii)—(iv)及归纳法假设，对$u\geqslant t_l$有

$$P\{\Delta_l,\ \tau^{r-1}\leqslant u,\ x^{r-1}(0)=i\}$$

$$=P\{\Delta_l,\tau^{r-1}\leqslant u,0<\sigma^{r-2}<\infty,x^{r-1}(0)=i\}$$
$$=P\{\Delta_l,\tau^{r-1}\leqslant u,0<\sigma^{r-2}<\infty\}\pi_i$$
$$=P\{\Delta_l\}(L_{jl}*L^{r-1-m-1})(u-t_l)\pi_i.$$

因此(20)左方等于

$$\sum_i\int_{t_l}^{s+t_l}\pi_iL_i(s+t_l-u)P\{\Delta_l\}d_u(L_{\cdot l}*L^{r-1-m-1})(u-t_l)$$

$$=P\{\Delta_l\}\int_0^s L(s-v)d_v(L_{\cdot l}*L^{r-1-m-1})(v)$$

$$=P\{\Delta_l\}(L_{\cdot l}*L^{r-m-1})(s).$$

得证(20)。

在(7)中对$j\in E$求和得

$$\sum_j p_{ij}(t)=1-L_i(t)+\int_0^t\sum_j\pi_j\{1-L_j(t-s)\}dK_i(s)$$

$$=1-L_i(t)+\int_0^t\Big\{\sum_j\pi_j-L(t-s)\Big\}dK_i(s)$$

$$=1-L_i(t)+\Big(\sum_j\pi_j\Big)K_i(t)-\sum_{n=0}^{\infty}(L_i*L^{n+1})(t)$$

$$=1-K_i(t)\Big(1-\sum_j\pi_j\Big).$$

由于(1)，至少有一个i使$K_i\neq 0$，故X不中断的充要条件是(10)成立，证毕。

注1 由于定理2中的过程$X=\{x(t,\ t<\sigma\}$的前面一部分是中断过程$X^0=\{x^0(t),\ t<\sigma^0\}$，而且满足：

$$P\{x(\tau^1)=j|\tau^1<\infty\}=\pi_j. \tag{21}$$

而且它们的转移概率有关系(7)。因此，我们称X为中断过程X^0的$\pi=\{\pi_j,\ j\in E\}D$型延拓过程。

注2 在(7)中取拉氏变换得

$$\psi_{ij}(\lambda)=\bar{\psi}_i(\lambda)+\xi_i(\lambda)\frac{\sum_k\pi_k\bar{\psi}_{kj}(\lambda)}{1-\sum_k\pi_k\xi_k(\lambda)}. \tag{22}$$

其中$\psi(\lambda)$和$\bar{\psi}(\lambda)$分别为$P(t)$和$\bar{P}(t)$的拉氏变换，而

$$\bar{\xi}_i(\lambda)=1-\lambda\sum_j\bar{\psi}_{ij}(\lambda)。\tag{23}$$

§3. D*型延拓

在上节中不要求Q矩阵有限，本节假定Q矩阵有限。

在D型延拓中，如果X^0的Q矩阵保守，从过程的样本轨道知，X^0与其D型延拓过程X有相同的Q矩阵。但如果X^0的Q矩阵非保守，则X与X^0的Q矩阵未必相同。为了保持延拓过程的Q矩阵不变，我们引进D^*型延拓。

定义1 设$X=\{x(t),t<\sigma\}\in\mathscr{X}_s$。称$\sigma(\omega)$为$X(\omega)$的$T$尾，如果$0<\sigma(\omega)<\infty$，且$\sigma(\omega)$是跳跃点(见定义6.8.1)，否则称$\sigma(\omega)$为$P$尾。

显然，如果过程$X^0=\{x^0(t),t<\sigma^0\}\in\mathscr{X}_s$的转移概率为$\bar{P}(t)$，则下列诸量

$$\left.\begin{aligned}&M_{ij}(t)=P_i\{x^0(t)=j,\ \sigma^0\text{为}P\text{尾}\},\\&N_{ij}(t)=P_i\{x^0(t)=j,\sigma^0\text{为}T\text{尾}\},\\&M_i(t)=\sum_j M_{ij}(t),\quad N_i(t)=\sum_j N_{ij}(t),\\&R_i(t)=M_i(0)-M_i(t)=P_i\{\sigma^0\leqslant t,\ \sigma^0\text{为}P\text{尾}\}.\end{aligned}\right\}\tag{1}$$

由$\bar{P}(t)$唯一决定。

仿引理2.1，我们有

引理1 设$\bar{P}(t)$满足(1.2.A—C)，对某个$i\in E$及$t>0$，有$R_i(t)>0$。分布π满足(2.2)。则存在概率空间$(\Omega,\mathscr{F},P)$及定义于其上的过程列$X^n=\{x^n(t),\ t<\sigma^n\}(n\geqslant0)$具有下列性质：

(i*) $X^n\in\mathscr{X}_s(n\geqslant0)$，且有相同的转移概率$\bar{P}(t)$。

(ii*) $\{\sigma^n=0\}\cup\{\sigma^n=\infty\}\cup\{\sigma^n\text{为}X^n\text{的}T\text{尾}\}\subset\{\sigma^{n+1}=0\}$。$(n\geqslant0)$

(iii*) $P\{x^{n+1}(0)=j\,|\Delta\}=\pi_j$，

$$P\{\sigma^{n+1}=0\mid\Delta\}=1-\sum_j \pi_j.$$

其中$\Delta=(0<\sigma^n<\infty, \sigma^n$是$X^n$的$P$尾)。

(iv*) 在条件($0<\sigma^n<\infty$, σ^n 是X^n的P尾)或$\{x^{n+1}(0)=i\}$之下，$X^m(m\leqslant n)$与$X^m(m>n)$条件独立。

定理2 设定义在同一概率空间上的过程列 $X^n=\{x^n(t), t<\sigma^n\}(n\geqslant 0)$具有引理1中的性质(i*)—(iv*)。按(2.5) (2.6)定义$X=\{x(t), t<\sigma\}$。

则 (i) $X\in\mathscr{H}_s$,且与$X^n(n\geqslant 0)$有相同的Q矩阵。

(ii) X的转移概率$P^*(t)=\{p_{ij}^*(t)\}$由下式给出:

$$p_{ij}^*(t)=\bar{p}_{ij}(t)+\int_0^t \pi_j(t-s)d_sK_i^*(t). \tag{2}$$

其中$\pi_j(t)$由(2.8)确定。K_i^*按(2.9)确定，但需将其中的$L_i(t)$换成(1)中的$R_i(t)$。

(iii) X不中断的充要条件是Q保守并且(2.10)成立。

证 仿定理2.1证明，但最后一结论需重新证明。由(2)得

$$\sum_j p_{ij}^*(t)=1-\{N_i(0)-N_i(t)\}-\left\{1-\sum_k \pi_k + \sum_k \pi_k N_k(0)\right\}K_i^*(t)+\int_0^t \sum_k \pi_k N_k(t-s)dK_i^*(s). \tag{3}$$

如果Q保守，显然$N_i(t)=0$，如果(2.10)成立，则由(3)得$\sum_j p_{ij}^*(t)=1$,即X不中断。反之。如果X不中断，则必定$N_i(t)=0$。因为不然的话，对某i有$N_i(t)=P_i(t<\sigma^0, \sigma^0$为$X^0$的$T$尾$\}>0$。依引理1(ii*)，在正概率集($t<\sigma^0, \sigma^0$为$X^0$的$T$尾)上，$\sigma=\sigma^0<\infty$，这与$X$不中断冲突。由$N_i(t)=0$及(3)得(2.10)$=0$成立。

Q必定保守，因为如果对某i有 $d_i=q_i-\sum_{j\neq i}q_{ij}>0$，则有

$$P_i\{0<\sigma^0<\infty, \ \sigma^0\text{是}X^0\text{的}T\text{尾}\}\geqslant\frac{d_i}{q_i}>0.$$

从而有$N_i(t)>0$，此不可能，证毕。

注1 在(2)中取拉氏变换得

$$\psi_{ij}{}^*(\lambda)=\overline{\psi}_{ij}(\lambda)+\xi_i(\lambda)\frac{\sum_k \pi_k\overline{\psi}_{kj}(\lambda)}{1-\sum_k \pi_k\xi_k(\lambda)}. \tag{4}$$

其中$\psi^*(\lambda)$和$\overline{\psi}(\lambda)$分别是$P^*(t)$和$\overline{P}(t)$的拉氏变换，而$\xi_i(\lambda)$是(1)中的$R_i(t)$的拉氏变换。

注2 我们称定理2中的过程$X\in\mathscr{H}_s$是过程$X^0\in\mathscr{H}_s$的πD^*型延拓过程。过程X^0和它的D^*型延拓过程有相同的Q矩阵。如果X^0满足向后方程组，则X也满足。

注3 当Q保守时，因没有T尾，故D^*型延拓与D型延拓一致。

§4. 杜勃过程

在D^*型延拓中，如果X^0或$\{\overline{p}_{ij}(t)\}$是熟知的最小Q过程。用Q的马亭流出边界B_e表示，(3.1)中的$R_i(t)$成为

$$L_i(t)=P_i\{\sigma^0\leqslant t,\ x(\sigma^0-0)\in B_e\}. \tag{1}$$

定理3.2便成为

定理1 设$\{f_{ij}(t)\}$为最小Q过程，对某个i及$t>0$，按(1)确定的$L_i(t)>0$。分布π满足(2.2)。设$X^n\in\mathscr{H}_s(Q)(n\geqslant 0)$为定义在同一概率空间上的最小$Q$过程，具有引理3.1中的性质(ii*)—(iv*)。则按(2.5)(2.6)定义的$X\in\mathscr{H}_1(Q)$，其转移概率$P(t)$由下式给出：

$$p_{ij}(t)=f_{ij}(t)+\int_0^t \pi_j(t-s)dK_i(s). \tag{2}$$

其中$\pi_j(t)=\sum_i \pi_i f_{ij}(t)$，而$K_i(t)$按(2.9)确定，但其中的$L_i(t)$应按(1)理解。$X$不中断的充要条件是$Q$保守并且(2.10)成立。

定义1 最小Q过程的πD^*型延拓过程，称为(Q,π)杜勃过程。

对(Q,π)杜勃过程$X=\{x(t),t<\sigma\}$,

$$P\{x(\tau)=i\,|x(\tau-0)\in B_e\}=\pi_i,i\in E. \tag{3}$$

$$P\{\sigma=\tau\ x(\tau-0)\in B_e\}=1-\sum_i\pi_i. \tag{4}$$

其中τ为第一个飞跃点. 而且(2)的拉氏变换为

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+\xi_i(\lambda)\frac{\sum_k\pi_k\phi_{kj}(\lambda)}{1-\sum_k\pi_k\xi_k(\lambda)}$$

$$=\phi_{ij}(\lambda)+\xi_i(\lambda)\frac{\sum_k\pi_k\phi_{kj}(\lambda)}{\left(1-\sum_k\pi_k\right)+\sum_k\pi_k(1-\xi_k(\lambda))}. \tag{5}$$

其中$\xi_i(\lambda)$为(1)中的$L_i(t)$的拉氏变换. 特别地, Q保守时,

$$\xi_i(\lambda)=1-\lambda\sum_i\phi_{ij}(\lambda). \tag{6}$$

§5. 广义D型延拓

D型延拓和D^*型延拓不考虑过程的边界. 本节我们将考虑依赖于过程的边界的延拓.

根据有限的 Q 矩阵Q, 按第六章的讨论, 我们可以确定Q的本质马亭边界B,流出边界B_e和消极边界B_p. H为非保守状态集.

假定最小Q过程$X=\{x(t),\ t<\tau\}$中断, 即对某i有$P_i\{\tau<\infty\}>0$, 或等价地$\mu\{H\cup B_e\}>0$, 其中测度μ按(6.12.2)确定.

显然, 对Borel集$\Gamma\subset H\cup B_e$, 下面诸量

$$L_i(\Gamma,\ t)=P_i\{x(\tau-0)\in\Gamma,\ \tau\leqslant t\},t\geqslant0. \tag{1}$$

$$h_i(\Gamma,\lambda)=E_i\{e^{-\lambda\tau},\ x(\tau-0)\in\Gamma\}$$

$$=\int_0^\infty e^{-\lambda t}d_tL_i(\Gamma,t),\ \lambda>0. \tag{2}$$

由Q唯一决定

设给一族分布$\Pi(a,\cdot)(a\in H\cup B_e)$,满足

$$\Pi(a, E)\leqslant 1, \quad \int_{H\cup B_e} \Pi(a,E)\mu(da) > 0. \tag{3}$$

引理1 存在概率空间$(\Omega, \mathscr{F}, P)$及定义在其上的最小Q过程$X^n=\{x^n(t), t<\sigma^n\}\in \mathscr{H}_s(Q)(n\geqslant 0)$，它们具有下列性质：

(i) $\{\sigma^n=0\}\cup\{\sigma^n=\infty\}\subset\{\sigma^{n+1}=0\}$。

(ii) 在集合$\{x^n(\sigma^n-0)\in H\cup B_e\}$上几乎有

$P\{x^{n+1}(0)=j\,|x^n(\sigma^n-0)\}=\Pi(x^n(\sigma^n-0),j)$, $j\in E$。

$P\{\sigma^{n+1}=0\,|x^n(\sigma^n-0)\}=1-\Pi(x^n(\sigma^n-0),E)$。

(iii) 对由(2.3)确定的Λ_m，

$$P\left\{\bigcap_{a=0}^{n+l}\Lambda_a\middle|x^{n+1}(0)=i\right\}=P\left\{\bigcap_{a=0}^{n}\Lambda_a\middle|x^{n+1}(0)=i\right\}$$

$$\cdot P\left\{\bigcap_{a=n+1}^{n+l}\Lambda_a\middle|x^{n+1}(0)=i\right\}, (i\in E, l\geqslant 1).$$

(iv) 在集合$\{x^n(\sigma^n-0)\in H\cup B_e\}$上几乎有

$$P\left\{\bigcap_{a=0}^{n+l}\Lambda_a\middle|x^n(\sigma^n-0)\right\}=P\left\{\bigcap_{a=0}^{n}\Lambda_a\middle|x^n(\sigma^n-0)\right\}$$

$$\cdot P\left\{\bigcap_{a=n+1}^{n+l}\Lambda_a\middle|x^n(\sigma^n-0)\right\},$$

其中Λ_a仍如(2.3)。

证 作独立乘积空间技巧，不难证明：存在概率空间$(\Omega,\mathscr{F},P)$，在其上可以定义一列最小过程$X^0=\{x^0(t), t<\sigma^0\}\in\mathscr{H}_s$，$X_i^n=\{x_i^n(t), t<\sigma_i^n\}\in\mathscr{H}_s(n\geqslant 1, i\in E)$以及取值于$\bar{E}$的随机变量$f^n(a)(n\geqslant 0, a\in H\cup B_e)$，它们具有下列性质：

1° $P\{x^0(0)\in E\}=P\{x_i^n(0)=i\}=1$。

2° $P\{f^n(a)=i\}=\Pi(a,i)(i\in E)$；

$P\{f^n(a)=\infty\}=1-\Pi(a,E)$。

3° 诸$X^0, X_i^n(n\geqslant 1, i\in E)$，$f^n(a)(n\geqslant 0, a\in H\cup B_e)$相互独立。

对 $\omega\in\{x^0(\sigma^0-0)\in H\cup B_e,\ f^0[x^0(\sigma^0-0)]=i\}$ $(i\in E)$，令 $x^1(t,\omega)=x_i^1(t,\omega)$，$t<\sigma^1(\omega)=\sigma_i^1(\omega)$；否则令 $\sigma^1(\omega)=0$。易见 $X^1=\{x^1(t),\ t<\sigma^1\}\in\mathscr{H}_s$ 是最小 Q 过程。故对 $\sigma^1>0$，可确定 $x^1(\sigma^1-0)$。对 $\omega\in\{x^1(\sigma^1-0)\in H\cup B_e, f^1[x^1(\sigma^1-0)]=i\}(i\in E)$，令 $x^2(t,\omega)=x_i^2(t,\omega)$，$t<\sigma^2(\omega)=\sigma_i^2(\omega)$；否则令 $\sigma^2(\omega)=0$。如此继续，我们得到一列过程 $X^n(n\geqslant 0)$。

仿照引理2.1的证明，可以证明现在的 $X^n\in\mathscr{H}_s(n\geqslant 0)$ 是最小 Q 过程且具有本引理的性质(i)——(iv)，证毕。

定理2 设定义在同一概率空间上的最小 Q 过程列 $X^n=\{x^n(t)$ $t<\sigma^n\}\in\mathscr{H}_s(n\geqslant 0)$ 具有引理 1 中的性质(i)—(iv)。按(2.5)(2.6)定义 $X=\{x(t),\ t<\sigma\}$。则

(i) 如果 $\int_{B_e}\Pi(a,E)\mu(da)=0$，则 $X\in\mathscr{H}_s$ 是最小过程，其 Q 矩阵与 X^0 的 Q 矩阵不同。如果 $\int_{B_e}\Pi(a,E)\mu(da)>0$，则 $X\in\mathscr{H}'_1$ 是一阶过程。

(ii) X 的转移概率 $\{P_{ij}(t)\}$ 由下式给出：

$$P_{ij}(t)=f_{ij}(t)+\sum_k\int_0^t f_{kj}(t-s)\,dK_{ik}(s). \tag{4}$$

而 $K_{ik}(t)$ 按下式确定：

$$\left.\begin{aligned}
T_{ij}^1(t)&=\int_{H\cup B_e}\Pi(a,j)L_i(da,t),\\
T_{ij}^{n+1}(t)&=\int_{H\cup B_e}\Pi(a,\ j)\sum_k[T_{ik}^n(\cdot)\cdot L_k(da,\cdot)](t),\\
K_{ij}(t)&=\sum_{n=1}^{\infty}T_{ij}^n(t).
\end{aligned}\right\}\tag{5}$$

$\cdot$ 表示卷积，即

$$[T_{ik}^{n}(\cdot)*L_k(\Gamma,\cdot)](t)=\int_0^t T_{ik}^{n}(t-s)L_k(\Gamma,ds). \tag{6}$$

将(4)变形后得

$$P_{ij}(t)=f_{ij}(t)+\int_{\Pi\cup B_e}\int_0^t \pi_j(a,t-s)K_i(da,ds). \tag{7}$$

其中$\pi_j(a,t)=\sum_k \Pi(a,k)f_{kj}(t)$. (8)

$$L_i^1(\Gamma,t)=L_i(\Gamma,t)[\text{见(1)式}]. \tag{9}$$

$$L_i^{n+1}(\Gamma,t)=\int_0^t\int_{\Pi\cup B_e} L_i^n(da,ds)\sum_k \Pi(a,k)L_k(\Gamma,t-s). \tag{10}$$

$$K_i(\Gamma,s)=\sum_{n=1}^{\infty} L_i^n(\Gamma,s). \tag{11}$$

虽然定理2的证明可以仿定理2.2进行，只不过书写更复杂罢了。但因§5§7中定理的证明与定理2更类似，然而我们却把证明省略了。因此，我们在这里仍给出定理2的简单证明线索。

证　只需证明X是以(4)为转移概率的齐次马氏过程即可，其余明显。

按(2.11)(2.12)确定A_l和$\Delta(m_1,\cdots m_h)=\Delta_h$，(2.13)仍成立。仿照(2.15)的证明，在现在的情形下，当$m_l=m_{l+1}$时，(2.15)仍然成立。

设$m_l<m_{l+1}$。简记$m_l=m, m_{l+1}=r$。则由引理1(iii)，

$$P\{\Delta_{l+1}\}=\sum_k P\{\Delta_l,t_l<\tau^r\leqslant t_{l+1}<\tau^r+\sigma^r,x^r(0)=k,$$
$$x^r(t_{l+1}-\tau^r)=j_{l+1}\}$$
$$=\sum_k\int_{t_l}^{t_{l+1}}P\{x^r(t_{l+1}-\tau^r)=j_{l+1}|\Delta_l,\tau^r=s,$$
$$x^r(0)=k\}d_sP\{\Delta_l,\tau^r\leqslant s,x^r(0)=k\}$$

$$= \sum_k \int_{t_l}^{t_{l+1}} f_{kj_{l+1}}(t_{l+1}-s)d_sP\{\Delta_l,\tau^r\leqslant s,x^r(0)=k\}. \tag{12}$$

如果能够证明，对$s\geqslant t_l$有

$$P\{\Delta_l,\ \tau^r\leqslant s,x^r(0)=k\}=T^{r-m}_{j_lk}(s-t_l). \tag{13}$$

则代回（12）中得

$$\begin{aligned}P\{\Delta_{l+1}\} &= \sum_k P\{\Delta_l\}\int_{t_l}^{t_{l+1}} f_{kj_{l+1}}(t_{l+1}-s)d_sT^{m_{l+1}-m_l}_{j_lk}(s-t_l)\\ &= P\{\Delta_l\}\sum_k\int_0^{t_{l+1}-t_l} f_{kj_{l+1}}(t_{l+1}-t_l-s)\,d_sT^{m_{l+1}-m_l}_{j_lk}(s).\end{aligned} \tag{14}$$

将(2.15)（14）代入(2.13)便得$P\{\Delta_{l+1}\}=P\{\Delta_l\}P_{j_lj_{l+1}}(t_{l+1}-t_l)$，这里$p_{ij}(t)$由(4)确定。因而X是齐次马氏过程。

为证(13)，即要证

$$P\{\Delta_l,\tau^r\leqslant s+t_l,\ x^r(0)=k\}=P\{\Delta_l\}T^{r-m}_{j_lk}(s),s\geqslant 0. \tag{15}$$

由引理1(iv)，上式左方等于

$$\begin{aligned}&\int_{H\cup B_c} P\{\Delta_l,x^{r-1}(\sigma^{r-1}-0)\in da,x^r(0)=k,\tau^r\leqslant s+t_l\}\\ &=\int_{H\cup B_c}\Pi(a,k)P\{\Delta_l,\ x^{r-1}(\sigma^{r-1}-0)\in da,\ \tau^r\leqslant s+t_l\}.\end{aligned} \tag{16}$$

先考虑$r=m+1$。则

$$\begin{aligned}&P\{\Delta_l,x^{r-1}(\sigma^{r-1}-0)\in da,\tau^r\leqslant s+t_l\}\\ &=\sum_i P\{\Delta_{l-1},x^m(0)=i,x^m(t_l-\tau^m)=j_l,\tau^m\leqslant t_l<\tau^m+\sigma^m,\\ &\qquad x^m(\sigma^m-0)\in da,\tau^m+\sigma^m\leqslant s+t_l\}.\end{aligned} \tag{17}$$

由引理1（iii），上式被加项等于

$$\int_0^{t_l} P\{x^m(t_l-u)=j_l,\ x^m(\sigma^m-0)\in da,$$

$$\sigma^m \leqslant s+t_l-u \mid x^m(0)=i\}\cdot d_u P\{\Delta_{l-1}, x^m(0)=i, \tau^m \leqslant u\}$$

$$=\int_0^{t_l} f_{ij_l}(t_l-u)\, L_{j_l}(da,s) d_u P\{\Delta_{l-1}, x^m(0)=i, \tau^m \leqslant u\}.$$

代回（17）中并逆转刚才的计算得(17)左方等于 $P\{\Delta_l\}L_{j_l}(da, s)$，从而代入（16）中知（13）对 $r=m+1$成立。类似的考虑用归纳法可得（13）对一切$r>m$成立，证毕。

记$G^n_{ij}(\lambda)$为$T^n_{ij}(t)$的拉氏变换，$h_i(\Gamma,\lambda)$如(2)。则由（5），

$$G^n_{ij}(\lambda)=\sum_k G^n_{ik}(\lambda) G^1_{ik}(\lambda).$$

于是从（4）得

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+\sum_k\Big(\sum_{n=1}^{\infty} G^n_{ik}(\lambda)\Big)\phi_{kj}(\lambda). \tag{18}$$

当Q保守时，上式即[1，公式(10.2.25)]。用矩阵符号，记$G(\lambda)=G^1(\lambda)$，则（18）成为

$$\psi(\lambda)=\phi(\lambda)+\sum_{n=1}^{\infty} G^n(\lambda)\phi(\lambda). \tag{19}$$

令 $$h^n_i(\Gamma, \lambda)=\int_0^{\infty} e^{-\lambda t} d_t L^n_i(\Gamma, t). \tag{20}$$

则从(9)(10)有

$$\left.\begin{aligned} & h^1_i(\Gamma,\lambda)=h_i(\Gamma,\lambda),\\ & h^{n+1}_i(\Gamma, \lambda)=\int_{\Pi \cup B_d} h^n_i(da, \lambda) \sum_k \Pi(a,k) h_k(\Gamma,\lambda). \end{aligned}\right\} \tag{21}$$

于是(7)成为

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+\int_{\Pi \cup B_d}\Big\{\sum_{n=1}^{\infty} h^n_i(da,\lambda)\Big\}\sum_k \Pi(a,k)\phi_{kj}(\lambda). \tag{22}$$

如果令

$$\left.\begin{aligned}&{}_0V(a,\ \Gamma,\lambda)=\begin{cases}0, a\bar{\in}\Gamma.\\1, a\in\Gamma.\end{cases}\\&V^n(a,\Gamma,\lambda)=\sum_k\Pi(a,k)h_k^n(\Gamma,\lambda).\end{aligned}\right\}\tag{23}$$

则有$V^{n+1}(a,\Gamma,\lambda)=\int\limits_{H\cup B_e}V^n(a,db,\lambda)V^1(b,\Gamma,\lambda)$, (24)

$$h_i^{n+1}(\Gamma,\ \lambda)=\int\limits_{H\cup B_e}h_i(da,\ \lambda)V^n(a,\Gamma,\lambda).\tag{25}$$

于是(22)成为

$$\psi_{ij}(\lambda)=\ \phi_{ij}(\lambda)+\int\limits_{H\cup B_e}h_i(da,\lambda)\int\limits_{H\cup B_e}\Big(\sum_{n=0}^{\infty}V^n(a,db,\lambda)\Big)$$

$$\times\Big(\sum_k\Pi(b,k)\phi_{kj}(\lambda)\Big).\tag{26}$$

当Q保守时，这正是Kunita[1,公式(10)]。

定义1 称定理2中的X为X^0的$\{\Pi(a,\cdot),\ a\in H\cup B_e\}$广义$D$型延拓过程。

显然，对定义1中的X^0和X，在集合$\{x(\sigma^0-0)\in H\cup B_e\}$上几乎有

$$P\{x(\sigma^0)=j|\ x(\sigma^0-0)\}=\Pi(x(\sigma^0-0),j),j\in E.$$

§6. 广义D*型延拓

当Q保守时，广义D型延拓保持Q矩阵不变。但Q非保守时未必。因此，我们引进广义D^*型延拓。

假定最小Q过程的流出边界B_e非空，即$\mu\{B_e\}>0$。给定一族分布$\Pi(a,\cdot)(a\in B_e)$满足$\Pi(a,E)\leqslant1$，$\int\limits_{B_e}\Pi(a,E)\mu(da)>0$。

$$\text{令}\quad \Pi(a,E)=0,a\in H.\tag{1}$$

于是§5的基本条件满足，因而引理5.1和定理5.2仍然有效。

定义1 设X^0为最小Q过程，称X^0的满足（1）的$\{\Pi(a,\cdot), a\in H\cup B_e\}$广义D型延拓过程X为$\{\Pi(a,\cdot), a\in B_e\}$广义$D^*$型延拓过程。

显然，当Q保守时，广义D型延拓与广义D^*型延拓一致。下面的定理是明显的。

定理1 最小Q过程X^c与其$\{\Pi(a,\ \cdot), a\in B_e\}$广义$D^*$型延拓过程X有相同的Q矩阵。更精确些，X满足向后方程组。而且在$\{x(\tau-0)\in B_e\}$上几乎有

$$P\{x(\tau)=j\,|\,x(\tau-0)\}=\Pi\{x(\tau-0),j\}. \tag{2}$$

§7. 瞬返过程的延拓

本节简要叙述瞬返过程的延拓，因证明与§5、§6相仿，故从略。

设 $X=\{x(t),\ t<\sigma\}\in\mathscr{X}_k$ 为k阶瞬返过程，其转移概率为${}_kP_{ij}(t)$，i阶流出边界为${}_iB_e$。代替 §5 中的$H\cup B_e$，我们要考虑${}_kH\cup{}_kB_e$，这里

$${}_kH=H+{}_0B_e+{}_1B_e+\cdots\cdots+{}_{k-1}B_e. \tag{1}$$

在${}_kH\cup{}_kB_e$上的测度为${}_k\mu$。假定X中断，即${}_k\mu\{{}_kH\cup{}_kB_e\}>0$。任给分布族${}_k\Pi(a,\ \cdot)(a\in{}_kH\cup{}_kB_e)$满足

$${}_k\Pi(a,\ E)\leqslant 1,\quad \int_{{}_kH\cup{}_kB_e}{}_k\Pi(a,\ E)\,{}_k\mu(da)>0. \tag{2}$$

引理1 存在概率空间$(\Omega,\mathscr{F},P)$，在其上可以定义一列过程$X^n=\{x^n(t),\ t<\sigma^n\}\in\mathscr{X}_k(n\geqslant 0)$，具有下列性质：

(i) $X^n(n\geqslant 0)$有相同的转移概率${}_kp_{ij}(t)$。

(ii) $(\sigma^n=0)\cup(\sigma^n=\infty)\subset(\sigma^{n+1}=0)$。

(iii) 在集合$\{x^n(\sigma^n-0)\in{}_kH\cup{}_kB_e\}$上几乎有

$$P\{x^{n+1}(0)=j\,|\,x^n(\sigma^n-0)\}={}_k\Pi(x^n(\sigma^n-0),\ j),$$

$$P\{\sigma^{n+1}=0|x(\sigma^n-0)\}=1-{}_k\Pi(x^n(\sigma^n-0),\ E).$$

(iv) 同引理5.1(iii)。

(v) 将引理5.1(iv)中的$H\cup B_e$修改成${}_kH\cup{}_kB_e$而得。

定理2 设定义在同一概率空间上的过程列$X^n\in\mathscr{X}_k(n\geqslant 0)$具有引理1中的性质(i)—(v)。按(2.5)(2.6)定义$X=\{x(t),\ t<\sigma\}$。则

(i) 如果$\int_{{}_kB_e}{}_k\Pi(a,\ E)_k\mu(da)=0$，则$X\in\mathscr{X}_k$，但$X$与$X^0$的$k$阶流出边界不相同。如果上面积分为正，则$X\in\mathscr{X}_{k+1}$。

(ii) X的转移概率${}_{k+1}P_{ij}(t)$由下式给出：

$$_{k+1}p_{ij}(t)={}_kp_{ij}(t)+\sum_l\int_0^t{}_kp_{lj}(t-s)d_s{}_kK_{il}(s).\tag{3}$$

或者
$$_{k+1}p_{ij}(t)={}_kp_{ij}(t)+\int_{{}_kH\cup{}_kB_e}\int_0^t{}_k\pi_j(a,t-s)_kK_i(da,ds).\tag{4}$$

其中
$$_k\pi_j(a,\ t)=\sum_i{}_k\Pi(a,\ i)_kp_{ij}(t).\tag{5}$$

而${}_kK_{ij}(t)$，${}_kK_i(\Gamma,t)$仍按(5.5)(5.6)(5.9)－(5.11)确定，但需作如下修改：$L_i(\Gamma,\ t)$，$\Pi(a,\ \cdot)$，H，B_e，$T_{ij}^n(t)$，$K_{ij}(t)$，$L_i^n(\Gamma,\ t)$，$K_i(\Gamma,\ t)$前面左下角应冠以k。而

$$_kL_i(\Gamma,\ t)=P_i\{x^0(\sigma^0-0)\in\Gamma,\ \sigma^0\leqslant t\},\tag{6}$$
$\Gamma\subset{}_kH\cup{}_kB_e$。

相应地，在(5.18)—(5.26)中作显然的左下角添加后，我们有

$$_{k+1}\psi(\lambda)={}_k\psi(\lambda)+\sum_{n=1}^\infty({}_kG(\lambda))^n{}_k\psi(\lambda).\tag{7}$$

$$_{k+1}\psi(\lambda)={}_k\psi(\lambda)+\int_{{}_kH\cup{}_kB_e}\left\{\sum_{n=1}^\infty{}_kh_i^n(da,\ \lambda)\right\}\sum_l{}_k\Pi(a,l)_k\psi_{lj}(\lambda).\tag{8}$$

$$
{}_{k+1}\psi_{ij}(\lambda) = {}_k\psi_{ij}(\lambda) + \int_{{}_kH \cup {}_kB_e} {}_kh_i(da,\ \lambda)
$$

$$
\times \int_{{}_kH \cup {}_kB_e} \Big(\sum_{n=0}^{\infty} {}_kV^n(a,\ db,\ \lambda)\Big)\Big(\sum_l {}_k\Pi(b,\ l)_k\psi_{lj}(\lambda)\Big). \tag{9}
$$

定义1 称定理1中的过程X为X^0的$\{{}_k\Pi(a,\ \cdot),\ a \in {}_kH \cup {}_kB_e\}$广义$D$型延拓过程。

对定义1中的X和X^0，在集合$\{x(\sigma^0-0) \in {}_kH \cup {}_kB_e\}$上几乎有

$$
P\{x(\sigma^0)=j \mid x(\sigma^0-0)\} = {}_k\Pi(x(\sigma^0-0),\ j). \tag{10}
$$

注意，X与X^0未必有相同的k阶流出边界，当然更谈不上有相同的Q了。因此有必要引进广义D^*型延拓过程。

假定k阶瞬返过程$X \in \mathscr{X}_k$满足${}_k\mu\{{}_kB_e\} > 0$。给定分布族${}_k\Pi(a,\ \cdot)(a \in {}_kB_e)$满足${}_k\Pi(a,\ E) \leqslant 1$，$\int_{{}_kB_e} {}_k\Pi(a,\ E)_k\mu(da) > 0$。

令：

$$
{}_k\Pi(a,\ E)=0,\ a \in {}_kH. \tag{11}
$$

于是本节开头的基本条件满足，因而引理1和定理2有效。

定义2 设$X^0 \in \mathscr{X}_k$。称X^0的满足(11)的$\{{}_k\Pi(a,\ \cdot),\ a \in {}_kH \cup {}_kB_e\}$广义$D$型延拓过程$X$为$X^0$的$\{{}_k\Pi(a,\ \cdot),\ a \in {}_kB_e\}$广义$D^*$型延拓过程。

定理3 k阶瞬返过程$X^0 \in \mathscr{X}_k$与其$\{{}_k\Pi(a,\ \cdot),\ a \in {}_kB_e\}$广义$D^*$型延拓过程$X$有相同的$l$级流出边界${}_lB_e(l \leqslant k)$。特别地，有相同的$Q$矩阵。$X$满足向后方程组。

§8. 关于非黏延拓

从现在开始我们转向V型延拓。D型延拓属于瞬返型：过程运动到无穷后，它从无穷立刻返回到有限状态。当然，过程也可以从无穷“慢慢地”返回到有限状态上来，这就是V型延拓。还可以有DV混合型延拓。

Chung [2] 研究马氏链的边界理论时，对过程的样本轨道进

行了细微的分析，引进了黏边界点和非黏边界点的概念，最后在有限流出的假设下，求出了过程的转移概率的解析表达式。但是，Chung[2]中没有将过程的样本轨道直接地构造出来。D型、V型和DV混合延拓过程，都是非黏返回过程，即其飞跃点可以按大小排列成序列的过程。以相邻的两个飞跃点为端点的左开右闭区间是飞跃区间即U区间。构造过程样本轨道在飞跃区间中的部分将是构造非黏返回过程的关键。

D型延拓过程在每个飞跃区间中的运动，是按照从有限状态出发的最小Q过程的运动方式进行的。因而描述跳跃情况的嵌入链是通常的离散参数马氏链$(x_n, n\geqslant 0)$，它有开始时刻和集中在状态空间E上的初始分布。但是，V型延拓过程在每个飞跃区间中，其运动是按从无穷出发的最小Q过程的运动方式进行的，描述跳跃情况的嵌入链应是$(x_n, -\infty<n<+\infty)$，然而它不是马氏链，也不是通常的平稳序列，它属于由Hunt引进的逼近马氏链。因此，逼近马氏链正是我们构造非黏返回过程的基石。但是，逼近马氏链是定义在测度空间上的，此空间甚至可以有无穷测度。

从现在起直至本章末，我们先介绍逼近马氏链及它的特征测度，进而研究逼近马氏链的发生时刻和中断时刻与特征测度的关系，然后以逼近马氏链作为嵌入链，构造飞跃区间中的样本轨道，即我们构造了所谓逼近最小Q过程，从而可以描述从无穷出发的最小Q过程。我们建立了流入族和逼近最小$\dot{Q}$过程之间的对应。最后，我们以逼近最小Q过程为基础构造一类非黏返回过程的样本轨道，即得到最小过程的DV型延拓过程，并求出了此类Q过程的转移概率的解析表达式，该表达式是通过流入族表示的。利用边界理论，还可以构造出更多的更复杂的非黏过程，即所谓广义DV型延拓过程和广义$(DV)^*$型延拓过程。

§9. 随机链和特征测度

随机链的概念由G.A.Hunt[1]引进。我们简述其定义和结

论．设E是一可数的集合，矩阵$\Pi=(\Pi_{ij},i,j\in E)$是子随机矩阵，即矩阵Π非负且行和不超过壹．当子随机矩阵的行和为壹时，称为随机矩阵．设$(\Omega,\mathscr{F},P)$是测度空间，即Ω是非空的抽象点集，$\mathscr{F}$是Ω的某些子集组成的σ代数，P是$\mathscr{F}$上的测度．我们强调指出：没有要求$(\Omega,\mathscr{F},P)$是全有限的测度空间．今后，N表示一切整数的集合．

定义1 设在$(\Omega,\mathscr{F},P)$上给定了三要素(x,α,β)：

(i) α 是取值于 $\{-\infty\}\cup N$ 的$\mathscr{F}$可测函数；β是取值于$N\cup\{+\infty\}$的F可测函数；对一切$\omega\in\Omega$，$\alpha(\omega)\leqslant\beta(\omega)$；

(ii) 对每个$\omega\in\Omega$及满足$\alpha(\omega)\leqslant n\leqslant\beta(\omega)$的$n\in N$，唯一确定$x(n,\omega)\in E$；而且对$n\in N$，$i\in E$，

$$\wedge(n,i)\equiv(x(n)=i)=(\omega;x(n,\omega)=i,\ \alpha(\omega)\leqslant n\leqslant\beta(\omega))\in\mathscr{F}.$$

我们将称(x,α,β)为准随机链．

对于准随机链(x,α,β)，含一切$\wedge(n,i)(n\in N,\ i\in E)$的最小$\sigma$代数，记为$\mathscr{F}(x,\alpha,\beta)$，称为由$(x,\ \alpha,\ \beta)$所产生的$\sigma$代数．含一切$\wedge(n,i)(n\leqslant\mathscr{K},\ i\in E)$的最小$\sigma$代数，记为$\mathscr{F}_k(x,\alpha,\beta)$．

定义2 设(x,α,β)是准随机链．如果对一切$n\in N$和$i\in E$，有$P[\wedge(n,i)]<+\infty$，称(x,α,β)为随机链．

因为 $\Omega=\bigcup\limits_{n\in N}\bigcup\limits_{i\in E}\wedge(n,i)$，因此，如果在测度空间$(\Omega,\mathscr{F},P)$上定义了随机链，则$\Omega$必是$\sigma$有限的．

可以对一切$n\in N$定义$x(n,\omega)$．例如，任取$\triangle\notin E$，$\theta\notin E$，$\triangle\neq\theta$．对$n<\alpha(\omega)$，定义$x(n,\omega)=\triangle$，对$n>\beta(\omega)$，定义$x(n,\omega)=\theta$．显然，集合

$$(x(n)=\triangle)=(n<\alpha),\quad(x(n)=\theta)=(\beta<n)$$

是$\mathscr{F}$可测集，但即使对于随机链(x,α,β)，它们的测度也未必有限．

定义3 设(x,α,β)是随机链．如果已知链的《现在》的条件下，它的《过去》和《将来》独立，即，对任意的k，m，$n\in N$，$k<$

$m<n$，以及$i_k, i_{k+1}, \cdots i_n \in E$，令

$$\wedge = \wedge(m, i_m), \quad \wedge' = \prod_{k\leq j\leq m} \wedge(j, i_j), \quad \wedge'' = \prod_{m<j\leq n} \wedge(j, i_j),$$

只要$P(\wedge)>0$，就有

$$\frac{P(\wedge'\wedge'')}{P(\wedge)} = \frac{P(\wedge')}{P(\wedge)} \cdot \frac{P(\wedge'')}{P(\wedge)},$$

我们称(x,α,β)为马氏链。

定义4　设(x,α,β)是马氏链。如果对任意$n\in N$，$i\in E$，有

$$P(x(n)=i,\ x(n+1)=j) = P(x(n)=i)\Pi_{ij},$$

称(x,α,β)为Π链。矩阵Π称为该链的(一步)转移矩阵。

定义5　设(x,α,β)是随机链，σ是定义在Ω上取值于$\{-\infty\}\cup N\cup\{+\infty\}$的$\mathscr{F}$可测函数。令

$$\Omega' = (\sigma\in N,\quad \alpha\leq\sigma\leq\beta), \quad \mathscr{F}' = \Omega'\cap\mathscr{F}, \quad P'(\wedge') = P(\wedge'),\ \wedge'\in\mathscr{F}'.$$

对$\omega\in\Omega'$，令

$$\gamma(\omega) = \beta(\omega) - \sigma(\omega), \quad y(n,\omega) = x(\sigma(\omega)+n, \omega).$$

如果$(y,0,\gamma)$是$(\Omega',\mathscr{F}',P')$上的$\Pi$链，我们称随机时刻$\sigma$将随机链$(x,\alpha,\beta)$引导到$\Pi$链$(y,0,\gamma)$。记为

$$f_\sigma(x,\alpha,\beta) = (y,0,\gamma).$$

定义6　设(x,α,β)是随机链。如果存在一列取值于$N\cup\{+\infty\}$的随机时刻α_n，$n\geq1$，使得在Ω上，几乎必然$\alpha_n\downarrow\alpha$，而且每个α_n将随机链(x,α,β)引导到某个Π链$f_{\alpha_n}(x,\alpha,\beta)$，则称$(x,\alpha,\beta)$为逼近$\Pi$链。

定义7　设(x,α,β)是随机链，σ是取值于$N\cup\{+\infty\}$的$\mathscr{F}$可测函数。如果对任意$n\in N$，$(\sigma\leq n)\in\mathscr{F}_n(x,\alpha,\beta)$，则称$\sigma$为链$(x,\alpha,\beta)$的不依赖将来的随机时刻，或称宽停时。

定义8　设(x,α,β)是逼近Π链。如果(x,α,β)是马氏链，而且将(x,α,β)引导到Π链的随机时刻序列α_n可以选取为链的宽停时序列，则称(x,α,β)为强逼近Π链。

记$\Pi^{(n)}=\Pi^n$，$G=\sum_{n=0}^{+\infty}\Pi^n$，则$G_{ij}\leq G_{jj}$。根据矩阵$\Pi$可以将$E$

中的状态分为常返状态和非常往状态．i 为 Π 常返的充要条件是 $G_{ii}=+\infty$。

定义9 设$\eta_j=(\eta_j,j\in E)$为E上的测度,即$0\leqslant\eta_j\leqslant+\infty(j\in E)$。当每个$\eta_j$有限时，称$\eta$是有限的；当$\sum\limits_j\eta_j<\infty$时，称$\eta$为全有限的。如果$\eta\Pi\leqslant\eta$，即 $\sum\limits_i\eta_i\Pi_{ij}\leqslant\eta_j(j\in E)$时，称$\eta$为$\Pi$过份测度。当$\eta\Pi=\eta$时，称$\eta$为$\Pi$调和测度。

定理1 设(x,α,β)是逼近Π链．令$C(i)$为只在i上有单位质量的测度，即$C_i(i)=1$．$C_j(i)=0(j\neq i)$，

$$\eta_j=\int_\Omega\sum_{\alpha<n\leqslant\beta}C_j[x(n)]dP=\sum_{n\in N}P(x(n)=j). \tag{1}$$

则$\eta=(\eta_j,j\in E)$是Π过份测度。称为Π链(x,α,β)的特征测度。

证 设$(x,0,\beta)$是Π链，初始分布为ν，则$(x,0,\beta)$确定的特征测度为$\eta=\nu G$，当然是Π过份测度．设(x,α,β)是逼近链，记Π链$f_{\alpha_n}(x,\alpha,\beta)$的特征测度为$\eta^n$，则$\eta^n$是$\Pi$过份的，即 $\eta^n\Pi\leqslant\eta^n$．但$\eta^n\uparrow\eta$，故$\eta\Pi\leqslant\eta$，即η是Π过份的．证完。

定理2 设(x,α,β)是逼近Π链，它的特征测度η是有限的。令 $\alpha'=-\beta$，$\beta'=-\alpha$，$x'(n)=x(-n)$，
则逆链(x',α',β')是逼近Q链．这里，当$\eta_j>0$时，

$$Q_{ji}=\eta_i\Pi_{ij}/\eta_j. \tag{2}$$

当$\eta_j=0$时，Q_{ji}可以任意选取，只要 Q_{ji} 非负，且行和不超过壹即可。

证 设D是E的有限子集，令τ为链(x,α,β)末离D的时刻，即

$$\tau=\begin{cases}\sup\{n;\ \alpha\leqslant n\leqslant\beta,\ x(n)\in D\},\\ -\infty,\ \text{如果上面的集合是空集}.\end{cases} \tag{3}$$

因此，$-\tau$是逆链(x',α',β')首达D的时刻．因 $\eta(D)<\infty$，故由(1)知$x(n)\in D$只对有穷多个n成立，因而$\tau\leqslant\beta$且$\tau<\infty$。

首先设(x,α,β)是Π链，且$\alpha=0$，初始分布为ν。按(6.3.12)确定$L_D(i)$，则(6.3.16)成立，即

$$P(x(\tau)=i_0,\ x(\tau-1)=i_1,\ \cdots,\ x(\tau-k)=i_k)$$
$$=\eta(i_k)\Pi(i_k,i_{k-1})\Pi(i_{k-1},i_{k-2})\cdots\Pi(i_1,i_0)L_D(i_0). \tag{4}$$

其中 $\eta=\nu G$。

(4)式对逼近Π链(x,α,β)也成立。实际上，设
$$f_{\alpha_n}(x,\alpha,\beta)=(x_n,0,\beta_n). \tag{5}$$

Π链$(x_n,0,\beta_n)$末离D的时刻为τ_n，特征测度为η^n。则$\eta^n\uparrow\eta$，且对几乎一切$\omega\in(\tau<\infty)$及充分大的n，有

$\alpha_n(\omega)+\tau_n(\omega)=\tau(\omega),x_n(\tau_n(\omega)-j)=x(\tau(\omega)-j)$。既然(4)对$(x_n,0,\beta_n)$成立，转向极限便知(4)对$(x,\alpha,\beta)$成立。

(4)用逆链表示即
$$P(x'(\varepsilon')=i_0,\ x'(\varepsilon'+1)=i_1,\cdots,x'(\varepsilon'+k)=i_k).$$
$$=\eta(i_0)L_D(i_0)Q(i_0,i_1)Q(i_1,i_2)\cdots Q(i_{k-1},i_k),$$

其中ε'是逆链(x',α',β')首达D的时刻。上式说明ε'将(x',α',β')引导到Q链。取$D=D_n\uparrow$个E时，$\tau(D_n)\uparrow\beta$，则$\varepsilon'(D_n)=-\tau(D_n)\downarrow\alpha'$。因此，$(x',\alpha',\beta')$是逼近$Q$链。证完。

定理3 设(x,α,β)是逼近Π链，其特征测度η有限。令$\alpha(D)$为链(x,α,β)首达E的有穷子集D的时刻，即
$$\alpha(D)=\begin{cases}\inf\ \{n:\alpha\leqslant n\leqslant\beta,\ x(n)\in D\},\\ +\infty,\ \text{如果上面的集合为空集.}\end{cases} \tag{6}$$

则$\alpha\leqslant\alpha(D)$，$-\infty<\alpha(D)$，$\alpha(D)$是宽停时，当$D\uparrow E$时有$\alpha(D)\downarrow\alpha$，而且$\alpha(D)$将$(x,\alpha,\beta)$引导到$\Pi$链
$$f_{\alpha(D)}(x,\alpha,\beta)=(x_D,0,\beta_D). \tag{7}$$

证 考虑(x,α,β)的逆链(x',α',β')末离D的时刻$\tau'(D)$，依定理2，$\tau'(D)\leqslant\beta'$，$\tau'(D)<+\infty$，当$D\uparrow E$时，$\tau'(D)\uparrow\beta'$。又显然$\alpha(D)=-\tau'(D)$，故$\alpha\leqslant\alpha(D)$，$-\infty<\alpha(D)$，当$D\uparrow E$时有$\alpha(D)\downarrow\alpha$。依定理2，$\alpha(D)$将$(x,\alpha,\beta)$引导到$\Pi$链。最后，按$\alpha(D)$的定义，易证$\alpha(D)$是$(x,\alpha,\beta)$的宽停时。证完。

注 定理3说明，对任何具有有限特征测度的逼近Π链(x,α,β)，总可以选取宽停时列$\alpha_n\downarrow\alpha$，使$f_{\alpha_n}(x,\alpha,\beta)$是$\Pi$链。但$(x,\alpha,$

β)未必是强逼近链，因为(x,α,β)本身未必是马氏链．

系 设(x,α,β)是逼近Π链，其特征测度η有限．记ν_D为(x,α,β)击中有限集D的分布，即

$$\nu_D(j)=P(x(\alpha(D))=j), \tag{8}$$

其中$\alpha(D)$由(6)确定．则ν_D的支撑集合含于D中，而且

$$\nu_D(j)=\eta(j)L_D^{\prime}(j), \tag{9}$$

其中

$L_D^{\prime}(i)=P_i(y_0\in D,\ y(n)\bar{\in}D$对$0<n\leqslant\delta)$，而$(y,0,\delta)$是$Q$链，$Q$按(2)确定．

证 在(4)中取$k=0$得

$$P(x(\tau)=j)=\eta(j)L_D(j).$$

将此式应用于逆链(x',α',β')得

$$P(x'(\tau')=j)=\eta(j)L_D^{\prime}(j),$$

其中τ'是(x',α',β')末离D的时刻，即(x,α,β)首达D的时刻，而$x'(\tau')=x(\alpha(D))$．上式化为(9)．证完

定理4 设η是有限的Π过份测度．指定E的有穷子集序列$D_n\uparrow E$．则存在测度序列ν^n具有下列性质

(i) ν^n全有限，其支撑集合含于D_n中；

(ii) $\nu^nG\leqslant\eta$，且在D_n上等号成立；

(iii) $\nu^nG\uparrow\eta$，$n\uparrow\infty$；

(iv) 设$n<m$．则ν^n是以ν^m为初始分布的Π链首达D_n的击中分布．即，如果$(x,0,\beta)$是Π链，初始分布为ν^m，首达D_n的时刻$\alpha(D_n)$将$(x,0,\beta)$引导到Π链$(x_n,0,\beta_n)$，则$(x_n,0,\beta_n)$的初始分布为ν^n．

(v) 存在测度空间$(\Omega,\mathscr{F},P)$及定义在其上的逼近Π链(x,α,β)，其特征测度与η重合，而按(8)确定的分布ν_{D_n}与ν^n重合．

这个定理的证明较烦琐，可参看 *Kemeny* 等的书[1,第十章§12]．我们指出，由(9)知，定理4中的ν^n由η和Π按下式确定：

$$\nu^n(j)=\eta(j)L_{D_n}^{\prime}(j). \tag{10}$$

§ 10. 逼近Π链的发生时刻和中断时刻

设(x,α,β)是准随机链，称α为链的发生时刻，β为链的中断时刻。注意，按定义9.2后面的说明，$x(n)$对一切$n\in N$有定义，但取值于$E\cup\{\triangle,\theta\}$。约定$C_j(\triangle)=C_j(\theta)=0$，$j\in E$。则逼近$\Pi$链$(x,\alpha,\beta)$确定的$\Pi$过份测度$\eta$为

$$\eta_j=\int_\Omega\sum_{n\in N}C_j[x(n)]dP=\int_\Omega\sum_{\alpha<n<+\infty}C_j[x(n)]dp,\quad j\in E \tag{1}$$

定理1 设(x,α,β)是逼近Π链，其特征测度为η。

(i) 如果Π是随机矩阵，则几乎必然$\beta=+\infty$；

(ii) 如果几乎必然$\alpha=-\infty$，则η是Π调和的；

(iii) 设η有限，其Riesz分解为

$$\eta=\eta^1+\eta^2,\quad \eta^1=\nu G,\quad \nu=\eta-\eta\Pi,\quad \lim_{l\to+\infty}\eta\Pi^l=\eta^2, \tag{2}$$

则过份量ν有表现：

$$\nu_j=P(-\infty<\alpha,\ x(\alpha)=j), \tag{3}$$

位势测度η^1有表现：

$$\eta^1_j=\int_{(-\infty<\alpha)}\sum_{n\in N}C_j[x(n)]dP, \tag{4}$$

调和测度η^2有表现：

$$\eta^2=\int_{(-\infty=\alpha)}\sum_{n\in N}C_j[x(n)]dP. \tag{5}$$

(iV) 设η有限。则η是位势当且仅当几乎必然$-\infty<\alpha$；η为调和的当且仅当几乎必然$-\infty=\alpha$。

证 设随机序列α_n，$n\geqslant1$将(x,α,β)引导到$\Omega_n=(\alpha_n\in N,\ \alpha\leqslant\alpha_n\leqslant\beta)$上的$\Pi$链$(x_n,0,\beta_n)$。

(i) 设Π是随机矩阵。周知，对任何Π链，其中断时刻几乎必然为$+\infty$。因此，在Ω_n上，几乎必然$\beta_n=\beta-\alpha_n=+\infty$，即$\beta=$

$+\infty$。因为$\Omega_n\uparrow\Omega$，故在Ω上几乎必然$\beta=+\infty$。

(ii) 因为

$$\eta_i=\int_\Omega\sum_{\alpha\leqslant k<+\infty}C_i[x(k)]dP$$

$$=\lim_{n\to+\infty}\int_{\Omega_n}\sum_{\alpha_n\leqslant k<+\infty}C_i[x(k)]dP$$

$$=\lim_{n\to+\infty}\int_{\Omega_n}\sum_{0\leqslant m<+\infty}C_i[x_n(m)]dP$$

$$=\lim_{n\to+\infty}\sum_{m=0}^{+\infty}P(x_n(m)=i),$$

故

$$\eta_i\Pi_{ij}=\lim_{n\to+\infty}\sum_{m=0}^{+\infty}P(x_n(m)=i)\Pi_{ij}$$

$$=\lim_{n\to+\infty}\sum_{0\leqslant m<+\infty}P(x_n(m)=i,\ x_n(m+1)=j)$$

$$=\lim_{n\to+\infty}\int_{\Omega_n}\sum_{0\leqslant m<+\infty}C_i[x_n(m)]C_j[x_n(m+1)]dP$$

$$=\lim_{n\to+\infty}\int_{\Omega_n}\sum_{\alpha_n\leqslant k<+\infty}C_i[x(k)]C_j[x(k+1)]dP$$

$$=\int_\Omega\sum_{\alpha\leqslant k<+\infty}C_i[x(k)]C_j[x(k+1)]dP.$$

对$i\in E$求和得

$$\sum_i\eta_i\Pi_{ij}=\int_\Omega\sum_{\alpha\leqslant k<+\infty}C_j[x(k+1)]dP$$

$$=\int_\Omega\sum_{\alpha+1\leqslant m<\infty}C_j[x(m)]dP. \tag{6}$$

如果几乎必然$\alpha=-\infty$，则上式右方等于η_j，因而η是调和的。

(iii) 如果η有限，则由(6)得

$$(\eta\Pi)_j=\eta_j-P(-\infty<\alpha,\ x(\alpha)=j). \tag{7}$$

由此得(3)。

类似于(6)，我们有

$$(\eta \Pi^{l})_{j}=\int_{\Omega}\sum_{\alpha+l\leqslant m<+\infty}C_{j}[x(m)]dP$$
$$=\int_{(-\infty<\alpha)}\sum_{\alpha+l\leqslant m<+\infty}C_{j}[x(m)]dP$$
$$+\int_{(-\infty=\alpha)}\sum_{m\in N}C_{j}[x(m)]dP. \tag{8}$$

注意

$$\eta_{j}=\int_{(-\infty<\alpha)}\sum_{\alpha\leqslant m<+\infty}C_{j}[x(m)]dP$$
$$+\int_{(-\infty=\alpha)}\sum_{m\in N}C_{j}[x(m)]dP<+\infty, \tag{9}$$

(8)成为

$$(\eta \Pi^{l})_{j}=\eta_{j}-\int_{(-\infty<\alpha)}\sum_{\alpha\leqslant m<\alpha+l}C_{j}[x(m)]dP$$
$$\uparrow \eta_{j}=\int_{(-\infty<\alpha)}\sum_{\alpha\leqslant m<+\infty}C_{j}[x(m)]dP,\quad l\uparrow+\infty.$$

结合(9)，我们得(5)，从而得(4)。

(iv) 当几乎必然$-\infty<\alpha$时，由(5)有$\eta^{2}=0$，从而$\eta=\eta^{1}$是位势．反之，如果η是有限位势，则$\eta\Pi^{l}\downarrow 0$，故(8)右方第二项必为零，即$\sum\limits_{m\in N}P(-\infty=\alpha,\ x(m)=i)=0$。对$i\in E$求和得几乎必然$-\infty<\alpha$。

如果几乎必然$-\infty=\alpha$，由(4)有$\eta^{1}=0$，因而$\eta=\eta^{2}$是调和的。反之，如果η是有限调和的，则由(7)得$P(-\infty<\alpha,\ x(\alpha)=j)=0$。对$j\in E$求和得$P(-\infty<\alpha)=0$。定理证完。

定理1说明，逼近Π链(x,α,β)可以分解为定义在$\Omega_{1}=(-\infty<\alpha)$和$\Omega_{2}=(-\infty=\alpha)$上的两个随机链，然而它们未必是逼近$\Pi$链．但我们很容易做到使得它们是逼近$\Pi$链．

定理2 设η是有限的Π过份测度，其Riesz分解为(2)。则存

在测度空间$(\Omega,\mathscr{F},P)$及定义在其上的逼近Π链(x,α,β)，它的特征测度η。而且，如果令$\Omega_1=(-\infty>\alpha)$，$\Omega_2=(-\infty=\alpha)$，在$\Omega_a$ $(a=1,2)$上，令$(x_a,\alpha_a,\beta_a)=(x,\alpha,\beta)$，则$(x_a, \alpha_a, \beta_a)(a=1,2)$是$\Omega_a$上的逼近$\Pi$链，它确定$\Pi$过份测使$\eta^a$。

证 依定理9.4，对$a=1$和2，存在测度空间$(\Omega_a,\mathscr{F}_a,P_a)$及定义在其上的逼近$\Pi$链$(x_a,\alpha_a,\beta_a)$，其特征测度为$\eta^a$。依定理1，可以认为$\Omega_1=(-\infty<\alpha_1)$，$\Omega_2=(-\infty=\alpha_2)$。

可以认为Ω_1与Ω_2无公共点。令$\Omega=\Omega_1\cup\Omega_2$，$\mathscr{F}=\{A,\ A\subset\Omega,\ A\cap\Omega_a\in\mathscr{F}_a,\ a=1,2\}$。易知$\mathscr{F}$是$\Omega$上的$\sigma$代数，且$\mathscr{F}_a\subset\mathscr{F}$。对$A\in\mathscr{F}$，令$P(A)=P_1(A\cap\Omega_1)+P_2(A\cap\Omega_2)$。对$\omega\in\Omega$，当$\omega\in\Omega_a$时，令$\alpha(\omega)=\alpha_a(\omega)$，$\beta(\omega)=\beta_a(\omega)$，$x(n,\omega)=x_a(n,\omega)$。则易见$(x,\alpha,\beta)$是定义在$(\Omega,\mathscr{F},P)$上的逼近$\Pi$链，具有定理中要求的性质。

§11. 嵌入链

称矩阵$A=(a_{ij},i,\ j\in E)$为Q矩阵，如果其对角线元素非正，有限，非对角线元素非负，有限，且A的行和非正，有限。当行和为零时，称Q矩阵A是保守的。设给定Q矩阵$Q=(q_{ij},i,\ j\in E)$，因而

$$0\leqslant q_{ij}<+\infty(i\neq j),\quad \sum_{j\neq i}q_{ij}\leqslant -q_{ii}<+\infty, \tag{1}$$

记$q_i=-q_{ii}$，称$d_i=q_i-\sum_{j\neq i}q_{ij}$为$i$的非保守量。回忆$C(i)$是只在$i$上有单位质量的测度。令[1]

$$\Pi_{ij}=\begin{cases}\dfrac{[1-C_j(i)]q_{ij}}{q_i} & \text{如}q_i>0,\\ 0, & \text{如}q_i=0.\end{cases} \tag{2}$$

称子随机矩阵$\Pi=(\Pi_{ij},\ i,j\in E)$为$Q$的嵌入矩阵。$\Pi$链称为$Q$的嵌

1) 为便于处理，这里的嵌入矩阵Π与(1.9.7)中的Π稍有不同。

入链。回忆$\theta\notin E$。记$E_\theta=E\cup\{\theta\}$。令$q_{i\theta}=d_i(i\in E)$，$q_\theta=-q_{\theta\theta}=0$，则$Q_\theta=(q_{ij},\ i,\ j\in E_\theta)$是保守的$Q$矩阵。$Q$的嵌入矩阵$\Pi$是$Q_\theta$的嵌入矩阵$\Pi_\theta=(\Pi_{ij},\ i,\ j\in E_\theta)$的子矩阵。显然，

$$\Pi_{i\theta}=\begin{cases}\dfrac{d_i}{q_i}, & \text{如}q_i>0,\\ 0, & \text{如}q_i=0,\end{cases}\qquad \Pi_{\theta j}=0,\ j\in E_\theta.$$

对$\lambda>0$，令

$$q_{ij}(\lambda)=q_{ij}(i\neq j),\ q_i(\lambda)=-q_{ii}(\lambda)=\lambda+q_i,$$

则$\Pi(\lambda)=(q_{ij}(\lambda),\ i,\ j\in E)$是非保守的$Q$矩阵，其嵌入矩阵记为$\Pi(\lambda)=(\Pi_{ij}(\lambda),\ i,\ j\in E)$，称为$Q$的$\lambda$-嵌入矩阵。$Q_\theta$的$\lambda$-嵌入矩阵记为$\Pi_\theta(\lambda)=(\Pi_{ij}(\lambda),\ i,j\in E_\theta)$。$\Pi(\lambda)$是$\Pi_\theta(\lambda)$的子矩阵，而且

$$\left.\begin{aligned}&\Pi_{ij}(\lambda)=\frac{[1-C_i(j)]q_{ij}}{\lambda+q_i},\ i,\ j\in E,\\ &\Pi_{i\theta}(\lambda)=\frac{d_i}{\lambda+q_i}(j\in E),\ \Pi_{\theta j}(\lambda)=0,\ j\in E_\theta\end{aligned}\right\}\tag{3}$$

注意，Π_θ的第θ行全为零，其余的行有行和为壹；而$\Pi_\theta(\lambda)$的第θ行全为零，其余的行有行和小于壹。因此，对于任何的逼近Π链或者逼近$\Pi_\theta(\lambda)$链(x,α,β)，如果链在某个时刻δ处于θ，即$x(\delta)=\theta$，则δ必是中断时刻，即$\beta=\delta$。

记
$$G(\lambda)=\sum_{n=0}^{\infty}[\Pi(\lambda)]^n,\tag{4}$$

$$\phi_{ij}(\lambda)=G_{ij}(\lambda)(\lambda+q_j)^{-1}.\tag{5}$$

则$\phi(\lambda)=\{\phi_{ij}(\lambda),i,\ j\in E\}$是熟知的Feller最小解$f(t)=\{f_{ij}(t),\ i,j\in E(t\geqslant0)$的拉普拉斯变换。因为$\lambda\sum\limits_j\phi_{ij}(\lambda)\leqslant1$，故$G(\lambda)<+\infty$，因而$E$中的状态都$\Pi(\lambda)$非常返。

引理1 设$(y,0,\delta)$是$\Pi_\theta(\lambda)$链，初始分布集中于E。设$\delta(E)$是链首出E的时刻：

$$\delta(E)=\sup\{n;\ y(n)\in E,\ 0\leqslant n\leqslant\delta\},$$

则 $$\delta(E)=\begin{cases}\delta, & 如\delta=+\infty，或\delta<+\infty，y(\delta)\overline{\in}E,\\ \delta-1, & 如\delta<+\infty，y(\delta)=\theta,\end{cases}\tag{6}$$

且$(y,0,\delta(E))$是$\Pi(\lambda)$链。

证 因$y(0)\overline{\in}E$，故$0\leqslant\delta(E)$。又因$\Pi_\theta(\lambda)$的第θ行全为零，故链只可能在有限的中断时刻δ到达θ，因而得(6)。因为$(y,0,\delta)$是$\Pi_\theta(\lambda)$链，故$(y,0,\delta(E))$是$\Pi(\lambda)$链是显然的。证完。

引理2 设(y,α,δ)是逼近$\Pi_\theta(\lambda)$链，且设$\delta(E)$是链首出E的时刻：

$$\delta(E)=\begin{cases}\sup\{n:\ y(n)\in E,\ \alpha\leqslant n\leqslant\delta\},\\ -\infty, & 如果上集合为空集。\end{cases}\tag{7}$$

则 $$\delta(E)=\begin{cases}-\infty, & 如\alpha=\delta，y(\delta)=\theta,\\ \delta-1, & 如\alpha<\delta<+\infty，y(\delta)=\theta,\\ \delta, & 如\delta=-\infty，或者\alpha\leqslant\delta<+\infty，y(\delta)\in E\end{cases}$$

且$(y,\alpha,\delta(E))$是$\Omega(E)=(-\infty<\delta(E))=(-\infty=\alpha)\cup(-\infty<\alpha,\ y(\alpha)\in E)$上的逼近$\Pi(\lambda)$链，

证 设随机时刻序列$\alpha_n\downarrow\alpha$，α_n将(y,α,δ)引导到$\Omega_n=(\alpha_n\in N,\ \alpha\leqslant\alpha_n\leqslant\delta)$上的$\Pi_\theta(\lambda)$链$(y_n,0,\delta_n)$。链$(y_n,0,\delta_n)$约束在$\Omega(E)\cap\Omega_n$上时，是初始分布集中在$E$的$\Pi_\theta(\lambda)$链。在$\Omega(E)$上，令$\alpha_n(E)=\alpha_n$。则在$\Omega(E)$上，$\alpha_n(E)\downarrow\alpha$，而且$\alpha_n(E)$将$\Omega(E)$上的随机链$(y,\alpha,\delta(E))$引导到$\Omega_n(E)\equiv(\alpha_n(E)\in N,\ \alpha\leqslant\alpha_n(E)\leqslant\delta)$上的链$(y_n,0,\delta(E))$。注意$\Omega(E)\cap\Omega_n=\Omega_n(E)$，$\delta_n(E)$正是$\Omega_n(E)$上的$\Pi_\theta(\lambda)$链$(y_n,0,\delta_n)$首出$E$的时刻。依引理1，$\Omega_n(E)$上的链$(y_n,0,\delta_n(E))$是$\Pi(\lambda)$链。所以$(y,\alpha,\delta(E))$是$\Omega(E)$上的逼近$\Pi(\lambda)$链。证完。

下面将指出：必要时扩大测度空间，可以将$\Pi(\lambda)$链或逼近$\Pi(\lambda)$链延拓而成为Π链或逼近Π链。为此，我们先引进牵涉随机链的条件分布和条件独立的概念。注意，下面的定义中的《分布》，是指定义在概率空间上的随机变量或随机过程，在它们的轨道空间上所导出的概率测度。例如，以q为参数的指数分布，是指分布函数

$$F(t)=\begin{cases}0, & \text{如}t<0,\\ 1-e^{-qt}, & \text{如}t\geqslant 0,\end{cases}$$

所产生的直线上概率测度。又例如，从i出发的Π链的分布，是指由相容的有限维分布族$\{R(i,i_1,i_2,\cdots,i_n)\}$所产生的无穷维空间$E\times E\times E\times\cdots$上的概率测度，这里

$$R(i,i_1,i_2,\cdots i_n)=\Pi_{ii_1}\Pi_{i_1i_2}\cdots\Pi_{i_{n-1}i_n}. \tag{8}$$

定义1 设$(\Omega,\mathscr{F},P)$是测度空间，$\mathscr{A}$是$\mathscr{F}$的子σ代数。设$\rho_n(n\geqslant 1)$为$\mathscr{F}$可测函数，$F_n(n\geqslant 1)$为直线上的分布。如果对任意$\Lambda\in\mathscr{A}$及直线上的Borel集$B_n(n\geqslant 1)$，有

$$P(\Lambda,\ \rho_1\in B_1,\cdots,\rho_n\in B_n)=P(\Lambda)F_1(B_1)\cdots F_n(B_n),$$

我们称在条件$\mathscr{A}$之下，$\rho_n(n\geqslant 1)$相互条件独立，ρ_n有条件分布F_n。

类似地，如果在$(\Omega,\mathscr{F},P)$上还定义了准随机链(z,ε,δ)或准随机链列$(z_n,\ \varepsilon_n,\ \delta_n)(n\geqslant 1)$，我们可以定义在条件$\mathscr{A}$之下，诸$\rho_n(n\geqslant 1)$，诸链$(z,\varepsilon,\delta)$，$(z_n,\varepsilon_n,\delta_n)(n\geqslant 1)$相互条件独立，以及在条件$\mathscr{A}$之下，$(z,\varepsilon,\delta)$的条件分布为某分布等等。

引理3 设ξ是有限的$\Pi(\lambda)$过份测度，则存在测度空间$(\Omega,\mathscr{F},P)$及定义在其上的逼近$\Pi_0(\lambda)$链(y,α,β)，使得如果$-\infty<\alpha$，则有$y(\alpha)\in E$，而且

$$\xi_j=\int_\Omega\sum_{\alpha\leqslant n\leqslant\delta}C_j[y(n)]dP,\quad j\in E. \tag{9}$$

证 依定理9.4，存在测度空间$(\Omega_1,\mathscr{F}_1,P_1)$及定义在其上的逼近$\Pi(\lambda)$链$(y_1,\alpha_1,\delta_1)$，它的特征测度为$\xi$。周知，存在概率空间$(\Omega_2,\mathscr{F}_2,P_2)$及定义在其上的一族独立随机变量$\xi_2(i)(i\in E)$，$\xi_2(i)$取0和1的概率分别为$\dfrac{\lambda}{\lambda+d_i}$和$\dfrac{d_i}{\lambda+d_i}$，$d_i$是$i$的非保守量。作乘积空间$(\Omega,\mathscr{F},P)=(\Omega_1\times\Omega_2,\mathscr{F}_1\times\mathscr{F}_2,P_1\times P_2)$。对$w=(w_1,w_2)\in\Omega$，令

$$\xi(i,w)=\xi_2(i,w_2),\quad \alpha(w)=\alpha_1(w_1),$$

$$\delta(E,w)=\delta_1(w_1);$$

$$y(n,w)=y_1(n,w_1),\quad \text{如果}\alpha(w)\leqslant n\leqslant\delta(E,w),$$

$$\delta=\begin{cases}+\infty, & \text{如果}\delta(E)=+\infty,\\ \delta(E)+\xi\{y[\delta(E)]\}, & \text{如果}\delta(E)<+\infty。\end{cases}$$

当$\delta(E)<+\infty$且$\xi\{y[\delta(E)]\}=1$时，令$y(\delta)=\theta$。则易证(y,α,δ)为所求。证完。

引理4 设ξ是有限的$\Pi(\lambda)$过份测度。则存在测度空间 $(\Omega,\mathscr{F},P)$，在其上定义了逼近$\Pi_\theta(\lambda)$链(y,α,δ)，准随机链列$(z_i,0,\varepsilon_i)(i\in E)$，以及一族$\mathscr{F}$可测函数 $S_u,u\in U$ (U是给定的$(-\infty,+\infty)$的子集)，使得

(i) 在$(-\infty<\alpha)$上有$y(\alpha)\in E$，而且(9)成立；

(ii) 在条件$\mathscr{A}=\mathscr{F}(y,\alpha,\delta)$之下，对每个$u\in U$，$S_u$的条件分布为指定的分布$F_u$；

(iii) 在条件$\mathscr{A}$之下，$(z_i,0,\varepsilon_i)$的条件分布是从i出发的Π链的分布；

(iV) 在条件$\mathscr{A}$之下，诸S_u，$u\in U$，$(z_i,0,\varepsilon_i)(i\in E)$相互条件独立。

证 利用引理3和独立乘积空间的技巧，即可证明。

定义2 设随机链(y,ε,γ)和(x,α,β)定义在同一测度空间上。如果$\varepsilon=\alpha$，$\gamma\leqslant\beta$，而且对满足$\varepsilon\leqslant n\leqslant\gamma$的$n\in N$，有$y(n)=x(n)$，则称$(x,\alpha,\beta)$是$(y,\varepsilon,\gamma)$的延拓，$(y,\varepsilon,\gamma)$是$(x,\alpha,\beta)$的前部。

定理5 设$\lambda>0$指定，$\zeta(\lambda)$是有限的$\Pi(\lambda)$过份测度。则存在测度空间$(\Omega,\mathscr{F},P)$及定义在其上的逼近$\Pi(\lambda)$链(y,α,γ)和逼近Π链(x,α,β)，使得(x,α,β)是(y,α,γ)的延拓。(y,α,γ)的特征测度是$\zeta(\lambda)$，(x,α,β)的特征测度为

$$\zeta_j=\zeta_j(\lambda)+\lambda\sum_i\zeta_i(\lambda)(\lambda+q_i)^{-1}G_{ij}\tag{10}$$

证 应用引理4.存在测度空间 $(\Omega,\mathscr{F},P)$ 及定义在其上的逼近$\Pi_\theta(\lambda)$链(y,α,δ)，使得

$$\zeta_j(\lambda)=\int_\Omega\sum_{\alpha\leqslant n<\delta}C_j[y(n)]dP,\quad j\in E。$$

必要时清洗$(-\infty<\alpha,\ y(\alpha)=\theta)$，故可以假设$(-\infty<\alpha,\ y(\alpha)=$

θ)是空集。在$(\Omega,\mathscr{F},P)$上还可以定义一列准随机链$(z_i,0,\varepsilon_i)$ $(i\in E)$，使在条件$\mathscr{F}(y,\alpha,\delta)$之下，诸$(z_i,0,\varepsilon_i)$ $(i\in E)$相互条件独立，并且在条件$\mathscr{F}(y,\alpha,\delta)$之下，$(z_i,0,\varepsilon_i)$的条件分布是从$i$出发的$\Pi$链的分布。

按(7)定义逼近$\Pi_\theta(\lambda)$链(y,α,δ)首出E的时刻$\delta(E)$，简记$\delta(E)$为γ，则$\Omega=(-\infty<\gamma)=(\alpha\leqslant\gamma)$。链$(y,\alpha,\gamma)$是逼近$\Pi(\lambda)$链，它的特征测度为$\zeta(\lambda)$。

令
$$\beta=\begin{cases}\gamma, & \text{如果}\gamma=+\infty\text{或}\gamma=\delta-1<+\infty,\\ \gamma+\varepsilon_{y(\delta)}, & \text{如果}\gamma=\delta<+\infty,\end{cases}\tag{11}$$
对$n\in N$，$\alpha\leqslant n\leqslant\beta$，定义
$$x(n)=\begin{cases}y(n), & \text{如果}n\leqslant\gamma,\\ z_{y(\delta)}(n-\delta), & \text{如果}\gamma<+\infty,\gamma<n\leqslant\beta.\end{cases}\tag{12}$$

显然，(x,α,β)取值于E，并且是(y,α,γ)的延拓，往证(x,α,β)是逼近Π链。

先假设逼近$\Pi_\theta(\lambda)$链(y,α,δ)是$\Pi_\theta(\lambda)$链，并且$\alpha=0$。往证按上面方式确定的$(x,0,\beta)$是Π链。即，设i_0，i_1，…，$i_n\in E$，$n\geqslant 1$，令
$$A_n\equiv P(x(0)=i_0,\cdots,x(n)=i_n),\tag{13}$$
要证$A_n=A_{n-1}\Pi_{i_{n-1}i_n}$。

按(8)定义$R(i_0,i_1,\cdots,i_n)$，类似定义
$$R(\lambda;\ i_0,i_1,\cdots,i_n)=\Pi_{i_0i_1}(\lambda)\Pi_{i_1i_2}(\lambda)\cdots\Pi_{i_{n-1}i_n}(\lambda),\tag{14}$$
注意$(y(0)=i_0)\subset(0\leqslant\gamma)$，$(y(n)=i_n)\subset(n\leqslant\gamma)\subset(n\leqslant\delta)$，故
$$\begin{aligned}A_n&=P(x(0)=i_0,\cdots,x(n)=i_n,n\leqslant\gamma)\\&\quad+P(x(0)=i_0,\cdots,x(n)=i_n,\gamma<n)\\&=P(y(0)=i_0,\cdots y(n)=i_n,\ n\leqslant\delta)\\&\quad+P(y(0)=i_0,\cdots y(\delta)=i_\delta,\ z_{y(\delta)}(1)=i_{\delta+1},\ \cdots,\\&\qquad z_{y(\delta)}(n-\delta)=i_n,\ \delta<n)\\&=P(y(0)=i_0)R(\lambda,i_0,i_1,\cdots i_n)\end{aligned}$$

$$+\sum_{k=0}^{n-1}P(y(0)=i_0,\ y(1)=i_1,\cdots,y(k)=i_k,\ \delta=k,$$
$$z_{i_k}(1)=i_{k+1},\cdots,z_{i_k}(n-k)=i_n),$$

由于条件独立性，$\sum_{k=0}^{n-1}$中被加项等于

$$P(y(0)=i_0,\ y(1)=i_1,\cdots,y(k)=i_k,$$
$$\delta=k)R(i_k,i_{k+1},\cdots i_n)$$
$$=P(y(0)=i_0)R(\lambda;i_0,i_1,\cdots,i_k)(1-\sum_{j\in E_\theta}\Pi_{i_kj}(\lambda))$$
$$\cdot R(i_k,i_{k+1},\cdots,i_n).$$

于是

$$A_n=P(y(0)=i_0)\{R(\lambda;\ i_0,i_1,\cdots i_n))+$$
$$+\sum_{k=0}^{n-1}R(\lambda;\ i_0,i_1,\cdots,i_k)\frac{\lambda}{\lambda+q_{i_k}}R(i_k,i_{k+1},\cdots,i_n)\}$$
$$=P(y(0)=i_0)\{R(\lambda;\ i_0,i_1,\cdots,i_{n-1})\Pi_{i_{n-1}\ i_n}(\lambda)$$
$$+R(\lambda;\ i_0,i_1,\cdots,i_{n-1})\lambda(\lambda+q_{i_{n-1}})^{-1}\Pi_{i_{n-1}i_n}$$
$$+\sum_{k=0}^{n-2}R(i_0,\cdots,i_k)\lambda(\lambda+q_{i_k})^{-1}R(i_k,\cdots,i_{n-1})\Pi_{i_{n-1}i_n}\}.$$

注意$\Pi_{i_{n-1}i_n}(\lambda)+\lambda(\lambda+q_{i_{n-1}})^{-1}\Pi_{i_{n-1}i_n}=\Pi_{i_{n-1}i_n}$，我们得$A_n=A_{n-1}\Pi_{i_{n-1}i_n}$.

又

$$\zeta_j=\int_\Omega\sum_{0\leqslant n\leqslant\beta}C_j[x(n)]dP$$
$$=\int_\Omega\sum_{0\leqslant n\leqslant\gamma}C_j[y(n)]dP+\int_{(\gamma<+\infty)}\sum_{\gamma<n\leqslant\beta}C_j[x(n)]dP$$
$$=\zeta_j(\lambda)+\sum_i\int_{(\delta<+\infty,\ y(\delta)=i)}\sum_{0\leqslant n\leqslant\varepsilon_i}C_j[z_i(n)]dP.$$

由条件独立性，$\sum$中被加项等于$P(\delta<+\infty,\ y(\delta)=i)G_{ij}$。而

$$P(\delta<+\infty,\ y(\delta)=i)=\sum_{n=0}^{+\infty}P(y(n)=i,\ \delta=n)$$

$$=\sum_{n=0}^{+\infty}P(y(n)=i)\frac{\lambda}{\lambda+q_i}=\lambda\zeta_i(\lambda)(\lambda+q_i)^{-1},\tag{15}$$

故得(10)。

现在设(y,α,δ)是逼近$\Pi_0(\lambda)$链，$\alpha_n\downarrow\alpha$，α_n将(y,α,δ)引导到$\Pi_0(\lambda)$链$(y_n,0,\delta_n)$。设α_n将(x,α,β)引导到$(x_n,0,\beta_n)$。令γ为链(y,α,δ)首出E的时刻。象从(y,α,δ)及$(z_i,0,\varepsilon_i)(i\in E)$出发，按(11)(12)得到$(y,\alpha,\gamma)$的延拓过程$(x,\alpha,\beta)$一样，从$(y_n,0,\delta_n)$及$(z_i,0,\varepsilon_i)$ $(i\in E)$出发，按(11)(12)而得到$(y_n,0,\gamma_n)$的延拓，此延拓正是$(x_n,0,\beta_n)$。这里γ_n是$(y_n,0,\delta_n)$首出E的时刻。由于在条件$\mathscr{F}(y,\alpha,\delta)$之下，$(z_i,0,\varepsilon_i)(i\in E)$相互条件独立，且条件分布是从$i$出发的$\Pi$链的分布，因而在条件$\mathscr{F}(y_n,0,\delta_n)$之下亦然。按照上面证明的事实，$(x_n,0,\beta_n)$是$\Pi$链。因此，$(x,\alpha,\beta)$是逼近$\Pi$链。

设ζ^n和$\zeta^n(\lambda)$分别是由$(x_n,0,\beta_n)$和$(y_n,0,\gamma_n)$确定的测度，则上面已证明：

$$\zeta_j^n=\zeta_j^n(\lambda)+\lambda\sum_i\zeta_i^n(\lambda)(\lambda+q_i)^{-1}G_{ij}.$$

因$\zeta^n\uparrow\zeta$，$\zeta^n(\lambda)\uparrow\zeta(\lambda)$，由上式得(10)。证完。

§12. 测度空间中的Q过程

我们要把定义在概率空间上的可列马尔科夫过程概念推广到定义在测度空间$(\Omega,\mathscr{F},P)$上。记T为$[0,+\infty)$或$(0,+\infty)$。设E有离散拓朴，并用单点$\infty\notin E$紧化E，记$\bar{E}=E\cup\{\infty\}$。

定义1 设$(\Omega,\mathscr{F},P)$是测度空间。σ是取值于$[0,+\infty]$的$\mathscr{F}$可测函数。对每个固定的$t\in T$及几乎一切$\omega\in(t<\sigma)$，$x(t,\omega)$确定并属于$\bar{E}$。如果对任意$t\in T$及$i\in E$，有$P(x(t)=\infty)=0$，而且

$$\triangle(t,i)\equiv(x(t)=i)=(\omega:\ x(t,\omega)=i,\ t<\sigma(\omega))\in\mathscr{F},\tag{1}$$

称$X=\{x(t),\ t\in T\cap[0,\sigma)\}$为准随机过程。如果对每个$t\in T$,

$i\in E$，有$P(\triangle(t,i))<+\infty$，称$X=\{x(t),\ t\in T\cap[0,\sigma)\}$为随机过程。包含一切$\triangle(t,i)(t\in T,\ i\in E)$的最小$\sigma$代数，称为由$X$产生的$\sigma$代数，记为$\mathscr{F}(X)$或$\mathscr{F}(x(t),\ t\in T\cap[0,\sigma))$。当$T=(0,+\infty)$时，随机过程$X=\{x(t),\ 0<t<\sigma\}$称为开的。

定义2 设$P(t)=(p_{ij}(t),\ i,\ j\in E)$，$t\geqslant 0$为标准广转移矩阵。设$X=\{x(t),\ t\in T\cap[0,\ \sigma)\}$为随机过程。如果对任意$n\geqslant 1$，$t_0\in T$，$t_0<t_1<\cdots<t_{n+1}$，$i_0,\ i_1,\cdots,i_{n+1}\in E$，有

$$P(x(t_0)=i_0,\cdots x(t_{n+1})=i_{n+1})$$
$$=P(x(t_0)=i_0,\cdots,x(t_n)=i_n)p_{i_n i_{n+1}}(t_{n+1}-t_n)。\tag{2}$$

称X为具转移概率矩阵$P(t)$的马尔科夫过程。如果$P(t)$以给定的Q矩阵Q为密度矩阵，称X为Q过程。如果$P(t)$是熟知的最小解，称X为最小过程。

定理1 设$X=\{x(t),\ t\in T\cap[0,\sigma)\}$是具有转移概率矩阵$P(t)$的马尔科夫过程。设$S$是$\mathscr{F}$可测函数，在条件$\mathscr{F}(X)$之下，$S$的条件分布是参数为壹的指数分布。对$\lambda>0$，令$p^{\lambda}_{ij}(t)=e^{-\lambda t}p_{ij}(t)$，$\sigma(\lambda)=min\left(\sigma,\ \dfrac{S}{\lambda}\right)$，则$X^{\lambda}=\{x(t),\ t\in T\cap[0,\sigma(\lambda))\}$是具转移概率矩阵$P^{\lambda}(t)=(p^{\lambda}_{ij}(t),i,j\in E)$的马尔科夫过程。特别地，如果$X$是最小$Q$过程，则$X^{\lambda}$是最小$Q(\lambda)$过程。

证 利用条件独立性条件，容易证得。

引理2 设给定全有限测度$\nu=(\nu_i,\ i\in E)$。则存在全有限测度空间$(\Omega,\mathscr{F},P)$及定义在其上的Π链$(x,\ 0,\ \beta)$和$\mathscr{F}$可测函数ρ^n_i $(n\geqslant 0,\ i\in E)$，使得$(x,0,\beta)$的初始分布为ν，在条件$\mathscr{F}(x,0,\beta)$之下，诸$\rho^n_i(n\geqslant 0,\ i\in E)$相互条件独立，且$\rho^n_i$的条件分布是以$q_i$为参数的指数分布。

注意，$(x,0,\beta)$的特征测度为$\zeta=\nu G$，而νG未必有限，因此，引理2不是引理11.4的直接推论。由于ν全有限，因此依通常的马尔科夫过程存在定理，存在定义在全有限测度空间上的Π链，它以ν为初始分布。由此仿引理11.4的证明即可证明引理2。

由最小过程的结构，下面的定理是明显的。

定理3 设ν，$(x,0,\beta)$，$\rho_i^n(n\geqslant 0, i\in E)$同引理2。对$0\leqslant n\leqslant\beta$，

令 $\rho^n=\rho^n_{x(n)}, \tau_0=0, \tau_n=\sum\limits_{i=0}^{n-1}\rho^i$，$\sigma=\tau_{\beta+1}$。对$0\leqslant t<\sigma$，令

$$X(t)=x(n), \quad 如\tau_n\leqslant t<\tau_{n+1}, \tag{3}$$

则$X=\{X(t), 0\leqslant t<\sigma\}$是最小$Q$过程，初始分布为$\nu$。

类似地，下面的引理4和定理5也是明显的。

引理4 设$\nu=(\nu_i, i\in E)$是全有限测度，则存在全有限的测度空间$(\Omega,\mathscr{F},P)$及定义在其上的 $\Pi_\theta(\lambda)$链$(y,0,\delta)$ 和$\mathscr{F}$可测函数ρ^θ，$\rho_i^n(n\geqslant 0, i\in E)$，使得$(y,0,\sigma)$的初始分布为$\nu$，在条件 $\mathscr{F}(y,0,\delta)$之下，诸ρ^θ，$\rho_i^n(n\geqslant 0, i\in E)$相互条件独立，且条件分布分别是以$\lambda$和$\lambda+q_i$为参数的指数分布。

定理5 设ν，$(y, 0, \delta)$，ρ^θ，$\rho_i^n(n\geqslant 0, i\in E)$ 同引理4。令$\delta(E)$为$\Pi_\theta(\lambda)$链$(y,0,\delta)$首出E的时刻。对$0\leqslant n\leqslant\delta(E)$，令

$$\rho^n=\rho^n_{y(n)}, \quad \tau_0=0, \quad \tau_{n+1}=\sum_{i=0}^{n}\rho^i$$

$$\sigma=\begin{cases}\sum\limits_{i=0}^{+\infty}\rho^i, & 如果\delta(E)=+\infty, \\ \tau_{\delta(E)+1}, & 如果\delta(E)<+\infty,\end{cases}$$

$$\sigma_\theta=\begin{cases}\sum\limits_{i=0}^{+\infty}\rho^i, & 如果\delta(E)=+\infty \\ \tau_{\delta(E)+1}, & 如果\delta(E)=\delta<+\infty, \\ \tau_{\delta(E)+1}+\rho^\theta, & 如果\delta(E)=\delta-1<+\infty.\end{cases}$$

对$0\leqslant t<\sigma_\theta$，令

$$X(t)=y(n), \quad 如果\tau_n\leqslant t<\tau_{n+1},$$

则$X=\{x(t), 0\leqslant t<\sigma\}$是最小$Q(\lambda)$过程，$X_\theta=\{x(t), 0\leqslant t<\sigma_\theta\}$是最小$Q_\theta(\lambda)$过程，初始分布都为$\nu$。

§ 13. 逼近最小Q过程

回忆E具有离散拓朴，"∞" 单点紧化E. 假定有限集$D_n \subset E$, $D_n \uparrow E$, $(n \uparrow +\infty)$.

定义 1 设$0<\sigma \leqslant +\infty$. 定义在$[0, \sigma)$上的函数$X=\{x(t), 0 \leqslant t<\sigma\}$称为U型跳跃函数，如果下列条件成立：

(i) $X(0) \in E \cup \{\infty\}$, $X(t) \in E$, $(0<t<\sigma)$, 且X右连续；

(ii) 对任意$[c, d] \subset (0, \sigma)$, X在$[c, d]$中只有有穷多个跳跃点；

(iii) 任意指定$i \in E$, 对任意$d \in (0, \sigma)$, X在$(0, d)$中只有有限个i区间。

定义 2 称$X=\{x(t), 0 \leqslant t<\sigma\}$为定义在测度空间$(\Omega, \mathscr{F}, P)$上的逼近最小$Q$过程，如果对任意$t \geqslant 0$, $i \in E$, 有

$$(x(t)=i) \in \mathscr{F}, \quad P(x(t)=i)<+\infty,$$

而且下列条件成立：

(i) (12.2) 式对$0 \leqslant t_0<t_1<\cdots<t_{n+1}$, i_0, i_1, $\cdots i_{n+1} \in E$成立，其中$P(t)$是最小解$P(t)$；

(ii) X的一切轨道都是U型跳跃函数。

(iii) 存在取值$[0, +\infty]$的随机时刻序列$\tau_n \downarrow 0$, 使得$X_n=\{x(\tau_n+t), 0 \leqslant t<\sigma-\tau_n\}$是定义在$\Omega_n=(\tau_n<\sigma)$上的最小$Q$过程，而且$X_n$的初始分布的支撑集含于$D_n$.

注意，最小Q过程是逼近最小Q过程，此时τ_n可取为0，或可取为首达D_n的时刻。将逼近最小Q过程限制在$\Omega'=(\sigma=0) \cup (\sigma>0, X(0) \neq \infty)$上考虑时，它就是最小$Q$过程。特别地，如果$P(X(0)=\infty)=0$时，逼近最小$Q$过程就是最小$Q$过程。但如果仍在$(\Omega, \mathscr{F}, P)$上考虑，而将参数限制在$(0, \infty)$中考虑，则$X'=\{X(t), 0<t<\sigma\}$是开最小$Q$过程。然而，逼近最小$Q$过程却未必是最小$Q$过程。因为对于逼近最小$Q$过程$X=\{x(t), 0 \leqslant t<\sigma\}$, 集合$(x(0)=\infty)$虽然$\mathscr{F}$可测，但其测度可能为无穷。但如果

$P(x(0)=\infty)=0$，则逼近最小Q过程就是最小Q过程。

定义 3 设$X=\{x(t),\ 0\leqslant t<\sigma\}$是逼近最小$Q$过程。令

$$\eta_j(\lambda)=\int_\Omega\int_0^\sigma e^{-\lambda t}C_j[x(t)]dP,\ \lambda>0,\ j\in E, \tag{1}$$

$$\eta_j=\int_\Omega\int_0^\sigma C_j[x(t)]dP,\ j\in E. \tag{2}$$

称η为X的特征测度，$\eta(\lambda)$为X的λ－特征测度。定理14.3将证明：$(\eta(\lambda),\ \lambda>0)$是流入族（见定义1.11.1），称为$X$的特征流入族。

定理 1 设τ_n是逼近最小Q过程$X=\{x(t),\ 0\leqslant t<\sigma\}$定义中的随机时刻序列，$\tau_n$将$X$引导到$\Omega_n=(\tau_n<\sigma)$上的最小$Q$过程$X^n=(X_n(t),\ 0\leqslant t<\sigma_n)$。则在$\Omega$上，对一切$t\in[0,\sigma)$，有

$$\lim_{n\to+\infty}X_n(t)=X(t),\ t\in[0,\sigma). \tag{3}$$

证 因$\tau_n\downarrow 0$，故$\sigma_n=\sigma-\tau_n\uparrow\sigma$。对$t\in[0,\sigma)$，由$X$的右连续性，

$$\lim_{n\to+\infty}X_n(t)=\lim_{n\to+\infty}X(\tau_n+t)=X(t)$$

§ 14. 流入族和逼近最小过程

数$\lambda>0$指定。设全有限测度$\eta(\lambda)$满足不等式

$$\lambda u-uQ\geqslant 0, \tag{1}$$

则$\zeta_j(\lambda)=\eta_j(\lambda)(\lambda+q_j)$是有限的$\Pi(\lambda)$过份测度。依定理9.4，存在全有限的测度$\nu^n(\lambda)$，使得

(i) $\nu^n(\lambda)$的支撑集含于D_n中；

(ii) $\nu^n(\lambda)G(\lambda)\uparrow\zeta(\lambda)$，$n\uparrow\infty$，$(\nu^n(\lambda)G(\lambda))_j=\zeta_j$，$j\in D_n$；

(iii) 对$m<n$，以$\nu^n(\lambda)$为初始分布的$\Pi(\lambda)$链首次到达D_m的击中分布是$\nu^m(\lambda)$。

注意(ii)等价于

$$\left.\begin{array}{l}\nu^n(\lambda)\phi(\lambda)\uparrow\eta(\lambda),\ n\uparrow+\infty,\\ (\nu^n(\lambda)\phi(\lambda))_j=\eta_j(\lambda),\ j\in D_n.\end{array}\right\} \tag{2}$$

定理1 设$\lambda>0$指定。全有限测度$\eta(\lambda)$满足(2)。则存在测度

空间$(\Omega,\mathscr{F},P)$ 及定义在其上的逼近最小$Q_\theta(\lambda)$过程$X=\{x(t),0\leqslant t<\sigma(\theta)\}$，使得

(i) 设τ_n是X首达D_n的时刻，则τ_n将X引导到初始分布为$\nu^n(\lambda)$的最小$Q_\theta(\lambda)$过程；

(ii) $$\eta_j(\lambda)=\int_\Omega\int_0^{\sigma(\theta)}C_j[X(t)]dtdP,\quad j\in E;\tag{3}$$

(iii) $$P(X(0)=\theta)=0,\quad P(x(0)=i)=\nu_i(\lambda),\tag{4}$$

这里 $$\nu(\lambda)=\lambda\eta(\lambda)\eta(\lambda)Q.\tag{5}$$

特别地，如果$\eta(\lambda)$是方程

$$\lambda u-uQ=0\tag{6}$$

的解，则在Ω上几乎必然$X(0)=\infty$。

证 因$\zeta_j(\lambda)=\eta_j(\lambda)(\lambda+q_j)(j\in E)$是有限的$\Pi(\lambda)$过份测度，依引理11.4，存在测度空间$(\Omega,\ \mathscr{F},\ P)$，在其上可以定义逼近$\Pi_\theta(\lambda)$链$(y,\ \alpha,\ \delta)$，使得

$$P(-\infty<\alpha,\ y(\alpha)=\theta)=0,P(-\infty<\alpha,y(\alpha)=i)=\nu_i(\lambda)$$

在$(-\infty=\alpha)$上几乎必然有

$$\lim_{n\to-\infty}y(n)=\infty,$$

又 $$\zeta_j(\lambda)=\int_\Omega\sum_{\alpha\leqslant n\leqslant\delta}C_j[y(n)]dP,\quad j\in E.\tag{7}$$

链$(y,\ \alpha,\ \delta)$首达D_n的击中分布是$\nu^n(\lambda)$。在$(\Omega,\ \mathscr{F},\ P)$上还可以定义一族$\mathscr{F}$可测函数$\rho_\theta$和$\rho_i^n(n\geqslant0,\ i\in E)$，使在条件$\mathscr{F}(y,\alpha,\delta)$之下，诸$\rho_\theta$，$\rho_i^n(n\geqslant0,\ i\in E)$相互条件独立，而且$\rho_\theta$和$\rho_i^n$的条件分布分别是以$\lambda$和$\lambda+q_i$为参数的指数分布。

设$\delta(E)$为链$(y,\ \alpha,\ \delta)$首出E的时刻，则$(-\infty<\delta(E))=(\alpha\leqslant\delta(E))=\Omega$。如果$\alpha\leqslant n\leqslant\delta(E)$，则令$\rho^n=\rho^n_{y(n)}$；如果$\delta(E)=\delta-1<+\infty$，则令$\rho^\delta=\rho_\theta$。往证

$$\int_\Omega\sum_{\alpha\leqslant n\leqslant\delta}\rho^n dP<+\infty.\tag{8}$$

实际上，$$\int_{\Omega}\sum_{\alpha<n<\delta}\rho^{n}dP$$

$$=\sum_{n\in N}\int_{(\alpha<n<\delta(E))}\rho^{n}_{y(n)}dP+\int_{(\delta(E)=\delta-1<+\infty)}\rho^{\delta}dP.$$

利用条件独立性，上式第一部分等于

$$\sum_{n\in N}\sum_{i\in E}\int_{(y(n)=i)}\rho^{n}_{i}dP=\sum_{n\in N}\sum_{i\in E}P(y(n)=i)(\lambda+q_i)^{-1}$$

$$=\sum_{i\in E}\sum_{n\in N}P(y(n)=i)(\lambda+q_i)^{-1}=\sum_{i\in E}\zeta_i(\lambda)(\lambda+q_i)^{-1}$$

$$=\sum_{i\in E}\eta_i(\lambda)<+\infty,$$

类似(27)的推导，第二部分等于

$$\int_{(\delta<+\infty,\ y(\delta)=\theta)}\rho_{\theta}dP=P(\delta<+\infty,\ y(\delta)=\theta)\lambda^{-1}$$

$$=\lambda^{-1}\sum_{i\in E}P(\delta(E)<+\infty,\ y(\delta(E))=i,\ y(\delta(E)+1)=\theta)$$

$$=\lambda^{-1}\sum_{i\in E}P(\delta(E)<+\infty,\ y(\delta(E))=i)\Pi_{i\theta}(\lambda)$$

$$=\lambda^{-1}\sum_{i\in E}\lambda\zeta_i(\lambda)(\lambda+q_i)^{-1}\Pi_{i\theta}(\lambda)$$

$$=\sum_{i\in E}\eta_i(\lambda)\Pi_{i\theta}(\lambda)\leqslant\sum_{i\in E}\eta_i(\lambda)<+\infty.$$

现在定义$X=(x(t),\ 0\leqslant t<\sigma(\theta))$。令

$$\tau_{\alpha}=0,\ \tau_n=\sum_{\alpha\leqslant K<n}\rho^{k},\ (\alpha<n<\delta),\ \sigma(\theta)=\tau_{\delta+1},\tag{9}$$

对$0\leqslant t<\sigma(\theta)$，令

$$\left.\begin{aligned}&x(t)=y(n),\ \text{如果}\tau_n\leqslant t<\tau_{n+1},\\&x(0)=\infty,\ \text{如果}-\infty=\alpha.\end{aligned}\right\}\tag{10}$$

往证$X=(x(t),\ 0\leqslant t<\sigma(\theta))$即为所求。

首先，(4)显然成立。对每个轨道X，定义1中的 (i)(ii) 成立(用E_{θ}代替E)，由 (7) 及$\zeta_j(\lambda)<+\infty$，几乎必然 $\sum_{\alpha\leqslant n<\delta}C_j[y(n)]$

$<+\infty$，从而定义1中的(iii)也成立。因此，X的轨道是U型跳跃函数。

其次，$j\in E$时，(3) 由下式得出：

$$\int_{\Omega}\int_{0}^{\sigma(\theta)} C_j[x(t)]dtdP=\int_{\Omega}\sum_{\alpha\leqslant n\leqslant\delta} C_j[y(n)]\rho^n dP$$

$$=\int_{\Omega}\sum_{\alpha\leqslant n\leqslant\delta} C_j[y(n)]\rho_j^n dP$$

$$=\sum_{n\in N}\int_{(y(n)=j)}\rho_j^n dP=\sum_{n\in N}P(y(n)=j)(\lambda+q_j)^{-1}$$

$$=\zeta_j(\lambda)(\lambda+q_j)^{-1}=\eta_j(\lambda).$$

再次，设α_n是逼近 $\Pi_\theta(\lambda)$ 链$(y,\ \alpha,\ \delta)$ 首达D_n的时刻，则X首达D_n的时刻为

$$b_n=\begin{cases}\tau_{\alpha_n}, & \text{如}\alpha_n<+\infty,\\ +\infty & \text{如}\alpha_n=+\infty.\end{cases}$$

设α_n将$(y,\ \alpha,\ \delta)$引导到$\Omega_n=(\alpha_n\leqslant\delta)$上的$\Pi_\theta(\lambda)$链$(y_n,\ 0,\ \delta_n)$。在$\Omega_n$上，令$\sigma_n=\sigma(\theta)-b_n$，对$0\leqslant t<\sigma_n$，令$x_n(t)=x(b_n+t)$，则$X_n=\{x_n(t),\ 0\leqslant t<\sigma_n\}$是定义在$\Omega_n$上的马氏链，其初始分布为$\nu^n(\lambda)$，由于$\alpha_n$是链 $(y,\ \alpha,\ \delta)$ 的不依赖于将来的时刻，因此，在条件$\mathscr{F}(y_n,\ 0,\ \delta_n)$ 之下，ρ^θ，$\bar{\rho}_i^m\equiv\rho_i^{\alpha_n+m}(m\geqslant0,\ i\in E)$ 是相互条件独立的，而且在条件$\mathscr{F}$ $(y_n,\ 0,\ \delta_n)$之下，ρ^θ 和 $\bar{\rho}_i^m$ 的条件分布分别是以λ和$\lambda+q_i$为参数的指数分布．按照定理12.5的方式，从$(y_n,\ 0,\ \delta_n)$，ρ^θ，$\bar{\rho}_i^m$ $(m\geqslant0,\ i\in E)$出发，我们得到与$(y_n$ $0,\ \delta_n)$同初始分布的最小$Q_\theta(\lambda)$过程，这个最小$Q_\theta(\lambda)$过程不是别的，正是X_n。而$(\alpha_n<+\infty)=(b_n<\sigma(\theta))$。因此，$X_n$是定义在$\Omega_n=(b_n<\sigma(\theta))$上的最小$Q_\theta(\lambda)$过程，$X_n$的初始分布的支撑集含于$D_n$。

最后，往证(12.2)对 $0\leqslant t_0<t_1<\cdots<t_{n+1}$，$i_0$，$i_1$，$\cdots$，$i_{n+1}\in E_\theta$及矩阵$Q_\theta(\lambda)$的最小解$P^\lambda(t)=(\bar{p}_{ij}^\lambda(t),\ i,\ j\in E_\theta)$成立。

实际上，因在$(-\infty<\alpha)$上有$y(\alpha)\in E$，故$\alpha_n\downarrow\alpha$，从而$\Omega_n=(\alpha_n$

$<+\infty)\uparrow\Omega$. 在Ω上，因$\sigma(\theta)<+\infty$，故$b_n=\tau_{\alpha_n}\downarrow 0$，从而$\sigma(\theta)-b_n\uparrow\sigma(\theta)$。于是对一切$t\in[0,\ \sigma(\theta))$，有$X_n(t)=x\ (b_n+t)\rightarrow x(t)$。所以

$$
\begin{aligned}
&(x(t_0)=i_0,\ \cdots,\ x(t_{m+1})=i_{m+1})\\
&=\lim_{n\to+\infty}\Omega_n\cap(X_n(t_0)=i_0,\ \cdots,\ X_n(t_{m+1})=i_{m+1})
\end{aligned}
$$

因而
$$
\begin{aligned}
&P(X(t_0)=i_0,\ \cdots,\ X(t_{m+1})=i_{m+1})\\
&=\lim_{n\to+\infty}P(\Omega_n,\ X_n(t_0)=i_0,\ \cdots,\ X_n(t_{m+1})=i_{m+1})\\
&=\lim_{n\to+\infty}P(\Omega_n,\ X_n(t_0)=i_0)\bar{P}^{\lambda}_{i_0i_1}(t_1-t_0)\cdots\bar{P}^{\lambda}_{i_mi_{m+1}}(t_{m+1}\\
&\quad -t_m)\\
&=P(X(t_0)=i)\bar{P}^{\lambda}_{i_0i_1}(t_1-t_0)\cdots\bar{P}^{\lambda}_{i_mi_{m+1}}(t_{m+1}-t_m).
\end{aligned}
$$

这样，$X=\{x(t),\ 0\leqslant t<\sigma(\theta)\}$是逼近最小$Q_\theta(\lambda)$过程，定理遂证完。

下面的定理2是显然的。

定理 2 设$\lambda>0$指定，$X=\{x(t),\ 0\leqslant t<\sigma(\theta)\}$是逼近最小$Q_\theta(\lambda)$过程。按(3)确定$\eta_j(\lambda)j\in E$。令$\sigma(E)$是$X$末离$E$的时刻，即

$$
\sigma(E)=\begin{cases}\sup\{t;\ \ 0\leqslant t<\sigma(\theta),\ x(t)\in E\},\\ -\infty,\ \text{如果上面集合为空集。}\end{cases}\tag{11}
$$

则$X_E=\{X(t),\ \ 0\leqslant t<\sigma(E)\}$是定义在$\Omega(E)=(-\infty<\sigma(E))$上的逼近最小$Q(\lambda)$过程，它的$\lambda$－特征测度是$\eta(\lambda)$。

定理 3 设$(\eta(\lambda),\ \lambda>0)$为流入族，数$\mu>0$指定。则存在测度空间$(\Omega,\ \mathscr{F},\ P)$及定义在其上的逼近最小$Q(\mu)$过程$Y=\{y(t),\ \ 0\leqslant t<\sigma(E)\}$和逼近最小$Q$过程$X=\{x(t),\ 0\leqslant t<\sigma\}$，使得

(i) X是Y的延拓，即$\sigma(E)\leqslant\sigma$且对$t\in[0,\ \sigma(E))$有$y(t)=x(t)$;

(ii) $P(y(0)=i)=P(x(0)=i)$，记为v_i，$i\in E$; 特别地，如果$v=0$，则几乎必然$y(0)=x(0)=\infty$;

(iii) 设τ_n为Y首达D_n的时刻，则$\tau_n\downarrow 0(n\uparrow+\infty)$，$\tau_n$引导$Y$和$X$到$\Omega_n=(\tau_n<+\infty)$上的最小$Q(\mu)$过程$Y_n$和最小$Q$过程$X_n$，初

始分布都是$\nu^{\pi}(\mu)$；

(iV) Y的特征测度是$\eta(\mu)$；

(V) X的特征流入族是$(\eta(\lambda),\ \lambda>0)$。

(Vi) X的特征测度是$(\eta\ (\lambda),\ \lambda>0)$的标准映象$\eta$(见(1.11.9))。

证 由定理1，存在测度空间 $(\Omega,\ \mathscr{F},\ P)$，在其上可以定义逼近最小$Q_{\theta}(\mu)$过程$Y_{\theta}=(y(t),\ \ 0\leqslant t<\sigma(\theta))$使$P\ (y(0)=\theta)=0$，$P(y(0)=i)=v_i$，而且

$$\eta_j(\mu)=\int_{\Omega}\int_0^{\sigma(\theta)}C_j[y(t)]dtdP,\ \ j\in E. \tag{12}$$

作独立乘积空间的技巧，还可以假定在$(\Omega,\ \mathscr{F},\ P)$上定义了一族准随机过程 $Z_i=(Z_i(t),\ \ 0\leqslant t<\varepsilon_i)(i\in E)$，使在条件$\mathscr{F}(Y_{\theta})$之下，诸$Z_i(i\in E)$相互条件独立，而且$Z_i$的条件分布是从$i$出发的最小Q过程的分布。

令$\sigma(E)$为Y_{θ}末离E的时刻。则显然$Y=(y(t),\ \ 0\leqslant t<\sigma(E))$是逼近最小$Q(\lambda)$过程，而且Y的特征测度为$\eta(\mu)$，即(iV)成立。

令

$$\sigma=\begin{cases}\sigma(E) & \text{如}\sigma(E)<\sigma(\theta)\text{或}\sigma(E)=+\infty,\\ \sigma(E)+\varepsilon_{y(\sigma(E)-0)} & \text{如}\sigma(E)=\sigma(\theta)<+\infty.\end{cases} \tag{13}$$

对$0\leqslant t<\sigma$，令

$$X(t)=\begin{cases}y(t), & \text{如果}0\leqslant t<\sigma(E),\\ Z_{y(\sigma(E)-0)}(t-\sigma(E)), & \text{如果}\sigma(E)=\sigma(\theta)<+\infty,\end{cases}$$

而且 $\quad\sigma(E)\leqslant t<\sigma.$ (14)

则X是Y的延拓，即(i)成立，而且(ii)成立，设X的特征测度是η，则

$$\eta_j=\int_{\Omega}\int_0^{\sigma}C_j[x(t)]dtdp=\int_{\Omega}\int_0^{\sigma(E)}C_j[y(t)]dtdp$$

$$+\int_{(\sigma(E)=\sigma(\theta)<+\infty)}\int_{\sigma(E)}^{\sigma}C_j[Z_{y(\sigma(E)-0)}\ (t-\sigma(E))dtdP$$

$$=\eta_j(\mu)+\sum_{i\in E}\int_{(\sigma(E)=\sigma(\theta)<+\infty,\,y(\sigma(E))=i)}\int_0^{\varepsilon_i}C_j[Z_i(u)]dudp.$$

由条件独立性及(11.5)，上式等于

$$\eta_j(\mu)+\sum_{i\in E}P(y(\sigma(E))=i,\ \sigma(E)=\sigma(\theta)<+\infty)\Gamma_{ij}$$

$$=\eta_j(\mu)+\sum_{i\in E}P(y(\sigma(\theta)=i)\Gamma_{ij}$$

$$=\eta_j(\mu)+\mu\sum_{i\in E}\eta_i(\mu)\Gamma_{ij}.$$

这正是$(\eta(\lambda),\ \lambda>0)$的标准映象(1.11.9)。得证(Vi)。

往证(iii)。设τ_n是Y首达D_n的时刻，则$\tau_n\downarrow 0$。设τ_n将Y_θ，Y和X分别引导到$\Omega_n=(\tau_n<+\infty)$上的$Y_{n\theta}$，$Y_n$和$X_n$。由定理1，$Y_n$是具初始分布为$\nu^n(\mu)$的最小$Q(\mu)$过程。显然$X_n$是$Y_n$的延拓，而且有相同的初始分布。因此，如果能证明$X_n$是最小$Q$过程，则得$X$是逼近最小$Q$过程。

正象从Y_θ和$Z_i(i\in E)$出发，按（13）（14）方式，我们得到X一样，在Ω_n上，从$Y_{n\theta}$和$Z_i(i\in E)$出发，按（13）（14）方式，我们得到X_n。因此，我们只需证明：假定Y_θ是最小$Q_\theta(\lambda)$过程，则按（13）（14）方式得到的X是最小Q过程。注意，在上述假设下，Y是最小$Q(\lambda)$过程。

设$i_0,\ i_1,\ \cdots i_{n+1}\in E$，$0\leqslant t_0<t_1<\cdots<t_{n+1}$。令

$$\Delta_k=\bigcap_{j=0}^{k}(x(t_j)=i_j),\ \Lambda_k=\bigcap_{j=0}^{k}(y(t_j)=i_j),$$

$$R_K(i,\ u)=\sum_{j=k}^{n+1}(Z_i(t_j-u)=i_j,\ 0\leqslant t_j-u\leqslant\varepsilon_i).$$

要证 $$P(\Delta_{n+1})=P(\Delta_n)\bar{P}_{i_n i_{n+1}}(t_{n+1}-t_n),\tag{15}$$

显然 $$P(\Delta_{n+1})=A_{n+1}+B_{n+1}+C_{n+1}.\tag{16}$$

其中 $A_{n+1}=P(\Delta_{n+1},\ t_{n+1}<\sigma(E))$，

$B_{n+1}=P(\Delta_{n+1},\ \sigma(E)\leqslant t_0)$，

$$C_{n+1}=\sum_{k=0}^{n}P(\Delta_{n+1},\ t_K<\sigma(E)\leqslant t_{K+1}).$$

因为 $(\Delta_{n+1},\ t_{n+1}<\sigma(E))=(\Lambda_{n+1},\ t_{n+1}<\sigma(E))=\Lambda_{n+1}$，

故 $A_{n+1}=P(\Lambda_{n+1})=P(\Lambda_n)\bar{P}^{\lambda}_{i_n i_{n+1}}(t_{n+1}-t_n)$

$$= A_n \overline{P}_{i_n i_{n+1}}(t_{n+1} - t_n) e^{-\lambda(t_{n+1} - t_n)}. \tag{17}$$

由条件独立性

$$B_{n+1} = \sum_{i \in E} P(y(\sigma(E) - 0) = i, \ R_0(i, \ \sigma(E))$$

$$= \sum_i \int_0^{t} \overline{P}_{i_0 i_1}(t_1 - t_0) \cdots \overline{P}_{i_n i_{n+1}}(t_{n+1} - t_n)$$

$$\cdot d_u P(y(\sigma(E) - 0)$$

$$= i, \ \sigma(E) \leqslant u, \ Z_i(t_0 - \sigma(E)) = i_0)$$

$$= \sum_i \int_0^{t} \overline{P}_{i_0 i_1}(t_1 - t_0) \cdots \overline{P}_{i_{n-1} i_n}(t_n - t_{n-1})$$

$$\cdot d_u P(y(\sigma(E) - 0)$$

$$= i, \ \sigma(E) \leqslant u, \ Z_i(t_0 - \sigma(E)) = i_0) \overline{P}_{i_n i_{n+1}}$$

$$\cdot (t_{n+1} - t_n)$$

$$= B_n \overline{P}_{i_n i_{n+1}}(t_{n+1} - t_n),$$

类似地计算

$$P(\Delta_{n+1}, \ t_K < \sigma(E) \leqslant t_{K+1}) = P(\Delta_n, \ t_K < \sigma(E) \leqslant t_{K+1})$$

$$\overline{P}_{i_n i_{n+1}}(t_{n+1} - t_n),$$

故
$$C_{n+1} = C_n \overline{P}_{i_n i_{n+1}}(t_{n+1} - t_n) + P(\Delta_{n+1}, \ t_n < \sigma(E)$$

$$\leqslant t_{n+1}).$$

于是为证 (15)，只要证

$$P(\Delta_{n+1}, \ t_n < \sigma(E) \leqslant t_{n+1}) = A_n \overline{P}_{i_n i_{n+1}}(t_{n+1} - t_n) \cdot$$

$$\cdot (1 - e^{-\lambda(t_{n+1} - t_n)}). \tag{18}$$

注意
$$L_{ki} \equiv P(y(\sigma(E) - 0) = i | y(0) = k)$$

$$= \sum_{m=0}^{+\infty} \Pi_{ki}^{(m)}(\lambda)[1 - \sum_{j \in E_\partial} \Pi_{ij}(\lambda)]$$

$$= G_{ki}(\lambda)(\lambda + q_i)^{-1}\lambda = \lambda \phi_{ki}(\lambda) = \lambda \int_0^{+\infty} e^{-\lambda t} \overline{P}_{ki}(t) dt,$$

$$L_{ki}(u) \equiv P(y(\sigma(E) - 0) = i, \ \sigma(E) \leqslant u | y(0) = k)$$

$$=L_{ki}-\sum_{l\in E}P(y(u)=l,\ u<\sigma(E),\ y(\sigma(E)-0)$$

$$=i\,|\,y(0)=k)=L_{ki}-\sum_{l\in E}\bar{P}^{\lambda}_{kl}(u)L_{li},$$

但是 $$\sum_{l\in E}\bar{P}^{\lambda}_{kl}(u)L_{li}=\sum_{l\in E}e^{-\lambda u}\bar{P}_{kl}(u)\lambda\int_{0}^{+\infty}e^{-\lambda t}\bar{P}_{li}(t)dt$$

$$=\lambda\int_{0}^{+\infty}e^{-\lambda(u+t)}\bar{P}_{ki}(u+t)dt.$$

$$=\lambda\int_{u}^{+\infty}e^{-\lambda t}\bar{P}_{ki}(t)dt,$$

所以 $$L_{ki}(u)=\lambda\int_{0}^{u}e^{-\lambda t}\bar{P}_{ki}(t)dt,$$

$$dL_{ki}(u)=\lambda e^{-\lambda u}\bar{P}_{ki}(u)du.$$

简记$\delta=t_{n+1}-t_{n}$。由条件独立性，

$$P(\Delta_{n+1},\ t_n<\sigma(E)\leqslant t_{n+1})$$

$$=\sum_{i}P(\Delta_{n+1},\ y(\sigma(E)-0)=i,\ t_n<\sigma(E)\leqslant t_{n+1})$$

$$=\sum_{i}P(A_n,\ y(\sigma(E)-0)=i,\ Z_i(t_{n+1}-\sigma(E))$$

$$=i_{n+1},\ t_n<\sigma(E)\leqslant t_{n+1})$$

$$=\sum_{i}\int_{t_n}^{t_{n+1}}\bar{P}_{ii_{n+1}}(t_{n+1}-u)d_uP(A_1,\ y(\sigma(E)-0)=i,$$

$$\sigma(E)\leqslant u)$$

$$=\sum_{i}\int_{t_n}^{t_{n+1}}\bar{P}_{ii_{n+1}}(t_{n+1}-u)d_uP(A_n)L_{i_ni}(u-t_n)$$

$$=A_n\sum_{i}\int_{0}^{\delta}\bar{P}_{ii_{n+1}}(\delta-u)duL_{i_1i}(u)$$

$$=A_n\sum_{i}\int_{0}^{\delta}\bar{P}_{ii_{n+1}}(\delta-u)\lambda e^{-\lambda u}\bar{P}_{i_ni}(u)du$$

$$=A_n\sum_{i}\int_{0}^{\delta}\lambda e^{-\lambda u}\bar{P}_{i_ni_{n+1}}(\delta)du$$

$$=A_n\overline{P}_{i_n i_{n+1}}(\delta)(1-e^{-\lambda\delta}).$$

得证(18)。

往证（V）。先设X是最小Q过程，初始分布为ν。则

$$\int_\Omega\int_0^\sigma e^{-\lambda t}C_j[x(t)]dtdp$$

$$=\int_0^{+\infty}e^{-\lambda t}\int_{(t<\sigma)}C_j[x(t)]dPdt$$

$$=\int_0^{+\infty}e^{-\lambda t}\sum_i\nu_i\overline{P}_{ij}(t)dt=\sum_i\nu_i\phi_{ij}(\lambda)$$

$$=[\nu\phi(\lambda)]_j.$$

现设X是逼近最小Q过程，则

$$\int_\Omega\int_0^\sigma e^{-\lambda t}C_j[x(t)]dtdp$$

$$=\lim_{n\to+\infty}\int_{\Omega_n}\int_0^{\sigma_n}e^{-\lambda t}C_j[x_n(t)]dtdp$$

$$=\lim_{n\to+\infty}[\nu^n(\mu)\phi(\lambda)]_j$$

但由$\phi(\lambda)$的预解方程

$$\phi(\lambda)=\phi(\mu)+(\mu-\lambda)\phi(\mu)\phi(\lambda)$$

及（2），我们有

$$\nu^n(\mu)\phi(\lambda)=\nu^n(\mu)\phi(\mu)+(\mu-\lambda)\nu^n(\mu)\phi(\mu)\phi(\lambda)$$

$$\longrightarrow\eta(\mu)+(\mu-\lambda)\eta(\mu)\phi(\lambda)=\eta(\lambda).$$

从而得(V)。定理证完。

§15. 全有限测度空间上的逼近最小Q过程

回忆定义7.5.1. 设$(\eta(\lambda),\ \lambda>0)$是流入族。如果

$$\lim_{\lambda\to+\infty}\lambda\eta(\lambda)1=M<+\infty,\tag{1}$$

则称（$\eta(\lambda)$，$\lambda>0$)为非黏流入族，当$M\leqslant1$时，流入族称为概率

的，这里$\eta(\lambda)1=\sum_j \eta_j(\lambda)$。

定理1 设$X=\{x(t),\ 0\leqslant t<\sigma\}$是定义在测度空间（$\Omega$，$\mathscr{F}$，$P$）上的逼近最小$Q$过程，对每个$\lambda>0$，$X$的$\lambda$特征测度是$\eta(\lambda)$。则（$\eta(\lambda)$，$\lambda>0$）是流入族。

证 仿定理14.3中证(V) 的方式进行。

定理2 设$X=\{x(t),0\leqslant t<\sigma\}$是定义在全有限测度空间($\Omega$，$\mathscr{F}$，$P$) 上的逼近最小$Q$过程，$X$的特征流入族为($\eta(\lambda)$，$\lambda>0$)。则（$\eta(\lambda)$，$\lambda>0$)是非黏的流入族。如果($\Omega$，$\mathscr{F}$,$P$)是概率空间，则($\eta(\lambda)$，$\lambda>0$)还是概率的流入族。

证
$$\lambda\eta(\lambda)1=\int_\Omega\int_0^\sigma \lambda e^{-\lambda t}dtdP$$
$$=\int_{(\sigma>0)}(1-e^{-\lambda\sigma})dp.$$
$$\uparrow P(\sigma>0)<+\infty,\ \lambda\uparrow\infty. \tag{2}$$

定理3 设($\eta(\lambda)$，$\lambda>0$) 是非黏流入族。则存在有限测度空间（Ω，$\mathscr{F}$，P）上的逼近最小Q过程$X=\{x(t),\ 0\leqslant t<\sigma\}$，它的特征流入族就是($\eta(\lambda)$，$\lambda>0$)。

证 依定理14.3，存在测度空间(Ω,$\mathscr{F}$，P)及其上的逼近最小Q过程X，它的特征流入族就是($\eta(\lambda)$，$\lambda>0$)。由($\eta(\lambda)$，$\lambda>0$)的非黏性及（2）有$P(\sigma>0)<+\infty$。清洗($\sigma=0$)后，我们可以假设$\Omega=(\sigma>0)$，因而(Ω，$\mathscr{F}$,P)是全有限测度空间。证完。

定理4 设($\eta(\lambda)$，$\lambda>0$)是流入族(相应地，概率的流入族)，则存在测度空间（相应地，概率空间)(Ω，$\mathscr{F}$，P）及定义在其上的逼近最小Q过程$X=\{x(t),\ 0\leqslant t<\sigma\}$使得
$$P(\sigma>0)=\lim_{\lambda\to\infty}\lambda\eta(\lambda)1 \tag{3}$$
$$\eta_j(\lambda)=\int_\sigma\int_0^\Omega e^{-\lambda t}C_j[x(t)]dtdp$$
$$=\int_0^{+\infty}e^{-\lambda t}H_j(t)dt,\ \lambda>0 \tag{4}$$
其中 $$H_j(t)=P(x(t)=j),\ t\geqslant 0 \tag{5}$$

是$\eta_j(\lambda)$，$\lambda>0$的反拉普拉斯变换。设（$\eta(\lambda),\lambda>0$）的Riesz分解为：

$$\eta(\lambda)=\alpha\phi(\lambda)+\bar{\eta}(\lambda) \tag{6}$$

则 $\alpha_i=P(x(0)=i), P(x(0)=\infty)=\lim\limits_{\lambda\to+\infty}\lambda\bar{\eta}(\lambda)1$ (7)

证 只需说明（7）第二式。实际上，

$$P(x(0)=\infty)=1-\sum_i P(x(0)=i)$$

$$=\lim_{\lambda\to+\infty}\lambda\eta(\lambda)1-\lim_{\lambda\to+\infty}\lambda\nu\phi(\lambda)1$$

$$=\lim_{\lambda\to+\infty}\lambda\bar{\eta}(\lambda)1. \tag{8}$$

§16. 非黏返回过程轨道的构造：DV型延拓和(DV)*型延拓

本节假设由矩阵Q确定的最小解$(\bar{P}_{ij}(t))$中断。设$\bar{X}=\{\bar{x}(t), t<\bar{\sigma}\}$是定义在概率空间$(\Omega, \mathscr{F}, P)$上的最小$Q$过程。则

$$L_i(t)=P_i(\bar{\sigma}\leqslant t)=1-\sum_j \bar{P}_{ij}(t). \tag{1}$$

$$R_i(t)=P_i(\bar{\sigma}\leqslant t, \bar{x}(\bar{\sigma}-0)=\infty). \tag{2}$$

由Q唯一确定，这里$P_i(\cdot)=P(\cdot\,|\,\bar{x}(0)=i)$。

下面恒设（$\eta(\lambda)$，$\lambda>0$）是给定的概率流入族。$H_i(t)$是（$\eta_j(\lambda)$，$\lambda>0$）的拉普拉斯反变换。令

$$H(t)=\sum_{i\in E}H_i(t),\quad H^0(t)=\begin{cases}0, & 如t<0,\\ 1, & 如t\geqslant 0,\end{cases}$$

$H^{n+1}=H^n*H$，$*$表示卷积

$$M_i=\sum_{n=0}^{+\infty}(L_i*H^n), \tag{3}$$

$$P_{ij}(t)=\bar{P}_{ij}(t)+\int_0^t H_j(t-s)dM_i(s). \tag{4}$$

引理1 存在概率空间（Ω，$\mathscr{F}$，P），在其上可以定义最小Q过程$X^0=\{x^0(t), 0\leqslant t<\sigma^0\}$和一列逼近最小$Q$过程$X^n=\{x^n(t),$

$0\leqslant t<\sigma^n\}$, $n\geqslant 1$, 每个$X^n(n\geqslant 1)$的特征流入族都是$(\eta(\lambda),\ \lambda>0)$。它们具有下列性质：

(i) $(\sigma_n=0)\cup(\sigma_n=+\infty)\subset(\sigma_{n+1}=0)$

(ii) 记 $\Delta_n=(0<\sigma_n<+\infty)$, $\Omega_n=(\sigma_n>0)$,

则 $$P(\Omega_{n+1}|\Delta_n)=M,$$

其中M由（15.1）确定。

(iii) 在条件Δ_n或Ω_{n+1}之下，诸$X^m(m\leqslant n)$和诸$X^m(m\geqslant n+1)$条件独立。

证 仿引理10.2.1证明。

定理2 对于引理1中的X^0和$X^n(n\geqslant 1)$,

令 $$\tau^0=0,\ \tau^{n+1}=\sum_{m=0}^{n}\sigma^m,\ \sigma=\sum_{m=0}^{+\infty}\sigma^m. \tag{5}$$

对$0\leqslant t<\sigma$, 令

$$X(t)=X^n(t-\tau^n)\quad 如果\tau^n\leqslant t<\tau^{n+1} \tag{6}$$

则$X=\{x(t),\ 0\leqslant t<\sigma\}$是马氏过程，其转移概率由（3）给出。

证 仿定理10.2.2证明。

注1 在（65）两边取拉普拉斯变换，我们得

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+Z_i(\lambda)\frac{\eta_j(\lambda)}{(1-M)+\lambda\eta(\lambda)1}. \tag{7}$$

其中 $Z(\lambda)=1-\lambda\phi(\lambda)1$.

注2 当Q保守，或者当$(\eta(\lambda),\ \lambda>0)$是流入解族时，定理2中的$X$是$Q$过程，但一般情况下未必。

注3. 定理2中的马氏过程X称为最小Q过程X^0的DV型延拓。设流入族$(\eta(\lambda),\ \lambda>0)$的Riesz分解为(15.6)。当$\eta(\lambda)=0$，则$X^0$的DV型延拓X化为§2中的D型延拓。当$a=0$时，我们称DV型延拓为V型延拓，此时X的飞跃区间除第一个外，全是$_\infty$U区间。

引理3 设对某个$i\in E$及$t>0$, 由(2)式确定的$R_i(t)>0$. 则存在概率空间$(\Omega,\ \mathscr{F},P)$及定义在其上的最小Q过程$X^0=\{x^0(t),\ 0\leqslant t<\sigma^0\}$和一列逼近最小$Q$过程$X^n=\{x^n(t),\ 0\leqslant t<\sigma^n\}$, $n\geqslant 1$,

每个$X^n(n\geqslant 1)$的特征流入族是$(\eta(\lambda),\ \lambda>0)$。它们满足下列条件：

(i) $(\sigma^n=0)\cup(\sigma^n=+\infty)\cup(X^n(\sigma^n-0)\in E)\subset(\sigma^{n+1}=0)$，$n\geqslant 0$；1)

(ii) 设$\Delta_n=(0<\sigma^n<+\infty,\ X^n(\sigma^n-0)=\infty)$，$\Omega_n=(\sigma_n>0)$，则

$$P(\Omega_{n+1}|\Delta_n)=M$$

其中M由(15.1)确定。

(iii) 在条件Δ_n或Ω_{n+1}之下，诸$X^m(m\leqslant n)$和诸$X^m(m\geqslant n+1)$条件独立。

证 仿引理10.2.1证明。

定理4 对于引理8中的X^0和X^n，按(5)(6)定义$X=\{x(t),\ 0\leqslant t<\sigma\}$。则$X$是$Q$过程，其转移概率为

$$p_{ij}(t)=\bar{p}_{ij}(t)+\int_0^t H_j(t-s)dN_i(s)。\tag{8}$$

其中
$$N_i=\sum_{n=0}^{+\infty}(R_i*H^n)。\tag{9}$$

证 仿定理10.2.2证明。

注1 在(8)两边取拉氏变换得

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+\zeta_i(t)\frac{\eta_j(\lambda)}{1-M+\lambda\eta(\lambda)1}。\tag{10}$$

其中$\zeta_i(\lambda)$是$R_i(t)$的拉普拉斯变换，即

$$\zeta(\lambda)=1-\lambda\phi(\lambda)1-\phi(\lambda)d。\tag{11}$$

其中$d=-Q1$。

注2 定理4中的X的第一个飞跃点τ即σ^0，而且如α按(15.6)确定，则

$$P(X(\tau)=i|\tau<\infty,\ X(\tau-0)=\infty)$$

1) 当$\sigma^n=0$时，$x^n(\sigma^n-0)$不确定。

$$
= \begin{cases} \alpha_i, & \text{如} i \in E, \\ \lim\limits_{\lambda \to +\infty} \lambda \eta(\lambda) 1, & \text{如} i = \infty, \end{cases} \tag{12}
$$

$$P(\sigma = \tau \mid \tau < \infty,\ X(\tau - 0) = \infty) = 1 - M,$$

$$P(\sigma = \tau \mid X(\tau - 0) \in H_e) = 1,$$

只要上面的条件概率有意义，H_e是非保守状态集。

注3. 定理4中的Q过程X称为最小Q过程X^0的$(DV)^*$型延拓。$(DV)^*$型延拓保持Q矩阵不变。和定理2的注3类似，当$\eta(\lambda) = 0$时，$(DV)^*$型延拓化为D^*型延拓；当$\alpha = 0$时，$(DV)^*$型延拓称为V^*型延拓。虽然D型延拓和D^*型延拓一般说来是不同的延拓，但V型延拓和V^*型延拓却是同一延拓。因此，我们只使用V^*型延拓一词。

§17. 广义DV型延拓和广义(DV)*型延拓

保留§5开头直至（5.2）的假定和记号，并记$A_e = H \cup B_e$。A_e上的Borel集合全体记为$\mathscr{B}_e$。对于（5.1）和（5.2）中的$L_i(\Gamma, t)$和$h_i(\Gamma, \lambda)$，有$\lim\limits_{t\to\infty} L_i(\Gamma, t) = \lim\limits_{\lambda \downarrow 0} h_i(\Gamma, \lambda)$，记此极限为$h_i(\Gamma)$。

假设给定$A_e \times \mathscr{B}_e$到$[0, 1]$中的映射$G(\cdot, \cdot)$，满足

(i) 当$a \in A_e$时，$G(a, \cdot)$是$\mathscr{B}_e$上的测度，$G(a, A_e) \leqslant 1$；

(ii) 当$\Gamma \in \mathscr{B}_e$时，$G(\cdot, \Gamma)$是$\mathscr{B}_e$可测函数。

还假设给定$A_e \times (0, \infty)$到巴拿赫空l间中的映射$\eta(\cdot, \cdot)$，满足

(i) 当$a \in A_e$时，$(\eta(a, \lambda), \lambda > 0)$是概率的流入族；

(ii) 当$\lambda > 0$时，$\eta(\cdot, \lambda)$是A_e到l的$\mathscr{B}_e$可测映射。

设$\eta_j(a, \lambda)$，$\lambda > 0$的拉普拉斯反变换为$H_j(a, t)$。令

$$W(a, \Gamma, \lambda) = \lambda \sum_j \eta_j(a, \lambda) h_j(\Gamma). \tag{1}$$

由引理1.11.4，存在极限

$$W(a, \Gamma, \lambda)\uparrow W(a, \Gamma), \ \lambda\uparrow\infty.$$

令

$$C(a, \Gamma, t)=W(a, \Gamma)-\sum_j H_j(a, t)h_j(\Gamma). \qquad (2)$$

下面的引理指出了$W(a, \Gamma)$及$C(a, \Gamma, t)$的概率意义。

引理1 指定a设逼近最小Q过程 $X=\{X(t), 0\leqslant t<\sigma\}$ 以$(\eta(a, \lambda), \lambda>0)$为特征流入族。则

$$P(X(\sigma-0)\in\Gamma)=W(a, \Gamma), \qquad (3)$$

$$P(X(\sigma-0)\in\Gamma, \sigma\leqslant t)=C(a, \Gamma, t). \qquad (4)$$

证。 由定义13.2后面的注意，$\{X(t), 0<t<\sigma\}$是开最小Q过程。故

$$P(t<\sigma, x(\sigma-0)\in\Gamma)=\sum_j P(X(t)=j, t<\sigma,$$

$$X(\sigma-0)\in\Gamma)=\sum_j H_j(a, t)h_j(\Gamma). \qquad (5)$$

于是在上式中令$t\downarrow 0$得

$$P(X(\sigma-0)\in\Gamma)=\lim_{t\downarrow 0}\sum_j H_j(a, t)h_j(\Gamma)$$

$$=\lim_{\lambda\to\infty}\lambda\sum_j\eta_j(a, \lambda)h_j(\Gamma)$$

$$=\lim_{\lambda\to\infty}W(a, \Gamma, \lambda)=w(a, \Gamma).$$

由(3)及(5)得(4)。证完。

注。 当$(\eta(a, \lambda), \lambda>0)$是一般的(即概率的或非概率的)流入族时，引理1也成立。

令

$$F(a, \Gamma, \lambda)=\int_0^\infty e^{-\lambda t}C(a, \Gamma, dt)$$

$$=\lambda\int_0^\infty e^{-\lambda t}C(a, \Gamma, t)dt. \qquad (6)$$

由(2)得

$$F(a,\ \Gamma,\ \lambda)=W(a,\ \Gamma)-W(a,\ \Gamma,\ \lambda). \tag{7}$$

定义

$$L_i^1(L,\ t)=L_i(\Gamma,\ t)(\text{见 (5.1) 式}), \tag{8}$$

$$L_i^{n+1}(\Gamma,t)=\int_0^t\int_{A_e}\int_{A_e}G(a,db)C(b,\Gamma,t-s)L_i^n(da,\ ds), \tag{9}$$

$$K_i(\Gamma,t)=\sum_{n=0}^{\infty}L_i^{n+1}(\Gamma,t), \tag{10}$$

$$p_{ij}(t)=f_{ij}(t)+\int_0^t\int_{A_e}\int_{A_e}G(a,db)H_j(b,t-s)K_i(da,ds), \tag{11}$$

其中$(f_{ij}(t))$是最小解。上面的$P(t)=(p_{ij}(t))$由Q及$G(\cdot,\cdot)$和$\eta(\cdot,\cdot)$唯一确定。

考虑到定理 15.4，利用独立乘积空间的技巧易得下面的引理。

引理2 存在概率空间$(\Omega,\ \mathscr{F},\ P)$，在其上可以定义最小$Q$过程$X^0=\{X^0(t),\ 0\leqslant t<\sigma^0\}$，逼近最小$Q$过程族$X_a^n=\{X^n(a,t),\ 0\leqslant t<\sigma^n(a)\}$，$a\in A_e$，$n\geqslant1$，以及取值于$A_e$的随机变量族$f^n(a)$，$a\in A_e$，$n\geqslant0$，使得

(i) $P(X^0(0)\in E)=1$；

(ii) X_a^n $(n\geqslant1)$的特征流入族是$(\eta(a,\ \lambda),\ \lambda>0)$；

(iii) $P(f^n(a)\in\Gamma)=G(a,\Gamma)$；

(iv) 诸量X^0，X_a^n $(a\in A_e,\ n\geqslant1)$，$f^n(a)(a\in A_e,\ n\geqslant0)$相互独立。

当$\sigma^0>0$且$X^0(\sigma^0-0)=a\in A_e$时，令$X^1(t)=X^1(f^0(a),t)$对$t<\sigma^1=\sigma^1(f^0(a))$；否则令$\sigma^1=0$。$X^1=\{X^1(t),\ 0\leqslant t<\sigma^1\}$显然是逼近最小$Q$过程。当$\sigma^1>0$且$X^1(\sigma^1-0)=a\in A_e$时，令$X^2(t)=X^2(f^1(a),t)$对$t<\sigma^2=\sigma^2(f^1(a))$；否则令$\sigma^2=0$。如此继续，我们得到最小$Q$过程$X^0$和一列逼近最小$Q$过程$X^n=\{X^n(t),$

$0\leqslant t<\sigma^n\}(n\geqslant 1)$。按(16.5)和(16.6)定义$X=\{X(t),\ 0\leqslant t<\sigma\}$。

定理3 X是时间齐次的马氏过程，其转移概率矩阵$P(t)=(p_{ij}(t))$由(11)给出，预解矩阵$\psi(\lambda)=(\psi_{ij}(\lambda))$由下式给出：

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+\int_{A_e}\int_{A_e}h_i(da,\ \lambda)\left[\sum_{n=0}^{\infty}V^n(a,db,\lambda)\right]\eta_j(b,\ \lambda), \tag{12}$$

其中

$$\left.\begin{aligned}&V^0(a,\ \Gamma,\ \lambda)=\begin{cases}0, & 如a\overline{\in}\Gamma,\\ 1, & 如a\in\Gamma,\end{cases}\\ &V^1(a,\ \Gamma,\ \lambda)=\int_{A_e}G(a,db)F(b,\Gamma,\lambda),\\ &V^{n+1}(a,\Gamma,\lambda)=\int_{A_e}V^n(a,db,\lambda)V^1(b,\Gamma,\lambda).\end{aligned}\right\} \tag{13}$$

证 第一个结论的证明可完全仿照定理5.2的证明进行，这里仅指出如何求出$p_{ij}(t)$以及引理2是如何被应用的。

很明显，

$$p_i(X(t)=j)=P_i(X^0(t)=j,\ t<\sigma^0)+\sum_{n=0}^{\infty}R_{ij}^{n+1}(t),$$

其中第一项等于$f_{ij}(t)$，而

$$\begin{aligned}R_{ij}^{n+1}(t)&=p_i(x^{n+1}(t-\tau^{n+1})=j,\ \tau^{n+1}\leqslant t<\tau^{n+1})\\&=\int_0^t\int_{A_e}\int_{A_e}p_i(\tau^{n+1}\in ds,\ X^n(\sigma^n-0)\in da,\\&\qquad f^n(a)\in db,\ X^{n+1}(b,\ t-s)=j,\ t-s<\sigma^{n+1})\end{aligned}$$

应用引理2，上式化为

$$R_{ij}^{n+1}(t)=\int_0^t\int_{A_e}\int_{A_e}G(a,\ db)H_j(b,\ t-s)L_i^{n+1}(da,ds),$$

其中

$$L_i^{n+1}(\Gamma,t)=p_i(\tau^{n+1}\leqslant t,\ X^n(\sigma^n-0)\in\Gamma). \tag{14}$$

因而

$$L_i^1(\Gamma,\ t)=L_i(\Gamma,\ t),$$

$$L_i^{n+1}(\Gamma,\ t)=\int_0^t\int_{A_e}\int_{A_e}p_i(\tau^n\in ds,$$
$$X^{n-1}(\sigma^{n-1}-0)\in da,\ f^{n-1}(a)\in db,$$
$$X^n(b,\ \sigma^n(b)-0)\in\Gamma,\ \sigma^n(b)\leqslant t-s)$$

再次应用引理2便得(9)，从而得 $p_i(X(t)=j)$ 等于由(11)确定的 $p_{ij}(t)$。

下面计算$p_{ij}(t)$的拉普拉斯变换 $\psi_{ij}(\lambda)$。在(11) 中两边取拉普拉斯变换得

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)$$
$$+\int_{A_e}\int_{A_e}G(a,db)\eta_j(b,\ \lambda)\ \sum_{n=0}^{\infty}h_i^{n+1}(da,\lambda), \tag{15}$$

其中

$$h_i^{n+1}(\Gamma,\ \lambda)=\int_0^{\infty}e^{-\lambda t}\Gamma_i^{n+1}(\Gamma,\ dt). \tag{16}$$

在(9)两边取拉普拉斯变换得

$$h_i^{n+1}(\Gamma,\lambda)=\int_{A_e}\int_{A_e}G(a,\ db)F(b,\Gamma,\lambda)h_i^n(da,\ \lambda). \tag{17}$$

由此可得

$$h_i^{n+1}(\Gamma,\ \lambda)=\int_{A_e}V^n(a,\Gamma,\lambda)h_i(da,\ \lambda). \tag{18}$$

实际上，上式对 $n=0$ 显然成立。设上式对 $n=m-1$ 成立，则

$$\int_{A_e}V^m(a,\Gamma,\lambda)h_i(da,\ \lambda)$$
$$=\int_{A_e}\left[\int_{A_e}V^{m-1}(a,db,\lambda)V^1(b,\Gamma,\lambda)\right]h_i(da,\ \lambda)$$
$$=\int_{A_e}\left[\int_{A_e}V^{m-1}(a,db,\lambda)h_i(da,\lambda)\right]V^1(b,\Gamma,\lambda),$$

$$=\int_{A_e} h_i^m(db,\lambda)V^1(b,\Gamma,\lambda) \quad \text{（依归纳法假设）}$$

$$=\int_{A_e}\int_{A_e} h_i^m(db,\lambda)G(b,dc)F(c,\Gamma,\lambda)$$

$$=h_i^{m+1}(\Gamma,\lambda)。\quad \text{（由(17)）}$$

即(18)对$n=m$成立。

将(18)代入(15)得(12)。证完。

注1 我们称定理3中的X为X^0的广义DV型延拓过程。广义DV型延拓可能改变Q矩阵，其原因是σ^0未必一定是X的第一个飞跃点，因为概率

$$p(X(\sigma^0)\in E|\sigma^0<\infty,\ X(\sigma^0-0)=a)。\tag{19}$$

对某个$a\in H$可以为正。

注2 设$\eta(a,\lambda)$的$Riesz$分解为

$$\eta(a,\ \lambda)=\alpha(a)\phi(\lambda)+\bar{\eta}\ (a,\ \lambda)。\tag{20}$$

为使广义DV型延拓保持Q矩阵不变，必须而且只需注1中的概率为零，即

$$\int_{Ae} G(a,\ db)\sum_i \alpha_i(b)=0,\ \ a\in He \tag{21}$$

或等价地

$$\int_{Ae} G(a,db)\alpha(b)=0,\ \ a\in He。\tag{22}$$

上述条件满足时的广义DV型延拓称为广义$(DV)^*$型延拓。显然，当Q保守时，广义$(DV)^*$型延拓与广义DV型延拓一致。对于广义$(DV)^*$型延拓，σ^0是X的第一个飞跃点，而且

$$p(X(\sigma^0)=i|\sigma^0<\infty,\ X(\sigma^0-0))$$

$$=\begin{cases}\int_{Ae} G(X(\sigma^0-0),\ db)\alpha_i(b), & \text{如 } i\in E\\ \int_{Ae} G(X(\sigma^0-0),\ db)M(b), & \text{如 } i=\infty\end{cases} \tag{23}$$

其中 $M(b)=\lim\limits_{\lambda\to\infty}\lambda\|\ \eta(b,\ \lambda)\|$。

注3 设$\alpha(b)=0$，$b\in Ae$。此时(22)成立，因而广义DV型延拓和广义$(DV)^*$型延拓一致，故Q矩阵保持不变。因此，我们称$\alpha(a)=0$，$a\in Ae$时的广义$(DV)^*$延拓为广义V^*型延拓。象定理16.2注3中指出的那样，我们只使用广义V^*延拓一词。对于广义V^*型延拓，σ^0是X的第一个飞跃点，且(22)成为

$$p(X(\sigma^0)=i\,|\,\sigma^0<\infty,\ X(\sigma^0-0))$$

$$=\begin{cases}0, & \text{如}\quad i\in E\\ \displaystyle\int_{Ae}G(X(\sigma^0-0),\ db)M(b), & \text{如}\quad i=\infty\end{cases}\tag{24}$$

其中 $M(a)=\lim\limits_{\lambda\to\infty}\lambda\,||\eta(a,\lambda)||$。

注4 设$\psi(\lambda)$是定理7.5.1中的Q过程(注意：本节中的H即第七章中的H_e)。对$a\in Ae$，令

$$\eta(a,\lambda)=\begin{cases}\eta^{a_i}(\lambda), & \text{如}\quad a\in\text{某}a_i\in J,\\ 0, & \text{如}\quad a\in\text{某}a_j\in A-J.\end{cases}\tag{25}$$

$$G(a,\ b)=\begin{cases}G^{a_ia_k}, & \text{如}a\in\text{某}a_i\in J\text{而}b\in\text{某}a_k\in J\\ 0, & \text{其它情形。}\end{cases}\tag{26}$$

则$G(\cdot,\cdot)$及$\eta(\cdot,\cdot)$满足本节开头的要求，易证对此$G(\cdot,\cdot)$及$\eta(\cdot,\cdot)$，由(12)确定的$\psi(\lambda)$与定理7.5.1中的$\psi(\lambda)$一致。因此，对于(25)(26)确定的$G(\cdot,\cdot)$及$\eta(\cdot,\cdot)$，引理2和定理3给出了定理7.5.1中的Q过程的轨道。

第五篇　生灭过程构造论：概率方法

第十一章　生灭过程的概率构造

§1. 引　　言

当$E=\{0,1,2,\cdots\cdots\}$，Q形如(5.1.1)时，过程$X\in\mathscr{X}_s(Q)$称为生灭过程．本章中，Q恒指生灭矩阵Q．第五章中已经用分析方法构造了全部生灭过程．

在第一章§3中已指出，王梓坤教授于1958年就保守的生灭过程提出了解决构造问题的方法，极限过渡法．这种方法的优点是：构造出来的过程，轨道结构清楚，概率意义明显．为了研究过程的性质，可以先对简单的过程研究清楚，然后再过渡到极限即可．因此，这一方法在理论上和实际上都有重要的潜在价值．例如杨向群[3]，[5]，侯振挺[2]．

这一方法的逻辑基础发表于王梓坤[1]．该文构造了一切不中断的生灭过程，但构造时分别使用了g_n变换和f_n变换来处理$S<\infty$和$S=\infty$的情形．王梓坤[2]对两种情形就不中断的过程作了统一的处理．王梓坤、杨向群[1，2]对两种情形并允许过程中断的情形作了统一的处理，本章内容取自该二文．

§2. 特征数的概率意义

采用第五章§2的特征数和记号．本节允许$a_0\geqslant 0$．设$X=$

$\{x(t),\ t<\sigma\}\in\mathscr{X}_s(Q)$，$\tau$为第一个飞跃点，$\tau_n$为第$n$个跳跃点。显然

$$x(\tau-0)=0\text{或}\infty。\tag{1}$$

而且对嵌入链$X_T=\{x(\tau_n),\ \tau_n<\tau\}$有$|x(\tau_n)-x(\tau_n-0)|=1$ $(\tau_n<\tau)$。

定理1 $$X_i^1\equiv P_i\{x(\tau-0)=0\}=\frac{a_0(z-z_i)}{a_0(z-z_0)+1},\tag{2}$$

$$X_i^2\equiv P_i\{x(\tau-0)=\infty\}=\frac{a_0(z_i-z_0)+1}{a_0(z-z_0)+1}.\tag{3}$$

约定$\frac{\infty}{\infty}=1$，$0\cdot\infty=0$。

证 由定理 6.8.2，X^1满足 $f_0=-a_0$，$f_i=0(i>0)$ 的方程(5.2.19)，故依引理5.2.5，

$$X_i^1=[a_0(z_i-z_0)+1]X_0^1-a_0(z_i-z_0).\tag{4}$$

类似地，X^2满足$f_i=0(i\geqslant0)$的方程(5.2.19)，故依引理5.2.5，

$$X_i^2=[a_0(z_i-z_0)+1]X_0^2.\tag{5}$$

注意$X^1+X^2=1$。由上式得，如果$X_0^2=0$，则$X^2=0$，从而$X^1=1$。如果$X_0^2>0$，则$X^2>0$，故在正概率集$\{x(\tau-0)=\infty\}$上，依鞅收敛定理，

$$X^2_{x(\tau_n)}=P\{x(\tau-0)=\infty\,|\,x(\tau_0),\ x(\tau_1),\ \cdots,\ x(\tau_n)\}$$
$\to1$.

故$X_i^2\to1(i\to\infty)$，从而 $X_i^1\to0(i\to\infty)$。于是从(4)、(5)得 $0=[a_0(z-z_0)+1]X_0^1-a_0(z-z_0)$，$1=[a_0(z-z_0)+1]X_0^2$，从而当$a_0z<\infty$时，

$$X_0^1=\frac{a_0(z-z_0)}{a_0(z-z_0)+1},\quad X_0^2=\frac{1}{a_0(z-z_0)+1}.$$

代入(4)(5)得(2)(3)对$a_0z<\infty$成立。按约定$\frac{\infty}{\infty}=1$，(2)(3)对$a_0z=\infty$也成立，证毕。

令 $\xi_i=\begin{cases}\inf\{t\mid x(t)=i,\ t<\tau\},\\ \infty,\text{如果上面的集合为空集。}\end{cases}$ (6)

显然$P_i\{\xi_n\uparrow\tau(i\leqslant n\uparrow\infty)\mid x(\tau-0)=\infty\}=1$。 (7)

定理2 对$i\leqslant k\leqslant n$，

$$P_k\{\xi_i<\xi_n\}=\frac{z_n-z_k}{z_n-z_i},\quad P_k\{\xi_n<\xi_i\}=\frac{z_k-z_i}{z_n-z_i}. \tag{8}$$

证 由定理6.8.2，$u_k=P_k\{\xi_i<\xi_n\}$满足 $f_i=1$，$f_k=0$ $(i<k\leqslant n)$的方程(4.4.11)。由定理5.2.3和(4.4.12)得(8)第一式。第二式由$P_k\{\xi_n<\xi_i\}+P_k\{\xi_i<\xi_n\}=1$得出。

定理3 对$i\leqslant k$，

$$P_k\{\xi_i<\tau\}=\frac{z-z_k}{z-z_i},\quad P_i\{\xi_k<\tau\}=\frac{a_0(z_i-z_0)+1}{a_0(z_k-z_0)+1}. \tag{9}$$

$$P_k\{\tau\leqslant\xi_i,\ x(\tau-0)=\infty\}=\frac{z_k-z_i}{z-z_i}. \tag{10}$$

证 因为在$\{x(0)=k\}$上，

$$\left.\begin{aligned}&(\xi_i<\xi_n)\uparrow\bigcup_{n=k+1}^{\infty}(\xi_i<\xi_n)=(\xi_i<\tau),\\&(\xi_n<\xi_i)\downarrow\bigcap_{n=k+1}^{\infty}(\xi_n<\xi_i)=\{\tau\leqslant\xi_i,\ x(\tau-0)=\infty\}\end{aligned}\right\} \tag{11}$$

在 (8) 中取极限得 (9) 第一式和 (10)。其次，$u_i=P_i(\xi_k<\tau)$ $(0\leqslant i\leqslant k)$满足方程 (5.2.13) 中$n=k$，$f_i=0$ $(i<k)$，$f_k=1$的情形。由引理5.2.4得 (9) 第二式。

定理4 设$i\leqslant k\leqslant n$。则

$$\begin{aligned}E_k\{\xi_i,\ \xi_i<\xi_n\}=&\frac{z_n-z_k}{z_n-z_i}\sum_{j=i+1}^{k-1}\frac{z_n-z_j}{z_n-z_i}(z_j-z_i)\mu_j\\&+\frac{z_k-z_i}{z_n-z_i}\sum_{j=k}^{n-1}\frac{z_n-z_j}{z_n-z_i}(z_n-z_j)\mu_j.\end{aligned} \tag{12}$$

$$E_k\{\xi_n,\ \xi_n<\xi_i\}=\frac{z_n-z_k}{z_n-z_i}\sum_{j=i+1}^{k-1}\frac{z_j-z_i}{z_n-z_i}(z_j-z_i)\mu_j$$

$$+\frac{z_k-z_i}{z_n-z_i}\sum_{j=k}^{n-1}\frac{z_j-z_i}{z_n-z_i}(z_n-z_j)\mu_j. \tag{13}$$

证 由定理6.8.2和定理2，$u_k=E_k\{\xi_i,\ \xi_i<\xi_n\}$满足当$f_i=f_n=0$，$f_k=\frac{z_n-z_k}{z_n-z_i}(i<k<n)$的方程（4.4.11）。由定理 5.2.3 和（4.4.12）得（12）。（13）可类似得到。

定理5 $E_k\{\tau,\ x(\tau-0)=\infty\}=\frac{z-z_k}{[a_0(z-z_0)+1]^2}$

$$+\frac{z-z_k}{a_0(z-z_0)+1}\sum_{j=1}^{k-1}\frac{a_0(z_j-z_0)+1}{a_0(z-z_0)+1}\mu_j$$

$$+\frac{a_0(z_k-z_0)+1}{a_0(z-z_0)+1}\sum_{j=k}^{\infty}\frac{a_0(z_j-z_0)+1}{a_0(z-z_0)+1}(z-z_j)\mu_j. \tag{14}$$

证 设$u_k=E_k\{\xi_n,\ x(\tau-0)=\infty\}$ $(k\leqslant n)$。
则u_k满足当$f_k=P_k\{x(\tau-0)=\infty\}(k<n)$，$f_n=0$的方程（5.2.13），依引理5.2.4，

$$E_k\{\xi_n,\ x(\tau-0)=\infty\}=\frac{z_n-z_k}{a_0(z_n-z_0)+1}\cdot\frac{1}{a_0(z-z_0)+1}$$

$$+\frac{z_n-z_k}{a_0(z_n-z_0)+1}\sum_{j=1}^{k-1}\frac{a_0(z_j-z_0)+1}{a_0(z-z_0)+1}\mu_j$$

$$+\frac{a_0(z_k-z_0)+1}{a_0(z_n-z_0)+1}\sum_{j=k}^{n-1}\frac{a_0(z_j-z_0)+1}{a_0(z-z_0)+1}(z_n-z_j)\mu_j. \tag{15}$$

注意（7），在上式中取极限得(14)。

定理6 设$a_0=0$。

(i) 令c_{kj}为从k出发经有限($\geqslant0$) 步到达j的概率，则

$$c_{kj}=P_k\{\xi_j<\tau\}=\begin{cases}1 & \text{如果} k\leqslant j,\\ \dfrac{z-z_k}{z-z_j}, & \text{如果} k>j.\end{cases} \tag{16}$$

(ii) 最小Q过程X常返的充要条件是$z=\infty$。如 $z=\infty$，则遍历的充要条件是$\sum\limits_{i=0}^{\infty}\mu_i<\infty$。

(iii) m_i，N_i，R由(5.2.4)—(5.2.7)确定，则

$$m_i=E_i\xi_{i+1},\quad N_i=E_i\tau,\quad R=E_0\tau. \tag{17}$$

(iv) $P_k\{\tau<\infty\}=1(k\in E)$的充要条件是$R<\infty$。

证 (i)由(9)得出。

(ii) 从0出发有穷($\geqslant 1$)步跳跃回到0的概率

$$f_0^*=\frac{b_0}{a_0+b_0}P_1(\xi_0<\tau)=\frac{z-z_1}{z-z_0}.$$

由此知$f_0^*=1$即最小过程常返的充要条件是$z=\infty$。

(iii) 当$a_0=0$时，在(15)中取$k=n-1$得$E_{n-1}\xi_n=m_{n-1}$。(14)式中$a_0=0$时就是(17)后二式。

(iv) 由(17)，$N_k\leqslant R$知，如$R<\infty$，则$P_k\{\tau<\infty\}=1$ ($k\in E$)。反之，如果对某个（从而一切）k有$P_k\{\tau<\infty\}=1$。则由定理6.8.2，$u_k(\lambda)=E_k(e^{-\lambda\tau})\neq 0$ ($\lambda>0$) 满足$D_ku^+(\lambda)=\lambda u(\lambda)$，故由引理5.2.5，

$$u_i(\lambda)=u_0(\lambda)+\lambda\sum_{j=0}^{i-1}(z_i-z_j)u_j(\lambda)\mu_j.$$

又显然$u_i(\lambda)\uparrow$，($i\uparrow$)，故$1\geqslant u_i(\lambda)\geqslant\lambda u_0(\lambda)\sum\limits_{j=0}^{i-1}(z_i-z_j)\mu_j$，$1\geqslant\lambda u_0(\lambda)R$，故$R<\infty$，证毕。

定理7 设$a_0=0$，$S=\infty$（见(5.2.7)）。

设$X=\{x(t),\ t<\sigma\}\in\mathscr{X}$，$(Q)$。$\xi_0$按(6)确定。则对$\lambda>0$，$E_i\{e^{-\lambda\xi_0}\}\downarrow 0(i\downarrow\infty)$。

证 因为从i出发经有限步到达0必经过$i-1$，故u_i (λ)

$=E_i\{e^{-\lambda\xi_0}\}\downarrow a\geqslant 0$ （$i\uparrow\infty$）。由定理6.8.2，u_0 （λ）$=1$，$D_x u_i^+(\lambda)$ $=\lambda u_i(\lambda)(i>0)$，即

$$u_{i-1}(\lambda)-u_i(\lambda)=\frac{b_i}{a_i}[u_i(\lambda)-u_{i+1}(\lambda)]+\frac{\lambda}{a_i}u_i(\lambda),\ i>0.$$

重复应用上式得

$$\begin{aligned}u_{i-1}(\lambda)-u_i(\lambda)&=\frac{b_ib_{i+1}\cdots b_{i+j+1}}{a_ia_{i+1}\cdots a_{i+j+1}}[u_{i+j+1}(\lambda)\\&\quad-u_{i+j+2}(\lambda)]\\&\quad+\lambda\left\{\frac{u_i(\lambda)}{a_i}+\sum_{l=0}^{j}\frac{b_ib_{i+1}\cdots b_{i+l}u_{i+l+1}(\lambda)}{a_ia_{i+1}\cdots a_{i+l}a_{i+l+1}}\right\}\\&\geqslant\lambda\left\{\frac{1}{a_i}+\sum_{l=0}^{j}\frac{b_ib_{i+1}\cdots b_{i+l}}{a_ia_{i+1}\cdots a_{i+l}a_{i+l+1}}\right\}a,\end{aligned}$$

令$j\to\infty$得　$u_{i-1}(\lambda)-u_i(\lambda)\geqslant\lambda e_i a(i>0)$，从

而　　$$1\geqslant u_0(\lambda)\geqslant u_0(\lambda)-u_j(\lambda)\geqslant\lambda\Big(\sum_{k=1}^{j}e_k\Big)a.$$

令$j\to\infty$得$1\geqslant\lambda Sa$。因$S=\infty$。故$a=0$，证毕。

定理8　设 $a_0=0$，$S=\infty$。则 $P\{\xi_{\infty h}<\sigma\}=0$，这里 $\xi_{\infty h}$ 按(9.4.1) 确定。换言之，每个Q过程$X\in\mathscr{X}_e(Q)$ 都纯自 E 流入，或者说，X不可能自“∞”流入。

证　先证P （$\xi_{\infty 0}<\sigma$）$=0$．用反证法．记 $\Omega_i=(\xi_{\infty i}<\sigma)$，设$P(\Omega_0)>0$。显然$\xi_{\infty i}\downarrow$，（$i\uparrow$）记$\Omega_i\uparrow\Omega_\infty(i\uparrow\infty)$。由引理 9.4.3及强马氏性，可以在概率空间（$\Omega_i$，$\Omega_i\mathscr{F}$，$P(\cdot|\Omega_i)$）上考虑过程$X_i=\{x(\xi_{\infty i}+t),\ t<\sigma-\xi_{\infty i}\}$。$X_i$ 与X有相同的转移概率矩阵，且$P\{x_i(0)=i|\Omega_i\}=1$。对X_i按(6)定义的随机变量记为ξ_0^i。因

$$u_i(\lambda)=E\{e^{-\lambda\xi_0^i}|\Omega_i\}\tag{18}$$

仅由初始分布和转移概率决定，故（18)确定的量与定理7中的量一致，因而$u_i(\lambda)\downarrow 0$　（$i\uparrow\infty$）。

另一方面，

$$u_i(\lambda)=\frac{1}{P(\Omega_i)}\int_{\Omega i}e^{-\lambda\xi_0^i}\,dP\geqslant\frac{1}{P(\Omega_\infty)}\int_{\Omega 0}e^{-\lambda\xi_0^i}dP,$$

但在Ω_0上，$\xi_0^i\leqslant\xi_{\infty i}<\infty$，故

$$u_i(\lambda)\geqslant\frac{1}{P(\Omega_\infty)}\int_{\Omega 0}e^{-\lambda\xi_{\infty 0}}dP>0.$$

这与$u_i(\lambda)\downarrow 0$矛盾。

于是$P(\xi_{\infty 0}<\sigma)=0$。对$i>0$，

$$P\{\xi_{\infty 0}<\sigma\}\geqslant P\{\xi_{\infty i}<\sigma\}\prod_{k=1}^{i}\frac{a_k}{a_k+b_k}.$$

故$P\{\xi_{\infty i}<\sigma\}=0$。

§3. 一个推广的邓肯(ДЫНКИН)引理

我们证明下面推广的邓肯引理，其中过程可以不必是生灭过程。

引理1 设$X=\{x(t),\ t<\sigma\}\in\mathscr{X}$，$\xi$为非负随机变量，$\theta$为推移算子，满足下面的条件：

(i) 对任意$s\geqslant 0$，$t\geqslant 0$，集合$A_s=\{\xi>s\}\in\mathscr{F}_s^0$，而且$A_{s+t}\subseteq A_s\cap\theta_s A_t$。

(ii) 存在正数$T>0$，$\alpha>0$使对一切$i\in E$有$P_i(A_T)\leqslant 1-\alpha$。

则对任意初始分布，ξ的各级矩$E\xi^l$有限，而且分布函数$P\{\xi\leqslant t\}$由其矩$E\xi^l(l=0,1,2,\cdots\cdots)$唯一决定。

证 因$A_s\in\mathscr{F}_s^0$，$\theta_s A_T\in\mathscr{F}_\infty^s$，由马氏性得

$$P_i\{A_{s+T}\}\leqslant P_i\{A_s\cap\theta_s A_T\}=\int_{A_s}P_{x(s)}(A_T)dP_i$$
$$\leqslant(1-\alpha)P_i(A_s).$$

于是$P_i(A_{nT})\leqslant(1-\alpha)^n$。由此得$P(A_{nT})\leqslant(1-\alpha)^n$，$P(\xi<\infty)=1$。这样

$$E\{\xi^l\}=\sum_{n=0}^{\infty}\int_{nT<\xi\leqslant(n+1)T}\xi^l dP\leqslant\sum_{n=0}^{\infty}\{(n+1)T\}^l P\{\xi$$

$$>nT\}\leqslant\sum_{n=0}^{\infty}\{(n+1)T\}^{l}(1-\alpha)^{n}<\infty.$$

其次，任取正数r使$e^{Tr}(1-\alpha)<1$，则

$$\sum_{l=0}^{\infty}\frac{E\{\xi\}^{l}}{l!}r^{l}\leqslant\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{\{(n+1)T\}^{l}r^{l}}{l!}(1-\alpha)^{n}$$

$$=\sum_{n=0}^{\infty}e^{(n+1)Tr}(1-\alpha)^{n}=\frac{e^{Tr}}{1-e^{Tr}(1-\alpha)}<\infty.$$

根据〔57，§15.4〕中一个定理，ξ的分布函数由其矩唯一决定，证毕。

§4. 不中断过程的常返性和遍历性

从现在起，我们恒考虑保守生灭过程，即$a_0=0$。由定理2.6，当且只当$R<\infty$时最小过程中断，故进一步设$R<\infty$。此时$P\{\tau<\infty\}=1$。

定理1 设$X=\{x(t),\ t<\sigma\}\in\mathscr{X}$，$(Q)$不中断，$\tau$为第一个飞跃点。令

$$\beta_1^n=\inf\{t\,|\,\tau\leqslant t<\sigma,\ x(t)\leqslant n\}.\qquad(1)$$

则对任意初始分布，β_1^n的各级矩$E[\beta_1{}^n]^l$ $(l\geqslant0)$有穷，且其分布$P\{\beta_1^n\leqslant t\}$由其矩$E[\beta_1^n]^l$ $(l\geqslant0)$唯一决定。

证 $P_i\{\tau<\infty\}=1$且X不中断，故存在$s>0$使

$$0<P_0(\tau<s)=P_0\{\tau<s,\ x(s)\in E\}=\sum_{j}P_0\{\tau<s,\ x(s)=j\}.$$

因此存在$j\in E$使$P_0(\tau<s,\ x(s)=j)>0$。但$P_{j0}(t)>0$ $(t>0)$故任取$t>0$有

$$\alpha\equiv P_0\{\beta_1^0\leqslant s+t\}\geqslant P_0\{\tau<s,\ x(s+t)=0\}$$

$$\geqslant P_0\{\tau<s,\ x(s)=j\}p_{j0}(t)>0.$$

但在集合$\{x(0)=i,\ \beta_1^n\leqslant s+t\}$上，对按（6.7.7）定义的$\eta_i^*$，有$\eta_{i+1}^*<\infty$，$\beta_1^0=\eta_{i+1}^*+\theta_{\eta_{i+1}^*}\beta_1^0$。于是

$$\{x(0)=i,\ \beta_1^0\leqslant s+t\}\subset\{x(0)=i,\ \eta_{i+1}^*<\infty,$$

$\theta_{n}.\ _{i+1}(\beta_1^0\leqslant s+t)\}$，从而$P_i\{\beta_1^0\leqslant s+t\}$随$i$增大而增加．故如取$T=s+t$，则$P_i(\beta_1^0\leqslant T)\geqslant\alpha>0$ $(i\in E)$．于是引理3.1中条件（ii）满足，条件(i)满足是明显的．依引理3.1，本定理对$n=0$成立．再注意$\beta_1^n\leqslant\beta_1^0$得证定理，证毕．

定理2 一切不中断过程$X\in\mathscr{X}_s(Q)$都常返而且遍历．更进一步，按（6.7.7）定义的η_i^*的各级矩$E[\eta_i^*]^l$有限，而且分布函数$p(\eta_i^*\leqslant t)$由其矩$E[\eta_i^*]^l$ $(l\geqslant 0)$唯一决定．

证 因$\eta_i^*\leqslant\beta_1^n(n\geqslant i)$，引用定理1即可．

§5. 两个引理

设$X=\{x(t),\ t<\sigma\}\in\mathscr{X}_s(Q)$．如果$X$中断，我们可以取$\Delta=-1$并按(6.6.6)方式将$X$化为不中断过程$\widetilde{X}=\{\widetilde{x}(t),t<\infty\}\in\mathscr{X}_s(\widetilde{Q})$，其中$\widetilde{Q}=(\widetilde{q}_{ij})$ $(i,\ j\in(-1)\cup E)$，且$\widetilde{q}_i=q_i$，$\widetilde{q}_{i-1}=\widetilde{q}_{-1-1}=\widetilde{q}_{-1j}=0(i,\ j\in E)$．今后我们将认为这么做了．

仍记τ为第一个飞跃点，τ_n为第n个跳跃点，ξ_i按（2.6）定义．对任意$0\leqslant\varepsilon\leqslant\infty$，令

$$f_\varepsilon(x)=\begin{cases}x, & \text{如果}0\leqslant x\leqslant\varepsilon,\\ \varepsilon, & \text{如果}x>\varepsilon.\end{cases}\tag{1}$$

引理1 对$k\geqslant i\geqslant 0$．令

$$H_{ki}^\varepsilon=E_k\left\{\sum_{0\leqslant\tau_j<\min(\xi_i,\tau)}f_\varepsilon(\tau_{j+1}-\tau_j)\right\}.\tag{2}$$

特别地

$$H_{ki}^\infty=E_k\{\min(\xi_i,\ \tau)\}.\tag{3}$$

则当$R<\infty$时，

$$H_{ki}^\varepsilon=\frac{z-z_k}{z-z_i}\sum_{j=i+1}^{k-1}(z-z_j)(1-e^{-(a_j+b_j)\varepsilon})\mu_j$$

$$+\frac{z_k-z_i}{z-z_i}\sum_{j=k}^{\infty}(z-z_j)(1-e^{-(a_j+b_j)\varepsilon})\mu_j\leqslant N_k.$$

(4)

$$\lim_{\varepsilon \downarrow 0} H^{\varepsilon}_{ki} = 0. \tag{5}$$

如果还有$S<\infty$，则

$$\lim_{k\to\infty} \frac{H^{\varepsilon}_{ki}}{C_{kt}} = \frac{1}{C_{i0}} \sum_{j=i+1}^{\infty} (z_j - z_i)(1 - e^{-(a_j+b_j)\varepsilon}) \tag{6}$$

$$\lim_{i\to\infty} \lim_{k\to\infty} \frac{H^{\varepsilon}_{ki}}{C_{k0}} = 0, \tag{7}$$

$$\lim_{\varepsilon\downarrow 0} \lim_{k\to\infty} \frac{H^{\varepsilon}_{ki}}{C_{k0}} = 0. \tag{8}$$

其中C_k按(2.16)确定。

证 设$i\leqslant k\leqslant n$。令

$$H^{\varepsilon}_{kin} = E_k\left\{ \sum_{0\leqslant \tau_j < \min(\zeta_i, \zeta_n)} f_\varepsilon(\tau_{\ +1} - \tau\ ) \right\}. \tag{9}$$

显然$H^{\varepsilon}_{kin} \uparrow H^{\varepsilon}_{ki} (n\uparrow\infty)$。易见$E_k f_\varepsilon(\tau_1) = \dfrac{1}{a_k+b_k}\left\{1 - e^{-(a_k+b_k)\varepsilon}\right\}$。

利用定理6.8.2，$u_k = H^{\varepsilon}_{kin}$满足$f_i = f_n = 0$，$f_k = 1 - e^{-(a_k+b_k)\varepsilon} (i<k<n)$ 的方程(4.4.11)，依定理5.2.3得

$$H^{\varepsilon}_{kin} = \frac{z_n - z_k}{z_n - z_i} \sum_{j=i+1}^{k-1} (z\ - z_i)(1 - e^{-(a_j+b_j)\varepsilon})\mu_j$$

$$+ \frac{z_k - z_i}{z_n - z_i} \sum_{j=k}^{n-1} (z_n - z_j)(1 - e^{-(a_j+b_j)\varepsilon})\mu_j. \tag{10}$$

令n→∞得(4)中等式。比较（5.2.6）便得(4)中不等式。

因$R<\infty$，利用控制收敛定理，从(4)得(5)。

因为$S<\infty$时，$\sum_{i=0}^{\infty}\mu_i<\infty$，而

$$\frac{1}{C_{k0}} \quad \frac{z_k - z_i}{z - z_i} \sum_{j=k}^{\infty} (z - z_j)(1 - e^{-(a_j+b_j)\varepsilon})\mu_j \leqslant z \sum_{j=k}^{\infty} \mu \ \to 0,$$

$(k\to\infty)$。

$$\frac{1}{C_{i0}}\sum_{j=i+1}^{\infty}(z-z_i)(1-e^{-(a_j+b_j)t})\mu_j\leqslant z\sum_{j=i+1}^{\infty}\mu_j\to 0,$$

$(i\to\infty)$。故从(4)得(6)，从而得(7)。利用控制收敛定理，从(6)得(8)，证毕。

引理2 设$X\in\mathscr{X}_s(Q)$为非最小Q过程，τ为第一个飞跃点，β_1^n按(4.1)确定。则

$$P\{\lim_{n\to\infty}\beta_1^n=\tau\}=1。\tag{11}$$

证 显然$\beta_1^n\downarrow(n\uparrow)$，故$\lim\limits_{n\to\infty}\beta_1^n\geqslant\tau$。另一方面，任取正数$\varepsilon_m\downarrow 0$。由定理6.6.2(ii)，$P\{x(\tau+\varepsilon_m)=\infty\}=0$。如果$\tau=\sigma$，当然$\lim\limits_{n\to\infty}\beta_1^n=\tau$。如果$\tau<\sigma$，则当$m$充分大时，$\tau+\varepsilon_m<\sigma$，且$x(\tau+\varepsilon_m)\in E$。于是当$n\geqslant x(\tau+\varepsilon_m)$时，$\beta_1^n\leqslant\tau+\varepsilon_m$，$\lim\limits_{n\to\infty}\beta_1^n\leqslant\tau+\varepsilon_m$。由$m$的任意性得$\lim\limits_{n\to\infty}\beta_1^n\leqslant\tau$，证毕。

§6. 特征数列

设$X=\{x(t),\ t<\sigma\}\in\mathscr{X}_s(Q)$为非最小过程。考虑定义9.5.1中的变换$g_n$。令$\beta_0^n=0$，$\tau_1^n$为$X$的第一个飞跃点，

$$\beta_1^n=\inf\{t\,|\,\tau_1^n\leqslant t<\sigma,\ x(t)\leqslant n\}。\tag{1}$$

约定空集的下确界为σ。设τ_{m-1}^n，β_{m-1}^n已定义好。如果$\beta_{m-1}^n=\sigma$，则定义$\tau_m^n=\beta_m^n=\sigma$，否则定义$\tau_m^n$为$\beta_{m-1}^n$后的第一个飞跃点，

$$\beta_m^n=\inf\{t\,|\,\tau_m^n\leqslant t<\sigma,\ x(t)\leqslant n\}。\tag{2}$$

变换W_{τ^n,β^n}即变换g_n，$X^n=W_{\tau^n,\beta^n}(X)$成为

$$X^n=g_n(X)。\tag{3}$$

我们有

$$g_n(X^{n+1})=X^n。\tag{4}$$

特别，如果X是(Q,π)杜勃过程，并且$\pi_j=0(j>n)$。则$g_n(X)=X$。

定理1 设$X\in\mathscr{X}_s(Q)$是非最小过程。则由(3)确定的$X^n=\{x^n(t),\ t<\sigma^n\}\in\mathscr{X}_s(Q)$是$(Q,V^n)$杜勃过程，满足(4)，这里

$V^n=(v_j^n,\ 0\leqslant j\leqslant n)$,

$$v_j^n=P\{x(\beta_1^n)=j\},\ (-1\leqslant j\leqslant n). \tag{5}$$

$$\left.\begin{aligned}&\text{满足}\,v_j^n=v_j^{n+1}\Big/\Big(\sum_{i=-1}^{n}v_i^{n+1}+v_{n+1}^{n+1}C_{n+1\,n}\Big),\quad -1\leqslant j<n.\\&v_n^n=\big(v_n^{n+1}+v_{n+1}^{n+1}C_{n+1\,n}\big)\Big/\Big(\sum_{i=-1}^{n}v_i^{n+1}+v_{n+1}^{n+1}C_{n+1\,n}\Big).\\&\sum_{j=-1}^{n}v_j^n=1,\quad \sum_{j=0}^{n}v_j^n>0.\end{aligned}\right\} \tag{6}$$

X不中断的充要条件是$v_{-1}^n=0$ $(n\geqslant 0)$。

证 注意β_n^n是马时，且$P\{x(\beta_n^n)>n\}=0$。往证由(5)确定的V^n满足(6)。

令$\bar{\beta}_1^n$为对过程$\bar{X}=\{x(\beta_1^{n+1}+t),\ t<\sigma-\beta_1^{n+1}\}$，按(1)方式确定的量。$A_n$为过程$\bar{X}$自$n+1$出发经有穷($\geqslant 0$)步跳跃到达$n$的事件，$\bar{A}_n$为$A_n$的对立事件。易验证$\{x(\beta_1^{n+1})=n+1\}$，$A_n$，$\bar{A}_n$都$\in\mathscr{F}_{\tau_{\frac{n+1}{2}}}$，$\{x(\bar{\beta}_1^n)=j\}\in\mathscr{F}'_{\tau_1^{n+1}}$。由定理6.17.1的系（注意，在生灭过程的情况，系中的条件独立性成为独立性），对

$-1\leqslant j\leqslant n$,

$$\begin{aligned}\Delta_j^n&\equiv P\{x(\beta_1^{n+1})=n+1,\ \bar{A}_n,\ \bar{x}(\bar{\beta}_1^n)=j\}\\&=P\{x(\beta_1^{n+1})=n+1,\ \bar{A}_n\}P\{\bar{x}(\bar{\beta}_1^n)=j\,|\,x(\beta_1^{n+1})=n+1\}\\&=P\{x(\beta_1^{n+1})=n+1\}P\{\bar{A}_n\,|\,x(\beta_1^{n+1})=n+1\}v_j^n\\&=v_{n+1}^{n+1}(1-C_{n+1\,n})v_j^n.\end{aligned}$$

于是
$$\begin{aligned}v_n^n&=P\{x(\beta_1^{n+1})=n\}+P\{x(\beta_1^{n+1})=n+1,\ A_n\}+\Delta_n^n\\&=v_n^{n+1}+v_{n+1}^{n+1}C_{n+1\,n}+v_{n+1}^{n+1}(1-C_{n+1\,n})v_n^n,\\v_j^n&=P\{x(\beta_1^{n+1})=j\}+\Delta_j^n=v_j^{n+1}+v_{n+1}^{n+1}(1-C_{n+1\,n})v_j^n,\end{aligned}$$

$(-1\leqslant j<n)$。

$$\left.\begin{aligned}&\text{从而}\,v_j^n=v_j^{n+1}/[1-v_{n+1}^{n+1}(1-C_{n+1\,n})],\quad -1\leqslant j<n.\\&v_n^n=(v_n^{n+1}+v_{n+1}^{n+1}C_{n+1\,n})/[1-v_{n+1}^{n+1}(1-C_{n+1\,n})].\end{aligned}\right\} \tag{7}$$

由上式知，或者对一切n有$P\{\beta_1^n<\sigma\}>0$，或者对一切n有

$P\{\beta_1^n=\sigma\}=0$。如后者成立，则由引理5.2有$P\{\tau=\sigma\}=1$，即X是最小过程，与定理假设不合、于是$\sum\limits_{j=0}^{n}v_j^n=P\{\beta_1^n<\sigma\}>0\ (n\geqslant 0)$。

往证$\sum\limits_{j=-1}^{n}v_j^n=1$。由此及(7)便得(6)。

当X不中断时，由定理4.1，$P\{\beta_1^n<\infty\}=1$。因此，$v_{-1}^n=0$，$\sum\limits_{j=0}^{n}v_j^n=1$。

当X中断时，由定理10.2.2，取$\pi_0=1$，$\pi_j=0\ (j>0)$。可以考虑X的πD型延拓过程$\bar{X}=\{\bar{X}(t),\ t<\infty\}$，即

$$\left.\begin{array}{l}\bar{X}(t)=x(t),\quad t<\sigma,\\ P\{\bar{x}(\sigma)=0\,|\,\sigma<\infty\}=1,\end{array}\right\}\tag{8}$$

令$\bar{\beta}_1^n$为对于$\bar{X}$的量，显然$\beta_1^n=\bar{\beta}_1^n$。既然$\bar{X}$不中断，$P\ (\bar{\beta}_1^n<\infty)=1$，因此$\sum\limits_{j=-1}^{n}v_j^n=P\{\beta_1^n<\infty\}=1$。

设$v_{-1}^n=0$，即$P\{\beta_1^n<\sigma\}=\sum\limits_{j=0}^{n}v_j^n=1$。由强马氏性，$P\ \{\beta_1^n<\beta_2^n<\cdots<\sigma\}=1$。此说明在$[0,\ \sigma(\omega))$中有无穷多个$j=j(\omega)\ (\leqslant n)$区间。由定理6.7.1，$P\{\sigma=\infty\}=1$，即$X$不中断。

往证$X^n=g_n(X)$是$(Q,\ V^n)$杜勃过程。令$X_m=\{x(\beta_m^n+t),\ t<\tau_{m+1}^n-\beta_m^n\}\ (m\geqslant 0)$。易见$X_m\in\mathscr{X}_s(Q)$是最小$Q$过程，具有引理10.3.1中的性质(ii*)。由$R<\infty$即$P\{\tau<\infty\}=1$知$\beta_m^n<\sigma$当且只当$0<\tau_{m+1}^n-\beta_m^n<\infty$。由定理6.17.1系，

$$\begin{aligned}&P\{x^{m+1}(0)=j\,|\,0<\tau_{m+1}^n-\beta_m^n<\infty\}\\ &=P\{x(\beta_{m+1}^n)=j\,|\,\beta_m^n<\sigma\}=v_j^n,\quad 0\leqslant j\leqslant n.\end{aligned}$$

因此性质(iii*)也满足。性质(iv*)对$\Delta=\{x^{m+1}(0)=i\}$满足可由X的强马氏性直接推出。不难验证，$X_m\ (m\leqslant k)$是$\mathscr{F}_{\tau_{k+1}^n}$-可测

的，而$X_m(m>k)$是$\mathscr{F}'_{\tau^n_{k+1}}$可测的。又已指出$\{0<\tau^n_{k+1}-\beta^n_k<\infty\}=(\beta^n_k<\sigma)$。由定理6.17.1系，性质（iv*）是满足的。依定理10.4.2，对X_m（$m\geqslant 0$）按（10.2.5）（10.2.6）确定的过程是(Q,V^n)杜勃过程。但这样的过程正是$X^n=g_n(X)$，证毕。

定理2 对任意$X=\{x(t),\ t<\sigma\}\in\mathscr{X}_s(Q)$，有$P\{\sigma<\infty\}=0$或1。

证 定理对最小过程或不中断过程X是对的。设X中断并且非最小。

由定理1，$P\{\beta^n_1=\sigma=\infty\}=1-\sum\limits_{j=-1}^{n}v^n_j=0$，即

$$P\{\beta^n_1<\sigma=\infty\}+P\{\sigma<\infty\}=1。\tag{9}$$

由强马氏性，

$$P\{\beta^n_1<\sigma=\infty\}=\sum_{j=0}^{n}v^n_jP_j\{\sigma=\infty\}$$

但$u_j\equiv P_j\{\sigma=\infty\}$满足方程$Qu=0$，故$u_j$为常数，即$P_j(\sigma=\infty)=P(\sigma=\infty)$与$j$无关。故(9)成为

$$\left(\sum^{n}v^n_j\right)P\{\sigma=\infty\}+P\{\sigma<\infty\}=1。$$

由定理1，$0<\sum\limits_{j=0}^{n}v^n_j<1$。因此从上式必定有$P\{\sigma=\infty\}=0$，$P\{\sigma<\infty\}=1$，证毕。

定理3 设$X=\{x(t),\ t<\sigma\}\in\mathscr{X}_s(Q)$为非最小$Q$过程，则存在非负数列$p$，$q$，$r_n(n\geqslant -1)$满足

$$\left.\begin{aligned}&p+q=1,\text{且}S=\infty\text{时}q=0。\\&r_n=0\,(n\geqslant 0),\text{如果}p=0。\\&0<\sum_{n=0}^{\infty}r_nN_n<\infty,\ \text{如果}p>0。\end{aligned}\right\}\tag{10}$$

使得

$$v^n_j=P\{x(\beta^n_1)=j\},\ -1\leqslant j\leqslant n。\tag{11}$$

可以表示为

$$\left.\begin{aligned} v_j^n &= \frac{X_n}{A_n} r_j, \ -1 \leqslant j < n. \\ v_n^n &= Y_n + \frac{X_n}{A_n} \sum_{l=n}^{\infty} r_l C_{ln}. \end{aligned}\right\} \tag{12}$$

这里

$$\left.\begin{aligned} A_n &= \sum_{l=0}^{\infty} r_l C_{ln}, d = \begin{cases} \frac{q}{p} A_0, \text{如} p>0, \\ 1, \text{如} p=0. \end{cases} \\ X_n &= \frac{A_n C_{n0}}{(r_{-1}+A_n)C_{n0}+d}, \\ Y_n &= \frac{d}{(r_{-1}+A_n)C_{n0}+d}, \\ \frac{X_n}{A_n} &= \frac{C_{n0}}{r_{-1}C_{n0}+1}, \ \text{如果} p=0. \end{aligned}\right\} \tag{13}$$

如果令η为β_1^0前的最后一个飞跃点，则

$$P\{x(\eta)=j\} = \begin{cases} \dfrac{r_{-1}}{r_{-1}+A_0+d}, & \text{如} j=-1 \\ \dfrac{r_j C_{j0}}{r_{-1}+A_0+d}, & \text{如} 0 \leqslant j < \infty, \\ \dfrac{d}{r_{-1}+A_0+d}, & \text{如} j=\infty. \end{cases} \tag{14}$$

p,q由X唯一决定．如$p>0$，则$r_n(n \geqslant -1)$除常数因子外由X唯一决定。如$p=0$，则$r_n(n \geqslant -1)$由X唯一决定。

证 分几步证明。

(一) 令 $R_n = \sum_{i=0}^{n-1} v_i^n C_{i0}, S_n = v_n^n C_{n0}, \Delta_n = R_n + S_n.$ (15)

则由(6)，

$$0 < \Delta_n = \frac{\Delta_{n+1}}{\delta_{n+1}}, \quad S_n = \frac{v_n^{n+1} C_{n0} + S_{n+1}}{\delta_{n+1}}, \tag{16}$$

$$\delta_{n+1} = \sum_{j=-1}^{n} v_j^{n+1} + v_{n+1}^{n+1} C_{n+1\,n}. \tag{17}$$

故$\frac{v_j^n}{\Delta_n}$与$n>j(\geq -1)$无关，而且存在极限；

$$\frac{S_n}{\Delta_n}\downarrow p\geq 0,\quad \frac{R_n}{\Delta_n}\uparrow q\geq 0. \tag{18}$$

如果$p=0$，取$r=1$，如果$p>0$，任取正数r。令

$$r_j=\frac{v_j^n}{\Delta_n}r,\ n>j\geq -1. \tag{19}$$

这样，我们得到非负数列$p,q,r_n(n\geq -1)$。

（二）显然$p+q=1$。又$p=0$时$r_n=0(n\geq 0)$。如果$p>0$，则至少存在一个$r_k>0(k\geq 0)$，故只需证

$$\sum_{n=0}^{\infty} r_n N_n<\infty. \tag{20}$$

注意$E\beta_1^0<\infty$。当X不中断时，此由定理4.1得出。当X中断时，存在不中断的Q过程$\bar{X}$满足(8)，故也有$E\beta_1^0=E\bar{\beta}_1^0<\infty$。于是$E\tau_2^0=E\beta_1^0+E\{\tau_2^0-\beta_1^0\}=E\beta_1^0+v_0^0E_0\tau=E\beta_1^0+v_0^0R<\infty$。

令 $M_i^n=\begin{cases}\tau_{i+1}^n-\beta_i^n, & \text{如果}\beta_i^n<\beta_1^0,\\ 0, & \text{反之}.\end{cases}$

显然 $\tau_2^0\geq\sum_{i=1}^{\infty}M_i^n$,

$$\infty>E\tau_2^0\geq\sum_{i=1}^{\infty}EM_i^n=\sum_{i=1}^{\infty}E\{\tau_{i+1}^n-\beta_i^n,\beta_i^n<\beta_1^0\}.$$

但 $E\{\tau_2^n-\beta_1^n,\ \beta_1^n<\beta_1^0\}=\sum^{n}v_j^nE_j\tau_1^n=\sum^{n}v_j^nN_j$,

$$E\{\tau_{i+1}^n-\beta_i^n,\ \beta_i^n<\beta_1^0\}=\sum^{n}P\{\beta_{i-1}^n<\beta_1^0,x(\beta_i^n)=j\}E_j\tau_1^n$$

$$=\sum_{j=1}^{n}\left\{\sum_{k=1}^{n}v_k^n(1-C_{k0})\right\}^{i-1}v_j^nN_j,$$

$$\text{故}\quad \infty>E\tau_2^0\geq\frac{\sum\limits_{j=1}^{n}v_j^nN_j}{1-\sum\limits_{k=1}^{n}v_k^n(1-C_{k0})}\geq\frac{\sum\limits^{n-k}v_j^nN_j}{v_{-1}^n+\Delta_n},$$

由(19)得

$$\frac{\sum_{j=1}^{n-1} r_j N_j}{r_{-1}+r} \leqslant E\tau_2^0 < \infty.$$

令$n\to\infty$得(20)。

(三) 往证(12)。

由(18)(19),

$$p = \lim_{n\to\infty}\frac{R_n}{\Delta_n} = \lim_{n\to\infty}\frac{1}{r}\sum_{j=0}^{n-1} r_i C_{i0} = \frac{A_0}{r},$$

$$\text{因此}\quad r = \begin{cases} \dfrac{A_0}{p}, \text{如} p>0, \\ 1, \text{如} p=0. \end{cases} \tag{21}$$

从(6) 用归纳法易得: $m>n$,

$$\left.\begin{aligned} & v_j^n = v_j^m/\delta_{m,n}, \quad -1 \leqslant j < n, \\ & v_n^n = \left(\sum_{j=n}^{m} v_j^m C_{jn}\right)\Big/\delta_{mn}, \\ & 0 < \Delta_n = \Delta_m/\delta_{mn}, \\ & \delta_{mn} = \sum_{j=-1}^{n} v_j^m + \sum_{j=n+1}^{m} v_j^m C_{jn}. \end{aligned}\right\} \tag{22}$$

由此及(18)(19),

$$\frac{v_n^n}{\Delta_n} = \frac{\sum_{j=n}^{m} v_j^m C_{jm}}{\Delta_m} = \frac{1}{r}\sum_{j=n}^{m-1} r_j C_{jn} + \frac{S_m}{\Delta_m C_{n0}},$$

从而当$m\to\infty$时有

$$\frac{v_n^n}{\Delta_n} = \frac{1}{r}\sum_{j=n}^{\infty} r_j C_{jn} + \frac{q}{C_{n0}}. \tag{23}$$

由(19)(23),

$$1 = \sum_{j=-1}^{n} v_j^n = \Delta_n\left(\frac{\sum_{j=-1}^{n-1} r_j}{r} + \frac{\sum_{j=n}^{\infty} r_j C_{jn}}{r} + \frac{q}{C_{n0}}\right),$$

因此

$$\Delta_n = \frac{rC_{n0}}{(r_{-1}+A_n)C_{n0}+qr}.$$

代回(19)(23)中并注意(21)便得(12)。

(四) 往证(14)。

如果X不中断，则$1=P\{\beta_1^n<\infty\}$. 由强马氏性，$P\{\beta_1^n<\beta_2^n<\cdots\}=1$，故对几乎一切$\omega\in\Omega$，$X(\omega)$在$[0,\lim\limits_{l\to\infty}\beta_l^n(\omega))$中有无穷多个$j(\omega)(\leqslant n)$区间，因而$P\{\lim\limits_{l\to\infty}\beta_l^n=\infty\}=0$。如果$X$中断，由定理2，$P\{\sigma<\infty\}=1$. 因为如果$\beta_l^n(\omega)<\sigma(\omega)$，则$\beta_l^n(\omega)<\tau_{l+1}^n(\omega)\leqslant\beta_{l+1}^n(\omega)$，故依定理6.7.1，$P\{$存在$l$使$\beta_l^n=\sigma\}=1$. 这样，不论$X$中断与否，恒有

$$P\{\lim_{l\to\infty}\beta_l^n=\sigma\}=1. \tag{24}$$

而且对几乎一切ω，存在唯一的l使$\beta_{l-1}^n<\eta\leqslant\beta_l^n\leqslant\beta_1^0$. 记此$l$为$l_n$，即

$$l_n=\min\{l\,|\,\beta_l^n\geqslant\eta\}, \tag{25}$$

$$\beta_{l_n}^n=\inf\{t\,|\,\eta\leqslant t<\sigma, x(t)\leqslant n\}. \tag{26}$$

如果对某个$n>j\geqslant-1$有$x(\beta_{l_n}^n)=j$，则必定$\beta_{l_n}^n=\eta$。因为如果$\eta<\beta_{l_n}^n$，按定义对$t\in(\eta,\beta_{l_n}^n)$有$n<x(t)<\infty$，而$x(\beta_{l_n}^n)=j<n$，由于生灭过程在跳跃点的跃度为1，这不可能。因此对$n>j\geqslant-1$，

$$\{x(\beta_{l_n}^n)=j\}\subset\{\beta_{l_n}^n=\eta, x(\eta)=j\}\subset\{x(\eta)=j\}. \tag{27}$$

又显然如果$x(\eta)=j$，则对$n>j$有$\beta_{l_n}^n=\eta$，故

$$\{x(\eta)=j\}=\lim_{n\to\infty}\{x(\beta_{l_n}^n)=j\}, (-1\leqslant j<\infty). \tag{28}$$

往证 $$\{x(\eta)=\infty\}=\lim_{n\to\infty}\{x(\beta_{l_n}^n)=n\}. \tag{29}$$

实际上，设$\omega\in$右方，则必定$\beta_{l_n}^n\downarrow\bar\eta\geqslant\eta$，从而由右连续性，$x(\bar\eta)=\lim\limits_{n\to\infty}x(\beta_{l_n}^n)=\lim\limits_{n\to\infty}n=\infty$. 但由$\eta$的定义，对任何$\eta<t<\beta_1^0$，$x(t)\in E$。因此$\bar\eta=\eta$，从而$x(\eta)=\infty$，即$\omega\in$左方。设$\omega\in$左方，则对任意$n$必定$x(\beta_{l_n}^n)=n$。因为不然由(27)将导致$x(\eta)=j\neq\infty$。这样，(29)成立。

对$0\leqslant j\leqslant n$，

$$P\{x(\beta_{l_n}^n)=j\}=\sum_{l=1}^{\infty}P\{x(\beta_l^n)=j, l_n=l\}$$

$$= \sum_{l=1}^{\infty} P\{\beta_{l-1}^{n} < \beta_{1}^{0}, x(\beta_{l}^{n}) = j, \beta_{l}^{n} \leqslant \beta_{1}^{0} < \tau_{l+1}^{n}\}$$

$$= \sum_{l=1}^{\infty} \left\{ \sum_{k=1}^{n} v_{k}^{n} (1 - C_{k0}) \right\}^{l-1} v_{j}^{n} C_{j0} = \frac{v_{j}^{n} C_{j0}}{v_{-1}^{n} + \Delta_{n}}, \tag{30}$$

$$P\{x(\beta_{l_n}^{n}) = -1\} = \sum_{l=1}^{\infty} P\{x(\beta_{l}^{n}) = -1, l_n = l\}$$

$$= \sum_{l=1}^{\infty} P\{\beta_{l-1}^{n} < \beta_{1}^{0}, x(\beta_{l}^{n}) = -1\}$$

$$= \sum_{l=1}^{\infty} \left\{ \sum_{k=1}^{n} v_{k}^{n} (1 - C_{k0}) \right\}^{l-1} v_{-1}^{n} = \frac{v_{-1}^{n}}{v_{-1}^{n} + \Delta_{n}}. \tag{31}$$

将(30)(31)代入(28)(29)，并注意(18)(19)得

$$P\{x(\eta) = j\} = \begin{cases} \dfrac{r_{-1}}{r_{-1} + r}, & \text{如} j = -1. \\ \dfrac{r_j C_{j0}}{r_{-1} + r}, & \text{如} 0 \leqslant j < \infty. \\ \dfrac{rq}{r_{-1} + r}, & \text{如} r = \infty. \end{cases}$$

再注意(21)，上式即(14)。

（五）设$S=\infty$。由定理2.8，$P\{\xi_{\infty 0} < \sigma\} = 0$。但$\{x(\eta) = \infty\} \subset \{\xi_{\infty 0} < \sigma\}$。故$p\{x(\eta) = \infty\} = 0$。由(13)(14)，$q = 0$。

（六）设还有非负数列$\bar{p}$，$\bar{q}$，$\bar{r}_n (n \geqslant -1)$使(10)—(14)成立。由(12)知：或者$p$与$\bar{p}$同时为零，因而$\bar{r}_n = r_n = 0 (n \geqslant 0)$，$q = \bar{q} = 1$，再由$v_{-1}^{n} = \dfrac{C_{n0}}{r_{-1} C_{n0} + 1} r_{-1} = \dfrac{C_{n0}}{\bar{r}_{-1} C_{n0} + 1} \bar{r}_{-1}$得$r_{-1} = \bar{r}_{-1}$；或者$p$与$\bar{p}$同时为正，根据(11)(12)可算出(18)，因而$p = \bar{p}$，$q = \bar{q}$。再由(19)得$r_j / r_k = v_{j}^{n} / v_{k}^{n} = \bar{r}_j / \bar{r}_k$，证毕。

定义1 称非负数列$p, q, r_n (n \geqslant -1)$为$Q$过程$X$的特征数列。

§7. 过程的概率构造

设给定非负数列$p, q, r_n (n \geqslant -1)$满足(6.10)。

引理1 根据(6.12)(6.13)定义的$v_j^n(-1 \leqslant j \leqslant n)$满足(6.6)。

证 当$p=0$时，$v_j^n = 0 (0 \leqslant j < n), v_{-1}^n = \frac{r_{-1} C_{n0}}{r_{-1} C_{n0} + 1}$,

$v_n^n = \frac{1}{r_{-1} C_{n0} + 1}$。直接验证知(6.6)成立。

设$p > 0$。直接验证知(6.6)最后一式成立。由(6.12)(6.13)可得

$$1 - v_{n+1}^{n+1}(1 - C_{n+1\,n}) = \sum_{j=-1}^{n} v_j^{n+1} + v_{n+1}^{n+1} C_{n+1\,n}$$

$$= \frac{X_{n+1}}{A_{n+1}} \sum_{j=-1}^{n} r_j + Y_{n+1} C_{n+1\,n} + \frac{X_n}{A_n} \sum_{j=n+1}^{\infty} r_j C_{jn}$$

$$= \frac{X_{n+1}}{A_{n+1}} \left\{ r_{-1} + A_n + \frac{Y_{n+1} A_{n+1} C_{n+1\,n}}{X_n} \right\}$$

$$= \frac{X_{n+1}}{A_{n+1}} \left(r_{-1} + A_n + \frac{d}{C_{n0}} \right) = \frac{X_{n+1}}{A_{n+1}} \cdot \frac{A_n}{X_n}, \tag{1}$$

$$v_n^{n+1} + v_{n+1}^{n+1} C_{n+1\,n} = \frac{X_{n+1}}{A_{n+1}} r_n + Y_{n+1} C_{n+1\,n}$$

$$+ \frac{X_{n+1}}{A_{n+1}} \sum_{l=n+1}^{\infty} r_l C_{ln}$$

$$= \frac{X_{n+1}}{A_{n+1}} \cdot \frac{A_n}{X_n} \left(Y_n + \frac{X_n}{A_n} \sum_{l=n}^{\infty} r_l C_{ln} \right). \tag{2}$$

从而可得(6.6)成立。

引理2 设$V^n = \{v_j^n, 0 \leqslant j \leqslant n\} (n \geqslant 0)$满足(6.6)。则存在概率空间$(\Omega, \mathscr{F}, P)$，在其上可以定义$(Q, V^n)$杜勃过程$X^n = \{x^n(t), t < \sigma^n\} \in \mathscr{X}_s(Q) (n \geqslant 0)$满足(6.4)。

证 固定一分布(v_i)作为初始分布。对每个n，存在概率空间$(\Omega_n, \mathscr{F}_n, P_n)$及定义于其上的$(Q, V^n)$ 杜勃过程$X_n = \{\bar{x}_n(t, \omega_n),$

$t<\bar{\sigma}_n(\omega_n)\}$，$(\omega_n\in\Omega_n)$。由定理6.1，对$m<n$，$Z_m=g_m(X_n)$是$(\Omega_n,\mathscr{F}_n,P_n)$上的$(Q,V^m)$杜勃过程．设$k\geqslant1$。对非负整数$n_i(1\leqslant i\leqslant k)$，非负实数$t_i\geqslant0(1\leqslant i\leqslant k)$及$j_i\in E(1\leqslant i\leqslant k)$，可以选取$n>\max(n_1,n_2,\cdots,n_k)$，定义$k$维分布

$$F_{n_1t_1\cdots n_kt_k}(j_1,\cdots,j_k)=P_n\{Z_{n_i}(t_i,\omega_n)=j_i,1\leqslant i\leqslant k\}.\tag{3}$$

易见此分布不依赖于n的选择，而且有穷维分布族$\{F_{n_1t_1,\cdots,n_kt_k}\}$是相容的。根据柯尔莫果洛夫定理(见王梓坤[1，§1.1定理1])，存在概率空间$(\Omega,\mathscr{F},p)$及定义于其上的过程列$X^n=\{x^n(t),t<\sigma^n\}$使

$$p\{x^{n_i}(t_i)=j_i,\ 1\leqslant i\leqslant k\}=F_{n_1t_1,\cdots,n_kt_k}(j_1,\cdots,j_k).\tag{4}$$

由此及(3)，$X^n\in\mathscr{X}_s(Q)$是(Q,V^n)杜勃过程．其次，按上面引用的定理，可取$\Omega=(\omega)$，$\omega=\omega(n,t)$是取值于E的二元函数$(n=0,1,2,\cdots,\ t\in[0,\sigma_n),\sigma_n\leqslant\infty)$，并且$x^n(t,\omega)=\omega(n,t)$，$\sigma^n(\omega)=\sigma_n$，从而(6.4)成立，证毕．

设$X^n=\{x^n(t),t<\sigma^n\}$为引理2的过程列．由(6.4)，$\sigma^n\leqslant\sigma^{n+1}$，故可令$\sigma^n\uparrow\sigma$。

按(6.1)(6.2)方式对X^k定义的量记为τ_m^{kn}，β_m^{kn}。由(6.4)，$\beta_m^{n0}\leqslant\beta_m^{n+1\,0}\leqslant\sigma^{n+1}$，故存在极限$\beta_m^0=\lim\limits_{n\to\infty}\beta_m^{n0}\leqslant\sigma$。由(6.24)，$\lim\limits_{m\to\infty}\beta_m^0\geqslant\lim\limits_{m\to\infty}\beta_m^{n0}=\sigma^n$．因此

$$P\{\lim_{m\to\infty}\beta_m^0=\sigma\}=1.\tag{5}$$

对$n>m$，令

$$L_{nm}^i=\begin{cases}\beta_i^{nm}-\tau_i^{nm},\text{如果}\beta_i^{nm}<\beta_1^{n0}\\0,\ \text{反之}.\end{cases}\tag{6}$$

$$T_i^{nm}=\sum_{\tau_1^n<\tau_{ij}^n<\beta_1^{nm}}f_i(\tau_{ij+1}^n-\tau_{ij}^n).\tag{7}$$

这里τ_{ij}^n是X^n过程第i个飞跃点后的第j个跳跃点，$f_i(x)$如(5.1)。由(6.4)易得

$$\left.\begin{array}{l} L_{nm}=\sum_{i=1}^{\infty} L_{nm}^{i} \leqslant L_{n+1\,m}, \\ T_{\varepsilon}^{n0} \leqslant T_{\varepsilon}^{n+1\,0},\ T_{\varepsilon_2}^{n0} \leqslant T_{\varepsilon_1}^{n0},\ (\varepsilon_1<\varepsilon_2). \end{array}\right\} \tag{8}$$

故可令 $L_{nm} \uparrow L_m(n \uparrow \infty), L_m \downarrow L(m \uparrow \infty)$,

$$T_{\varepsilon}^{n0} \uparrow T_{\varepsilon}(n \uparrow \infty), T_{\varepsilon} \downarrow T(\varepsilon \downarrow 0). \tag{9}$$

引理3 $P\{L=0\}=p\{T=0\}=1$.

证 令 $\bar{\beta}_i^{nm}=inf\{t \mid t \geqslant \tau_{i0}^n, x^n(t) \leqslant m\}$.

$$\text{考虑 } \bar{L}_{nm}^i=\begin{cases} min(\tau_{i+1\,0}^n, \bar{\beta}_i^{nm})-\tau_{i0}^n, & \text{如果} \tau_{i0}^n<\beta_1^{n0}, \\ & \text{而且} m<x^n(\tau_{i0}^n). \\ 0, & \text{反之}. \end{cases}$$

$$\bar{T}_{\varepsilon i}^{n0}=\begin{cases} \sum\limits_{\tau_{i0}^n \leqslant \tau_{ij}^n<min(\tau_{i+1,1}^n;\, \bar{\beta}_i^{n0})} f_{\varepsilon}(\tau_{i\,j+1}^n-\tau_{ij}^n), & \text{如果} \tau_{i0}^n<\beta_1^{n0}. \\ 0, & \text{反之}. \end{cases}$$

依L_{nm}及T_{ε}^{n0}的定义有

$$L_{nm}=\sum_{i=1}^{\infty} \bar{L}_{nm}^i, \quad T_{\varepsilon}^{n0}=\sum_{i=1}^{\infty} \bar{T}_{\varepsilon i}^{n0}.$$

但依引理5.1及(6.11)(6.12),

$$\begin{aligned} EL_{nm} &= \sum_{i=1}^{\infty} E\bar{L}_{nm}^i=\sum_{i=1}^{\infty} \sum_{j=m+1}^{n} P\{\tau_{i0}^n<\beta_1^{n0}, \\ & x^n(\tau_{i0}^n)=j\} E\{min(\tau_{i+1\,0}^n, \bar{\beta}_i^{nm})-\tau_{i0}^n \mid x^n(\tau_{i0}^n)=j\} \\ &= \sum_{i=1}^{\infty} \sum_{j=m+1}^{n}\left\{\sum_{k=1}^{n} v_k^n(1-C_{k0})\right\}^{i-1} v_j^n H_{jm}^{\infty} \\ &= \frac{\sum\limits_{j=m+1}^{n} v_j^n H_{jm}^{\infty}}{v_{-1}^n+\sum\limits_{k=0}^{n} v_k^n C_{k0}} \\ &= \frac{\sum\limits_{j=m+1}^{n-1} r_j H_{jm}^{\infty}+\left(\dfrac{d}{C_{n0}}+\sum\limits_{l=n}^{\infty} r_l C_{ln}\right) H_{nm}^{\infty}}{r_{-1}+A_0+d}, \end{aligned}$$

$$ET_i^{\varepsilon 0}=\sum_{i=1}^{\infty}ET_{ii}^{\varepsilon 0}=\sum_{i=1}^{\infty}\sum_{k=1}^{n}P\{\tau_{i0}^{n}<\beta_1^{n0},x^n(\tau_{i0}^{n})$$

$$=k\}E\left\{\sum_{\tau_{i0}^{n}\leqslant\tau_{ij}^{n}<\min(\tau_{i+10}^{n},\bar{\beta}_i^{n0})}f_{\varepsilon}(\tau_{ij+1}^{n}-\tau_{ij}^{n})x_n(\tau_{i0}^{n})\right.$$

$$\left.=k\right\}=\sum_{i=1}^{\infty}\sum_{k=1}^{n}\left\{\sum_{j=1}^{n}v_j^n(1-C_{j0})\right\}^{i-1}v_k^nH_{k0}^{\varepsilon}$$

$$=\frac{\sum_{k=1}^{n}v_k^nH_{k0}^{\varepsilon}}{v_{-1}^{n}+\sum_{j=0}^{n}v_j^nC_{j0}}$$

$$=\frac{\sum_{k=1}^{n-1}r_kH_{k0}^{\varepsilon}+\left(\frac{d}{C_{n0}}+\sum_{l=n}^{\infty}r_lC_{ln}\right)H_{n0}^{\varepsilon}}{r_{-1}+A_0+d}.$$

由(5.2.6) 可验证$C_{ln}N_n\leqslant N_l$ ($l\geqslant n$)。由引理5.1及 (6.10) (6.12)，$H_{jm}^{\varepsilon}\leqslant N_j$，$\sum_{j=0}^{\infty}r_jN_j<\infty$，$\lim_{\varepsilon\downarrow 0}H_{k0}^{\varepsilon}=0$；如$S=\infty$，则$d=0$；如$S<\infty$，则$\lim_{m\to\infty}\lim_{n\to\infty}\frac{H_{nm}^{\infty}}{C_{n0}}=\lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\frac{H_{n0}^{\varepsilon}}{C_{n0}}=0$。因此由上面二式得

$$EL=\lim_{m\to\infty}\lim_{n\to\infty}EL_{nm}$$

$$=\frac{\lim_{m\to\infty}\sum_{j=m+1}^{\infty}r_jH_{jm}^{\infty}+\lim_{m\to\infty}\lim_{n\to\infty}\left(\frac{d}{C_{n0}}+\sum_{l=n}^{\infty}r_lC_{ln}\right)H_{nm}^{\infty}}{r_{-}+A_0+d}=0,$$

$$ET=\lim_{\varepsilon\downarrow 0}\lim_{n\to 0}ET_{\varepsilon}^{n0}$$

$$=\frac{\lim_{\varepsilon\downarrow 0}\sum_{k=1}^{\infty}r_kH_{k0}^{\varepsilon}+\lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\left(\frac{d}{C_{n0}}+\sum_{l=n}^{\infty}r_lC_{ln}\right)H_{n0}^{\varepsilon}}{r_{-1}+A_0+d}=0。$$

由此得证引理，证毕。

定理4 对引理2中的(Q,V^n)杜勃过程列$X^n=\{x^n(t),\ t<\sigma^n\}$，其强极限过程$X=\{x(t),t<\sigma\}$存在。$X\in\mathscr{X}_S(Q)$是非最小过程。

X是满足(6.11)的唯一Q过程，亦即X是具特征数列p，q，$r_n(n\geqslant -1)$的唯一Q过程。

证 （一）显然$\sigma^n(\omega)\uparrow\sigma(\omega)$存在。往证对几乎一切$\omega\in\Omega$，按勒贝格测度$L$而言，对几乎一切$t\in[0,\sigma(\omega))$，$x^n(t,\omega)$收敛于$E$中的状态，对例外的$t\in[0,\sigma(\omega))$，$x^n(t,\omega)$收敛到$\infty$。

不失一般性，设对一切$\omega\in\Omega$，$X^n(\omega)$在任意有限区间$[0,t)$ $(t\leqslant\sigma(\omega))$中只有有限个$i$区间($i\in E$，$n\geqslant 0$)。如果向左平移每个$X^n$的每个常值区间（即泛指的$i$区间），而且每个区间平移的距离不大于$\varepsilon$，则在$[0,\beta_1^{n,0}(\omega))$中使$x_n(t,\omega)\neq x_m(t,\omega)$ $(n>m)$的点t所成的区间总长不超过$\varepsilon+T_\varepsilon^n(\omega)<\varepsilon+T_\varepsilon(\omega)$。固定$k$，取$n>m>l(>k)$，由于$\beta_1^{n,0}(\omega)>\beta_1^{k,0}(\omega)$得

$$L\{t\mid t\in[0,\beta_1^{k,0}(\omega)),\ x_n(t,\omega)\neq x_m(t,\omega)\}$$
$$\leqslant L_l(\omega)+T_{L_l(\omega)}(\omega)。\tag{10}$$

令$\Omega_0=\{L_l+T_{L_l}\downarrow 0,\ l\uparrow\infty\}$。由引理3，$P\{\Omega_0\}=1$。固定$\omega\in\Omega_0$，由(10)知$x_n(t,\omega)$在$[0,\beta_1^{k,0}(\omega))$中依测度$L$收敛，故存在子列$n_i\to\infty$使$x_{n_i}(t,\omega)$在$[0,\beta_0^{k,0}(\omega))$中对几乎一切$t$收敛。固定一个收敛点$t_0$，由于$Doob$过程不取“$\infty$”值，故存在$M\in E$使$x_{n_i}(t_0,\omega)\to M(i\to\infty)$。由于$E$离散，故有正数$N$使

$$x_{n_i}(t_0,\omega)=M,\ (i\geqslant N)。\tag{11}$$

今证存在正数N'使$n>N'$时$x_n(t_0,\omega)=M$，从而$x_n(t_0,\omega)$收敛。因为不然，必存在$m_i\to\infty$使

$$x_{m_i}(t_0,\omega)\neq M\tag{12}$$

由此式及(11)，并根据$g_m(X^n)=X^m(m<n)$知在$[0,t_0]$中，X^M有无穷多个M区间，此与证明开头的假设矛盾。

于是设$\omega\in\Omega_0$，则对几乎一切$t\in[0,\beta_1^{k,0}(\omega))$，$x_n(t,\omega)$收敛于$E$中的状态。令$k\to\infty$得同样结论对$[0,\beta_1^0(\omega))$中成立。同样地可证同样结论对$[0,\beta_i^0(\omega))$中成立。由(5)知对几乎一切$t\in[0,\sigma(\omega))$，$x^n(t,\omega)$收敛于$E$中的状态。对于例外的$t\in[0,\sigma(\omega))$，如果$x^n(t,\omega)$不收敛到$\infty$，则必存在二子列$n_i$，$m_i$及$M\in E$使(11)和(12)成立，同样会导致矛盾。

（二）往证$P\{x(t)=\infty\}=0$，$(t\geqslant 0)$。

由（一），$L\{t\,|x(t,\ \omega)=\infty\}=0$。由Fubini定理，存在集$T$，$L(T)=0$，使对$t\bar{\in}T$时，$P\{x(t)=\infty\}=0$。显然$0\bar{\in}T$。

设$t_0\in T$。则$t_0>0$。可取t_1使$t_1\bar{\in}T$，$t_0-t_1\bar{\in}T$。于是

$$
\begin{aligned}
&P\{x(t_0)\geqslant N\}=\lim_{n\to\infty}P\{x^n(t_0)\geqslant N\}\\
&=\lim_{n\to\infty}E\{Px^n{}_{(t_1)}[x^n(t_0-t_1)\geqslant N]\}\\
&=E\{\lim_{n\to\infty}Px^n{}_{(t_1)}[x^n(t_0-t_1)\geqslant N]\}\\
&=E\{\lim_{n\to\infty}Px_{(t_1)}[x^n(t_0-t_1)\geqslant N]\}\\
&=E\{Px_{(t_1)}[x(t_0-t_1)\geqslant N]\}.
\end{aligned}
$$

令$N\to\infty$得 $P\{x(t_0)=\infty\}=E\{P_{x(t_1)}[x(t_0-t_1)=\infty]\}=0$。

（三）由（一）、（二）易知$X\in\mathscr{X}_e$且是非最小Q过程。

（四）往证X满足(6.11)。事实上，按(6.1)(6.2)对X^k定义β_1^{kn}。由(6.4)，$\beta_1^{kn}\uparrow(n\leqslant k\uparrow)$。易见有$\lim\limits_{k\to\infty}\beta_1^{kn}=\beta_1^n$，$\lim\limits_{k\to\infty}x^n(\beta_1^{kn})=x(\beta_1^n)$。而且由(6.4)，$x^k(\beta_1^{kn})=x^n(\beta_1^{nn})$。这样，对$0\leqslant j\leqslant n$，

$$P\{x(\beta_1^n)=j\}=P\{x^n(\beta_1^{nn})=j\}=v_j^n.$$

从而上式对$j=-1$也成立。

（五）设$\bar{X}=\{\bar{x}(t),\ t<\bar{\sigma}\}\in\mathscr{X}_e(Q)$也满足(6.11)。由定理9.5.2系2，$\bar{X}$是$(Q,\ V^n)$杜勃过程$\bar{X}^n=g_n(\bar{X})$的强极限，故$\bar{X}$与$X$都以$(Q,\ V^n)$杜勃过程的转移概率$p_{ij}^n(t)$的极限为转移概率，即$\bar{X}$和$X$有相同的转移概率，因而是同一过程，证毕。

§8. 小　结

定理1　设$X\in\mathscr{X}_e(Q)$是非最小Q过程，则它的特征数列p，q，$r_n(n\geqslant -1)$满足(6.10)。

反之，给定一列非负数列p，q，$r_n(n\geqslant -1)$满足(6.10)。则存在唯一的非最小过程$X\in\mathscr{X}_e(Q)$，以p，q，$r_n(n\geqslant -1)$为特征数列。而且可以选取X是一列$(Q,\ V^n)$杜勃过程列X^n的强极限，其中$V^n=(v_j^n,\ 0\leqslant j\leqslant n)$根据$p$，$q$，$r_n(n\geqslant -1)$按(6.12)(6.13)确

定。

X不中断的充要条件是$r_{-1}=0$。X满足向前方程组的充要条件是$p=0$。

最后一句话还需要证明。

如果X满足向前方程组，由定理9.4.4，位于$[\tau, \sigma)$中的U区间都是$_\infty U$区间，故（6.14）中的$\mathrm{P}\{x(\eta)=j\}=0(j\in E)$，从而$r_j=0(j\geqslant 0)$，于是$p=0$。反之，设$p=0$。由（6.14）得$p\{\xi_{k0}<\sigma\}=0(k\in E)$，其中$\xi_{kj}$按(9.4.1)确定。又对$j\in E$，

$$\mathrm{P}\{\xi_{kj}<\sigma\}\prod_{i=1}^{j}\frac{a_i}{a_i+b_i}\leqslant \mathrm{P}\{\xi_{k0}<\sigma\}.$$

故由引理9.4.2，$\mathrm{P}\{\xi_{EE}<\sigma\}\leqslant \sum\limits_{kj\ j\in E} P\{\xi_{kj}<\sigma\}=0$。由定理9.4.4，$X$满足向前方程组。

第十二章　两种生灭过程构造论的关系

§1. 引　　言

在第五章和第十一章中，我们已经用分析方法和概率方法分别构造了保守($a_0=0$)的生灭过程。自然要问，它们之间有什么联系。我们完满地解决了这一问题(杨向群[3])。

在本章中，我们假定$a_0=0$，边界点z正则或流出。并且记(5.5.1)的$X^2(\lambda)$为$X(\lambda)$。$X_i(\lambda)=E_i\{e^{-\lambda\tau}\}$，$\tau$为$Q$过程的第一个飞跃点。

§2. 对应定理

定理1　设非最小过程$X\in\mathscr{X}_s(Q)$的特征数列为p，q，$r_n(n\geqslant-1)$，则X的预解算子为

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+X_i(\lambda)$$

$$\times\frac{\sum\limits_k r_k\phi_{kj}(\lambda)+dzX_j(\lambda)\mu_j}{r_{-1}+\sum\limits_k r_k(1-X_k(\lambda))+dz\lambda\sum\limits_k X_k(\lambda)\mu_k} \tag{1}$$

其中d按(11.6.13)确定。

证　设$\psi^n_{ij}(\lambda)$为(Q,V^n)杜勃过程X^n的预解算子，则依(10.4.5)，

$$\psi^n_{ij}(\lambda)=\phi_{ij}(\lambda)+X_i(\lambda)\frac{\sum\limits_{k=0}^{n}v_k^n\phi_{kj}(\lambda)}{1-\sum\limits_{k=0}^{n}v_k^n+\sum\limits_{k=0}^{n}v_k^n(1-X_k(\lambda))}. \tag{2}$$

记上式中分式为$H_j^n(\lambda)$。将(11.6.12)代入$H_j^n(\lambda)$中，并注意$\sum\limits_{k=-1}^{n}v_k^n=1$得

$$H_j^n(\lambda)=$$

$$\frac{\frac{X_n}{A_n}\sum\limits_{k=0}^{n-1}r_k\phi_{kj}(\lambda)+\left(Y_n+\frac{X_n}{A_n}\sum\limits_{l=n}^{\infty}r_lC_{ln}\right)\phi_{nj}(\lambda)}{\frac{X_n}{A_n}r_{-1}+\frac{X_n}{A_n}\sum\limits_{k=0}^{n-1}r_k(1-X_k(\lambda))+\left(Y_n+\frac{X_n}{A_n}\sum\limits_{l=n}r_lC_{ln}\right)(1-X_n(\lambda))}$$

$$=\frac{\sum\limits_{k=0}^{n-1}r_k\phi_{kj}(\lambda)+\left(\frac{d}{C_{n0}}+\sum\limits_{l=n}^{\infty}r_lC_{ln}\right)\phi_{nj}(\lambda)}{r_{-1}+\sum\limits_{k=0}^{n-1}r_k(1-X_k(\lambda))+\left(\frac{d}{C_{n0}}+\sum\limits_{l=n}^{\infty}r_lC_{ln}\right)(1-X_n(\lambda))}$$

$$=\frac{\sum\limits_{k=0}^{n-1}r_k\phi_{kj}(\lambda)+\left(d+\sum\limits_{l=n}^{\infty}r_lc_{l0}\right)\frac{\phi_{nj}(\lambda)}{c_{n0}}}{r_{-1}+\sum\limits_{k=0}^{n-1}r_k(1-X_k(\lambda))+\left(d+\sum\limits_{l=n}^{\infty}r_lc_{l0}\right)\frac{1-X_n(\lambda)}{C_{n0}}}.$$

注意$A_0=\sum\limits_{l=0}^{\infty}r_lC_{l0}<\infty$，又$z$流出时$d=0$。由引理5.9.1和5.9.2，

$$\lim_{n\to\infty}H_j^n(\lambda)=\frac{\sum\limits_k r_k\phi_{kj}(\lambda)+dzX_j(\lambda)\mu_j}{r_{-1}+\sum\limits_k r_k(1-X_k(\lambda))+dz\lambda\sum\limits_k X_k(\lambda)\mu_k}.$$

因为$X=\lim\limits_{n\to\infty}X^n$，故$\psi_{ij}(\lambda)=\lim\limits_{n\to\infty}\psi^n_{ij}(\lambda)$，因而定理成立，证毕。

依定理5.6.1及定理5.9.3和5.9.4， 每个非最小Q过程 $\psi(\lambda)$有下列表现：

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+X_i(\lambda)\frac{\sum_k a_k\phi_{kj}(\lambda)+DX_j(\lambda)\mu_j}{c+\sum_k a_k(1-X_k(\lambda))+D\lambda\sum_k X_k(\lambda)\mu_k}. \tag{3}$$

其中行矢量$\alpha\geqslant 0$满足(5.9.5)，常数$D\geqslant 0$且z流出时$D=0$。并且$\sum_k a_k\phi_{kj}(\lambda)+DX_j(\lambda)\mu_j\neq 0$。常数$c\geqslant 0$。

我们指出：除常数因子不计外，矢量α，常数c和D由过程唯一决定。实际上可设α，c，D和$\bar{\alpha}$，$\bar{c}$，$\bar{D}$对应同一过程，则

$$\frac{\alpha\phi(\lambda)+DX(\lambda)\mu}{A_\lambda}=\frac{\bar{\alpha}\phi(\lambda)+\bar{D}X(\lambda)\mu}{\bar{A}_\lambda} \tag{4}$$

其中$A_\lambda=c+[\alpha,\ 1-X(\lambda)]+D\lambda[X(\lambda)\mu,\ 1]$，$\bar{A}_\lambda$为相应于$\bar{\alpha}$,$\bar{c}$,$\bar{D}$的量。右乘$\lambda I-Q$到(4)两边得

$$\frac{\alpha}{A_\lambda}=\frac{\bar{\alpha}}{\bar{A}_\lambda}\text{从而}\frac{D}{A_\lambda}=\frac{\bar{D}}{\bar{A}_\lambda}.$$

从而 $K=\dfrac{A_\lambda}{\bar{A}_\lambda}>0$且与$\lambda$无关。于是$\alpha=K\bar{\alpha}$，$D=K\bar{D}$，再代入(4)得$c=K\bar{c}$。

定义1 称(3)中的过程为$(Q,\ \alpha,\ c,\ D)$过程。

定理2 $(Q,\ \alpha,\ c,\ D)$过程的特征数列p，q，$r_n(n\geqslant -1)$是：

$$\left.\begin{aligned}
&r_{-1}=c,\quad r_n=a_n(n\geqslant 0),\\
&p=\begin{cases}0, & \text{如}\alpha=0,\\ \dfrac{A_0z}{A_0z+D}. & \text{如}\alpha\neq 0.\end{cases}\\
&q=\begin{cases}1, & \text{如}\alpha=0,\\ \dfrac{D}{A_0z+D}. & \text{如}\alpha\neq 0.\end{cases}
\end{aligned}\right\} \tag{5}$$

其中 $A_0=\sum_{l=0}^{\infty}r_lC_{l0}$。

证 比较(1)和(3)便知r_n与α_n相差常数因子，不妨认为 $r_n=\alpha_n(n\geqslant 0)$，故$r_{-1}=c$，$dz=D$。由此得(5)。

§3. 过程在第一个飞跃点的性质

由于对应定理，用分析方法构造出来的 $(Q,\ \alpha,\ c,\ D)$过程的运动情况，也就清楚了，而且可以计算它的某些概率的量。

定理1 设$X\in\mathscr{X}_s(Q)$为$(Q,\ \alpha,\ c,\ D)$过程。β_1^*按(11.6.1)确定。则概率$v_j^n=p\{x(\beta_1^*)=j\}(-1\leqslant j\leqslant n)$按下面方式计算：

$$\left.\begin{aligned}&v_{-1}^n=\frac{X_n}{A_n}c,\quad v_j^n=\frac{X_n}{A_n}\alpha_j(0\leqslant j<n),\\&v_n^n=Y_n+\frac{X_n}{A_n}\sum_{l=n}^{\infty}\alpha_lc_{ln}.\end{aligned}\right\}\tag{1}$$

其中

$$\left.\begin{aligned}&A_n=\sum_{l=0}^{\infty}\alpha_lc_{ln},\quad d=\begin{cases}\dfrac{D}{z},&\text{如}\alpha\neq 0,\\1,&\text{如}\alpha=0.\end{cases}\\&X_n=\frac{A_nc_{n0}}{(c+A_n)c_{n0}+d},\quad Y_n=\frac{d}{(c+A_n)c_{n0}+d}.\\&\frac{X_n}{A_n}=\frac{c_{n0}}{cc_{n0}+d},\quad\text{如果}\alpha=0.\end{aligned}\right\}\tag{2}$$

证 由定理2.2及定理11.6.3得出。

定理2 设$X\in\mathscr{X}_s(Q)$为$(Q,\ \alpha,\ c,\ D)$过程，τ为第一个飞跃点。称$D>0$或$[\alpha,\ 1]=\infty$为情形A，称$D=0$，$[\alpha,\ 1]<\infty$为情形B。则对$0\leqslant i<\infty$，

$$P\{x(\tau)=i\}=\begin{cases}0,&\text{情形}A,\\\dfrac{\alpha_i}{c+[\alpha,\ 1]},&\text{情形}B.\end{cases}$$

$$P\{\tau=\sigma<\infty\}=P\{x(\tau)=-1\}=\begin{cases}0, & \text{情形}A,\\ \dfrac{c}{c+[\alpha,\ 1]}, & \text{情形}B.\end{cases}$$

$$P\{x(\tau)=\infty\}=\begin{cases}1, & \text{情形}A,\\ 0, & \text{情形}B.\end{cases}$$

证 由(11.5.11)及X的右连续性，

$$\{x(\tau)=i\}=\lim_{n\to\infty}\{x(\beta_1^n)=i\},\quad -1\leqslant i<\infty.$$

由定理1，

$$P\{x(\tau)=i\}=\lim_{n\to\infty}\frac{X_n}{A_n}a_i,\quad i\geqslant 0.$$

$$P\{x(\tau)=-1\}=P\{\tau=\sigma<\infty\}=\lim_{n\to\infty}\frac{X_n}{A_n}c.$$

但$\dfrac{X_n}{A_n}=\dfrac{zc_{n0}}{z(c+A_n)c_{n0}+D}$。如$D>0$，显然$\dfrac{X_n}{A_n}\to 0$。如$D=0$，

则
$$\frac{X_n}{A_n}=\frac{1}{c+A_n}\to\frac{1}{c+[\alpha,\ 1]}.\tag{3}$$

实际上，因$A_n=\sum\limits_{l=0}^{n}a_l+\sum\limits_{l=n}^{\infty}a_lc_{ln}$。故如$[\alpha,\ 1]=\infty$，（3）显然成立。如$[\alpha,\ 1]<\infty$，则（3）由$\sum\limits_{l=n}^{\infty}a_lc_{ln}\leqslant\sum\limits_{l=n}^{\infty}a_l\to 0$ 得出。再注意

$P\{x(\tau)=\infty\}=1-\sum\limits_{i=-1}^{\infty}P\{x(\tau)=i\}$而得证定理，证毕。

第六篇　和构造论相联系的马氏过程的性质

第十三章　生灭过程的性质

§1. 引　　言

用概率方法构造出来的生灭过程，其轨道结构清楚，每一个生灭过程都是一列杜勃过程的强极限。因此为了研究Q过程的性质，只要先研究杜勃过程的性质，然后过渡到极限即可。我们将对首回时间的分布来执行这一程序。假定$a_0=0$，并采用第五章的记号，如增加解$u(\lambda)$和下降解$v(\lambda)$等，仍记$X(\lambda)=X^2(\lambda)=\dfrac{u(\lambda)}{u(z,\lambda)}$。

§2. 最小过程的一些精细结果

设$X=\{x(t),\ t<\sigma\}\in\mathscr{X}_s(Q)$，$\tau$为第一个飞跃点，$\tau_1$为第一个跳跃点。令

$$\eta_i=\begin{cases}inf\{t\,|\,\tau_1\leqslant t<\sigma,\ x(t)=i\}\\ \sigma,\ \text{如上集合为空集。}\end{cases}\tag{1}$$

为首回i的时刻。王梓坤[2，§5.2]已指出：存在$h=h(j)>0$，当$\lambda>-h$时，$E_k\{e^{-\lambda\eta_j}\}(k<j)$有穷，而且是方程

$$\left.\begin{aligned}&D_\mu u_k^+=\lambda\mu_k,\ k<j.\\&u_j=1.\end{aligned}\right\}\tag{2}$$

的唯一解．矩${}_1N^l_{kj}=E_k\{\eta^l_j\}=E_k\{\eta^l_j,\ \eta_j<\tau\}(k<j)$ 有穷，满足关系

$$\left.\begin{aligned}&{}_1N^l_{kj}=l\sum_{i=k}^{j-1}(z_{j+1}-z_j)\sum_{s=0}^{i}{}_1N^{l-1}_{sj}\mu_s,\ (k<j)\\&{}_1N^0_{kj}=1.\end{aligned}\right\}\tag{3}$$

又$N^l_k=E_k\{\tau^l\}$满足

$$\left.\begin{aligned}&N^l_k=l\sum_{i=k}^{\infty}(z_{i+1}-z_i)\sum_{s=0}^{i}N^{l-1}_s\mu_s\leqslant l!R^l,\ k\geqslant 0.\\&N^0_k=1.\end{aligned}\right\}\tag{4}$$

因此，实际上

$$E_k\{e^{-\lambda\eta_j}\}=\frac{u_k(\lambda)}{u_j(\lambda)},\ (k<j),$$

$$E_k\{e^{-\lambda\tau}\}=X_k(\lambda),\ \lambda>0.\tag{5}$$

$$E_k\{e^{-\lambda\eta_j}\}=\sum_{l=0}^{\infty}(-\lambda)^l\frac{{}_1N^l_{kj}}{l!},\ (|\lambda|<h,\ k<j).\tag{6}$$

当$R<\infty$时，

$$E_k\{e^{-\lambda\tau}\}=\sum_{l=0}^{\infty}(-\lambda)^l\frac{N^l_k}{l!},\ |\lambda|<\frac{1}{R}.\tag{7}$$

今设$i\leqslant k<j$，$i+1<j$，令

$${}_1\varphi_{kij}(\lambda)=E_k\{e^{-\lambda\eta_i},\ \eta_i<\eta_j\},$$

$${}_2\varphi_{kij}(\lambda)=E_k\{e^{-\lambda\eta_j},\ \eta_j<\eta_i\}.\tag{8}$$

$${}_1N^l_{kij}=E_k\{\eta^l_i,\ \eta_i<\eta_j\},\quad {}_2N^l_{kij}=E_i\{\eta^l_j,\ \eta_j<\eta_i\}.\tag{9}$$

$${}_1\varphi_{ki}(\lambda)=E_k\{e^{-\lambda\eta_i},\ \eta_i<\tau\},\ {}_2\varphi_{ki}(\lambda)=E_k\{e^{-\lambda\tau},\ \tau\leqslant\eta_i\}.\tag{10}$$

$${}_1N^l_{ki}=E_k\{\eta^l_i,\ \eta_i<\tau\},\quad {}_2N^l_{ki}=E_k\{\tau^l,\ \tau\leqslant\eta_i\}.\tag{11}$$

显然$j\uparrow\infty$时$P_k\{\eta_j\uparrow\tau\}=1$，且对$i\leqslant k$，$j\uparrow\infty$时，

$${}_1\varphi_{kij}(\lambda)\uparrow{}_1\varphi_{ki}(\lambda),\ {}_1\varphi_{kij}(\lambda)+{}_2\varphi_{kij}(\lambda)\uparrow{}_1\varphi_{ki}(\lambda)+{}_2\varphi_{ki}(\lambda).\tag{12}$$

$${}_1N^l_{kij}\uparrow{}_1N^l_{ki},\quad {}_1N^l_{kij}+{}_2N^l_{kij}\uparrow{}_1N^l_{ki}+{}_2N^l_{ki}.\tag{13}$$

定理1．(i) 存在$h=h(j)>0$，当$\lambda>-h$时，${}_a\varphi_{kij}(\lambda)$ $(a=1,$

2，$i\leqslant k<j$)有穷，一切矩$_aN^l_{kij}$($a=1$，2，$l\geqslant 0$，$i\leqslant k<j$)有穷，满足关系式

$$\left.\begin{aligned}
{}_aN^l_{kij} &= l\left\{\frac{z_j-z_k}{z_j-z_i}\sum_{s=i+1}^{k}(z_s-z_i)\,{}_aN^{l-1}_{sij}\mu_s\right.\\
&\quad\left.+\frac{z_k-z_i}{z_j-z_i}\sum_{s=k+1}^{j-1}(z_j-z_s)\,{}_aN^{l-1}_{sij}\mu_s\right\},\quad i<k<j.\\
{}_1N^0_{kij} &= \frac{z_j-z_k}{z_j-z_i},\quad {}_2N^0_{kij}=\frac{z_k-z_i}{z_j-z_i},\quad i<k<j.
\end{aligned}\right\}\tag{14}$$

$$\left.\begin{aligned}
{}_1N^l_{iij} &= \frac{1}{q_i}\{l\,{}_1N^{l-1}_{iij}+a_{i1}N^l_{i-1i}+b_{i1}N^l_{i+1ij}\},\\
{}_1N^0_{iij} &= \frac{a_i}{q_i}+\frac{b_i}{q_i}\,\frac{z_j-z_{i+1}}{z_j-z_i}
\end{aligned}\right\}\tag{15}$$

$$\left.\begin{aligned}
{}_2N^l_{iij} &= \frac{1}{q_i}\{l\,{}_2N^{l-1}_{iij}+b_{i2}N^l_{i+1ij}\},\\
{}_2N^0_{iij} &= \frac{b_i}{q_i}\,\frac{z_{i+1}-z_i}{z_j-z_i}.
\end{aligned}\right\}\tag{16}$$

$${}_a\varphi_{kij}(\lambda)=\sum_{l=0}^{\infty}(-\lambda)^l\frac{{}_aN^l_{kij}}{l!},\quad (i\leqslant k<j,\ |\lambda|<h).\tag{17}$$

$$\left.\begin{aligned}
{}_1\varphi_{kij}(\lambda) &= \frac{u_j(\lambda)v_k(\lambda)-u_k(\lambda)v_j(\lambda)}{u_j(\lambda)v_i(\lambda)-u_i(\lambda)v_j(\lambda)},\\
{}_2\varphi_{kij}(\lambda) &= \frac{u_k(\lambda)v_i(\lambda)-u_i(\lambda)v_k(\lambda)}{u_j(\lambda)v_i(\lambda)-u_i(\lambda)v_j(\lambda)}.
\end{aligned}\right\}(\lambda>0,\ i<k<j)\tag{18}$$

$$\left.\begin{aligned}
{}_1\varphi_{iij}(\lambda) &= \frac{1}{\lambda+q_i}\left(a_i\frac{u_{i-1}(\lambda)}{u_i(\lambda)}+b_{i1}\varphi_{i+1ij}(\lambda)\right),\\
{}_2\varphi_{iij}(\lambda) &= \frac{1}{\lambda+q_i}b_{i2}\varphi_{i+1ij}(\lambda).
\end{aligned}\right\}(\lambda>0)\tag{19}$$

(ii)

$$
{}_1\varphi_{ki}(\lambda)=\frac{\nu_k(\lambda)}{\nu_i(\lambda)},\ {}_2\varphi_{ki}(\lambda)=X_k(\lambda)-X_i(\lambda)\frac{\nu_k(\lambda)}{\nu_i(\lambda)},\ (i<k,\lambda>0). \tag{20}
$$

$$
\left.\begin{aligned}
{}_1\varphi_{ii}(\lambda)&=\frac{1}{\lambda+q_i}\left(a_i\frac{u_{i-1}(\lambda)}{u_i(\lambda)}+b_i\frac{\nu_{i+1}(\lambda)}{\nu_i(\lambda)}\right),\lambda>0.\\
{}_2\varphi_{ii}(\lambda)&=\frac{b_i}{\lambda+q_i}\left(X_{i+1}(\lambda)-X_i(\lambda)\frac{\nu_{i+1}(\lambda)}{\nu_i(\lambda)}\right),\lambda>0.
\end{aligned}\right\} \tag{21}
$$

$$
\left.\begin{aligned}
{}_aN^l_{ki}&=l\Bigg\{\frac{z-z_k}{z-z_i}\sum_{s=i+1}^{k}(z_s-z_i)\,{}_aN^{l-1}_{si}\mu_s\\
&\quad+\frac{z_k-z_i}{z-z_i}\sum_{s=k+1}^{\infty}(z-z_s)\,{}_aN^{l-1}_{si}\mu_s\Bigg\}\leqslant l!R^{l-1}N_k\leqslant lR^l,\\
&\quad(i<k,\ a=1\text{或}R<\infty\text{时}a=2).\\
{}_1N^0_{ki}&=\frac{z-z_k}{z-z_i},\quad {}_2N^0_{ki}=\frac{z_k-z_i}{z-z_i},\quad(i<k).
\end{aligned}\right\} \tag{22}
$$

$$
\left.\begin{aligned}
{}_1N^l_{ii}&=\frac{1}{q_i}\{l\,{}_1N^{l-1}_{ii}+a_i\,{}_1N^l_{i-1\,i}+b_i\,{}_1N^l_{i+1\,i}\}\leqslant l!R^l,\\
{}_1N^0_{ii}&=\frac{a_i}{q_i}+\frac{b_i}{q_i}\frac{z-z_{i+1}}{z-z_i}.
\end{aligned}\right\} \tag{23}
$$

$$
\left.\begin{aligned}
{}_2N^l_{ii}&=\frac{1}{q_i}\{l\,{}_2N^{l-1}_{ii}+b_i\,{}_2N^l_{i+1\,i}\}\leqslant l!R^l,\\
{}_2N^0_{ii}&=\frac{b_i}{q_i}\cdot\frac{z_{i+1}-z_i}{z-z_i}.
\end{aligned}\right\} \tag{24}
$$

当$R<\infty$时，

$$
{}_a\varphi_{ki}(\lambda)=\sum_{l=0}^{\infty}(-\lambda)^l\frac{{}_aN^l_{ki}}{l!}\left(i\geqslant k,|\lambda|<\frac{1}{R}\right). \tag{25}
$$

证。(i)因为$\lambda>-h$时，$E_k\{e^{-\lambda\eta_j}\}$ $(k>j)$，$E_k\{\eta^l_j\}$ $(k<j)$有穷，故${}_a\varphi_{kij}(\lambda)$及${}_aN^l_{kij}$有穷，而且由 *Wilks*〔1，第14页〕，

$$
{}_aN^l_{kij}=(-1)^l\frac{d^l}{d\lambda^l}\,{}_a\varphi_{kij}(\lambda)\Big|_{\lambda=0} \tag{26}
$$

并且(17)成立。

利用强马氏性易得(19)。在(19)两边乘$(\lambda+q_i)$，微分l次并注意(26)及(5)(6)，便得(15)和(16)中的第一式。又显然

$$ {}_1N^0_{ij}=P_i\{\eta_i<\eta_j\}=\frac{a_i}{q_i}P_{i-1}(\eta_i<\eta_j)+\frac{b_i}{q_i}+P_{i+1}(\eta_i<\eta_i), $$

$$ {}_2N^0_{ij}=\frac{b_i}{q_i}P_{i+1}(\eta_j<\eta_i). $$

于是(15)和(16)中第二式自(11.2.8)得出。

应用强马氏性可得${}_a\varphi_{kij}(\lambda)(i<k<j,\ \lambda>0)$满足方程：

$$ \left.\begin{aligned} &u_i=\frac{1-(-1)^a}{2},\\ &(\lambda+a_k+b_k)u_k-a_ku_{k-1}-b_ku_{k+1}=0,\quad(i<k<j)\\ &u_j=\frac{1+(-1)^a}{2}. \end{aligned}\right\} $$

解上方程得(18)。微分上式l次并注意(26)得$u_k={}_aN^l_{kij}$ $(i<k<j,$ $l\geqslant1)$满足$f_i=0, f_k=l\,{}_aN^{l-1}_{kij}$，$(i<k<j)$，$f_j=0$的方程(4.4.11)，依定理5.2.3和(4.4.12)得(14) 第一式。第二式由 (11.2.8) 得出。

(ii) 由(12)，在(18)、(19)中令$j\uparrow\infty$而得(20)(21)。

当$R<\infty$时，由(7)，对$k\geqslant i$有

$$ {}_1N^l_{ki}+{}_2N^l_{ki}=E_k\{min(\eta_i,\tau)\}^l\leqslant E_k\tau^l\leqslant l!R^l<\infty. $$

故(25)成立

(22)第二式由(11.2.16)得出。由(13)，(23)、(24) 只需对(i)中相应的式子取极限而得。在(14)第一式中令$j\uparrow\infty$，依单调收敛定理得(22)对$a=1$成立以及 (22) 中用${}_1N^l_{ki}+{}_2N^l_{ki}$，${}_1N^{l-1}_{ii}+{}_2N^{l-1}_{ii}$代替${}_1N^l_{ki}$，${}_1N^{l-1}_{ii}$后仍成立，而且有${}_1N^l_{ki}+{}_2N^l_{ki}\leqslant l!R^l<\infty$。从而得(22)对$a=2$也成立，证毕。

定理2. 最小解常返的充要条件是$z=\infty$。更精确些，

$$ \int_0^\infty f_{ij}(t)dt=\lim_{\lambda\downarrow0}\phi_{ij}(\lambda)=\Gamma_{ij}=\begin{cases}(z-z_j)\mu_j,\text{如}j\geqslant i.\\(z-z_i)\mu_j,\text{如}j<i.\end{cases} \tag{27} $$

证. 由(5.3.2)得

$$u_i(\lambda)\downarrow 1,\ (\lambda\downarrow 0).\tag{28}$$

由(5.3.4)得$\nu_i(\lambda)\to\sum_{j=i}^{\infty}(z_{j+1}-z_j)=z-z_i$，$(\lambda\downarrow 0)$.故得(27)。

定理3. 设$z=\infty$。则最小解遍历的充要条件是$\sum_{i=0}^{\infty}\mu_i<\infty$。

证. 由(11.2.17)，${}_1N_{i-1i}^1=m_{i-1}$。当$z=\infty$时，由(22)(23)，${}_1N_{ii}^0=1$，

$$E_i\eta_i={}_1N_{ii}^i=\frac{1}{q_i}\{1+a_im_{i-1}+b_{i\,1}N_{i+1i}^1\},$$

$${}_1N_{i+1i}^1=(z_{i+1}-z_i)\sum_{s=i+1}^{\infty}u_s.$$

因此$E_i\{\eta_i\}<\infty$等价于$\sum_{s=0}^{\infty}u_s<\infty$，证毕。

定理4. 设$R<\infty$，$S<\infty$。则

$$\lim_{n\to\infty}\frac{{}_1\varphi_{nj}(\lambda)}{z-z_n}=\frac{X_0(\lambda)}{\nu_j(\lambda)},\ \lambda>0.\tag{29}$$

$$\lim_{n\to\infty}\frac{1-{}_2\varphi_{nj}(\lambda)}{z-z_n}=\lambda\sum_k X_k(\lambda)\mu_k+\frac{X_0(\lambda)X_j(\lambda)}{\nu_j(\lambda)}.\ \lambda>0.\tag{30}$$

证. 由引理5.9.1和5.9.2，从(20)便得(29)(30)。

§3. 过程的不变测度

设$R<\infty$。如果过程常返，它必不中断。反之，由定理11.4.2，不中断过程常返而且遍历。我们将求出其不变测度。

定理1 设X为$(Q,a,0,D)$过程。则其不变测度

$$\pi_j=\lim_{t\to\infty}p_{ij}(t)=\frac{\sum_k a_k\Gamma_{kj}+D\mu_j}{\sum_k a_kN_k+D\sum_k\mu_k}\tag{1}$$

而且　$$\mathrm{m}_{ii}=E_i\eta_i=\frac{\sum\limits_k a_k\Gamma_{ki}+D\mu_i}{q_i\left(\sum\limits_k a_k N_k+D\sum\limits_k \mu_k\right)}\tag{2}$$

这里Γ_{kj}由(2.27)确定，N_k由(5.2.6)确定。

证．由(6.7.10)，(1)中极限存在．由滔泊(Tauber)定理(Hardy〔1，定理98〕)，$\lim\limits_{\lambda\downarrow 0}\lambda\psi_{ij}(\lambda)=\lim\limits_{t\to\infty}p_{ij}(t)$。

由于$P_i\{\tau<\infty\}=1$，$X_i(\lambda)=E_i\{e^{-\lambda\tau}\}\uparrow 1\ (\lambda\downarrow 0)$。又$\dfrac{1-X_i(\lambda)}{\lambda}\sum\limits_j\phi_{ij}(\lambda)\uparrow\sum\limits_j\Gamma_{ij}=N_i,(\lambda\downarrow 0)$。注意$z$流出时$D=0$，正则时$\sum\limits_k u_k<\infty$。故由(12.2.3)得证(1)。由(1)及(6.7.10)得(2)。

定理2．设$X\in\mathscr{X}_t(Q)$为$(Q,\ a,\ 0,\ D)$过程，π为不变测度，函数f，g满足$\sum\limits_i|f(i)|\pi_i<\infty$，$\sum\limits_i|g(i)|\pi_i<\infty$，$\sum\limits_i g(i)\pi_i\neq 0$，则

$$P\left\{\lim_{t\to\infty}\frac{\int_0^t f[x(u)]du}{\int_0^t g[x(u)]du}=\frac{\sum\limits_i f(i)\pi_i}{\sum\limits_i g(i)\pi_i}\right\}=1。\tag{3}$$

证。见李漳南、吴荣〔1，定理3.1〕。

§4.　首回时的分布

定理1。设$X\in\mathscr{X}_t(Q)$为$(Q,\ a,\ 0,\ D)$过程，η_j如(2.1)。则对$\lambda>0$，

$$E_i\{e^{-\lambda\eta_j}\}=\frac{u_i(\lambda)}{u_j(\lambda)}，\ (i<j)。\tag{1}$$

$$E_i\{e^{-\lambda\eta_i}\}=1-m(\lambda)(\lambda+q_i)^{-1}\Big\{m(\lambda)\phi_{ii}(\lambda)+X_i(\lambda)$$

$$\cdot\left[\sum_k a_k\phi_{ki}(\lambda)+DX_i(\lambda)u_i\right]\Bigg\}^{-1}. \tag{2}$$

$$E_i\{e^{-\lambda\eta_j}\}=\Big\{m(\lambda)\phi_{ij}(\lambda)+X_i(\lambda)\left[\sum_k a_k\phi_{kj}(\lambda)+DX_j(\lambda)\mu_j\right]\Big\}$$

$$\times\Big\{m(\lambda)\phi_{jj}(\lambda)+X_j(\lambda)\left[\sum_k a_k\phi_{kj}(\lambda)+DX_j(\lambda)u_j\right]\Big\}^{-1}$$

$$(i>j). \tag{3}$$

其中 $$m(\lambda)=\sum_k a_k(1-X_k(\lambda))+D\lambda\sum_k X_k(\lambda)\mu_k. \tag{4}$$

或等价地

$$E_i\{e^{-\lambda\eta_j}\}={}_1\varphi_{ij}(\lambda)+{}_2\varphi_{ij}(\lambda)\Big\{\sum_{k=0}^{j}a_k\frac{u_k(\lambda)}{u_j(\lambda)}+\sum_{k=j+1}^{\infty}a_k\frac{\nu_k(\lambda)}{\nu_j(\lambda)}$$

$$+D\frac{X_j(\lambda)}{\nu_j(\lambda)}\Big\}\Big\{\sum_{k=0}^{j}a_k+\sum_{k=j+1}^{\infty}a_k\Big[1-X_k(\lambda)$$

$$+X_j(\lambda)\frac{X_0(\lambda)}{\nu_j(\lambda)}\Big]\Big\}^{-1},\ (i\geqslant j) \tag{5}$$

其中${}_a\varphi_{ij}(\lambda)$由(2.20)(2.21)确定。

证。(1)即(2.5)·按熟知的公式 Chung 〔1，第192–193页〕，

$$p_{ii}(t)=e^{-q_it}+\int_0^t p_{ii}(t-u)dp_i(\eta_i\leqslant u),$$

$$p_{ij}(t)=\int_0^t p_{jj}(t-u)dP_i(\eta_j\leqslant u),\ \ i\neq j.$$

取拉氏变换，注意(12.2.3)并经过简单整理即得(2)(3)。

今考虑(Q,V^n)杜勃过程。令X的第h个飞跃点为σ_h、对$i\geqslant j$，由常返性$P_i\{\eta_j<\infty\}=1$。由杜勃过程的结构知，

$$E_i\{e^{-\lambda\eta_j}\}=E_i\{e^{-\lambda\eta_j},\ \eta_j<\sigma_1\}+\sum_{h=1}^{\infty}E_i\{e^{-\lambda\eta_j},\ \sigma_h\leqslant\eta_j<\sigma_{h+1}\}$$

$$={}_1\varphi_{ij}(\lambda)+\sum_{h=1}^{8}E_i\{e^{-\lambda\sigma_1},\ \sigma_1\leqslant\eta_j\}\Big[\sum_{k=j+1}^{\infty}v_kE_k(e^{-\lambda\sigma_1},$$

$$\left.\sigma_j\leqslant\eta_j)\right]^{-1}\cdot\left\{\sum_{k=0}^{j-1}v_k E_k(e^{-\lambda\eta_j})+v_j\right.$$

$$\left.+\sum_{k=j+1}^{\infty}v_k E_k(e^{-\lambda\sigma_j},\ \eta_j\leqslant\sigma_j)\right\}$$

$$={}_1\varphi_{ij}(\lambda)+{}_2\varphi_{ij}(\lambda)\left\{\sum_{k=0}^{j}v_k\frac{u_k(\lambda)}{u_j(\lambda)}+\sum_{k=j+1}^{\infty}v_k\frac{\nu_k(\lambda)}{\nu_j(\lambda)}\right\}$$

$$\times\left\{1-\sum_{k=j+1}^{\infty}v_{k\,2}\varphi_{kj}(\lambda)\right\}^{-1}$$

$$={}_1\varphi_{ij}(\lambda)+{}_2\varphi_{ij}(\lambda)\left\{\sum_{k=0}^{j}v_k\frac{u_k(\lambda)}{u_j(\lambda)}+\sum_{k=j+1}^{\infty}v_k\frac{v_k(\lambda)}{\nu_j(\lambda)}\right\}$$

$$\cdot\left\{\sum_{k=0}^{j}v_k+\sum_{k=j+1}^{\infty}v_k(1-{}_2\varphi_{kj}(\lambda))\right\}^{-1}. \qquad (6)$$

对于一般的$(Q,\ a,\ 0,\ D)$过程X，它是不断的$(Q,\ V^n)$杜勃过程X^n的强极限·对X和X^n分别定义η_i和η_i^n，则$\eta_i^n\uparrow\eta_i$。

将(12.3.1)中的V^n代入(6)并令$n\to\infty$，注意(5.9.2)(5.9.3)(2.29)(2.30)及(2.20)，经简单计算后便得(5)，证毕。

第十四章　常返性和遍历性

§1. 引　言

过程的常返性和遍历性对于研究转移概率在无穷的渐近性质是十分有用的，在研究过份函数和零壹律时（王梓坤[2,5,8,9]）在研究可逆性时（钱敏、侯振挺[1]），亦非常重要。马氏过程的状态分类的一般研究已总结于 Chung [1,II § 10]。吴立德[1]和 Rupert [1]研究了最小Q过程的状态分类，找出了过程常返或遍历的联系于Q矩阵的充要条件。

本章研究Q过程的状态分类，紧密地结合构造论。此时分类，不仅依赖于Q，还依赖于过程的构造，因而对不同的构造需不同的处理。本章内容取自杨向群[11]。

§2. 两个引理

状态i关于$\psi(\lambda)$常返或遍历，将说i为$\psi(\lambda)$常返或遍历。如果一切状态i都常返或遍历，则说过程常返或遍历。显然，i为$\psi(\lambda)$常返当且只当$\lim\limits_{\lambda\downarrow 0}\psi_{ii}(\lambda)=\infty$。由(6.7.9)及滔泊(Tauber)定理，$i$为$\psi(\lambda)$遍历当且只当$\lim\limits_{\lambda\downarrow 0}\lambda\psi_{ii}(\lambda)=\pi_i>0$。而且按（6.7.8）定义的$m_{ii}$，有

$$\pi_i=\frac{1}{q_im_{ii}}. \tag{1}$$

下面$i\xrightarrow{\psi(\lambda)}j$表示关于过程$\psi(\lambda)$，$i$可到达$j$，并且恒有$i\xrightarrow{\psi(\lambda)}i$。

设$X \in \mathscr{X}_s(Q)$，τ_1和τ分别为第一个间断点和飞跃点。

引理1 当$\lambda \downarrow 0$时，

$$\phi_{ij}(\lambda) \uparrow \Gamma_{ij} = \sum_{n=0}^{\infty} \Pi^n_{ij} \frac{1}{q_j} \leqslant \Gamma_{jj}. \tag{2}$$

$$\xi_i(\lambda) \equiv E_i\{e^{-\lambda\tau}\} \uparrow \xi_i = P_i\{\tau < \infty\}. \tag{3}$$

$$\frac{1-\xi_i(\lambda)}{\lambda} \uparrow N_i = \sum_j \Gamma_{ij} = E_i\tau. \tag{4}$$

其中(Π_{ij})如(1.9.7)，

$$\xi_i(\lambda) = 1 - \lambda \sum_j \phi_{ij}(\lambda). \tag{5}$$

证 (2)式即(1.10.28)。至于$\Gamma_{ij} \leqslant \Gamma_{jj}$，由定理6.2.1得出，仿定理6.12.3的证明可得

$$\lambda \sum_j \phi_{ij}(\lambda) = 1 - E_i\{e^{-\lambda\tau}\}. \tag{6}$$

比较(5)得$\xi_i(\lambda) = E_i\{e^{-\lambda\tau}\}$，并且得(3)。最后，

$$\frac{1-\xi_j(\lambda)}{\lambda} = \sum_j \phi_{ij}(\lambda) = E_i\left\{\int_0^{\tau} e^{-\lambda t}\, dt\right\}. \tag{7}$$

由此得(4)

引理2 设X为$\psi(\lambda)$过程，i为$\psi(\lambda)$常返。则$P_i\{\sigma = \infty\} = 1$。设$i$为$\psi(\lambda)$常返而$\phi(\lambda)$非常返，则$P_i\{\tau < \infty\} = 1$。

证 第一结论是明显的，因为i为$\psi(\lambda)$常返，则在$\{x(0) = i\}$上，X无穷多次回到i。而依定理6.7.3，第n次在i停留的时间ρ^n_i $(n \geqslant 0)$彼此独立。故$\sigma \geqslant \sum_{n=0}^{\infty} \rho^n_i = \infty$。

由定理6.7.4，如果i为$\psi(\lambda)$常返，则

$P_i\{X(\omega)$在$[0, \infty)$中有无穷多个i区间$\} = 1$，

如果i为$\phi(\lambda)$非常返，则

$P_i\{X(\omega)$在$[0, \tau(\omega))$中只有有穷多个i区间$\} = 1$。

所以$P_i\{\tau < \infty\} = 1$。

§3. 杜勒过程

定理1 设$\psi(\lambda)$为(Q,π)杜勒过程，i为$\phi(\lambda)$非常返，则i为$\psi(\lambda)$常返的充要条件是

$$\xi_i=1,\quad \sum_k \pi_k\xi_k=1,\ \text{存在}k\text{使}\pi_k>0,\ k\overset{\varphi(\lambda)}{\Longrightarrow}i. \tag{1}$$

如果i为$\psi(\lambda)$常返，则i遍历当且只当

$$\sum_k \pi_k N_k<\infty. \tag{2}$$

注 因$\sum\limits_k \pi_k\leqslant 1$，$\xi_k\leqslant 1$．故如(1)成立，必定$\sum\limits_k \pi_k=1$，即$\psi(\lambda)$不中断。

证 由假设$\Gamma_{ii}<\infty$。由引理2.1，在(10.4.5)中令$\lambda\downarrow 0$得

$$\lim_{\lambda\downarrow 0}\psi_{ii}(\lambda)=\Gamma_{ii}+\xi_i\frac{\sum\limits_k \pi_k\Gamma_{ki}}{1-\sum\limits_k \pi_k\xi_k}. \tag{3}$$

设i为$\psi(\lambda)$常返，故(3)左方为无穷．依引理2.2，$\xi_i=1$。其次必定$\sum\limits_k \pi_k\Gamma_{ki}\leqslant\left(\sum\limits_k \pi_k\right)\Gamma_{ii}<\infty$，且为正。因为不然的话，(3)成为$\lim\limits_{\lambda\downarrow 0}\psi_{ii}(\lambda)=\Gamma_{ii}<\infty$，矛盾．既然$\sum\limits_k \pi_k\Gamma_{ki}>0$，故存在$\pi_k>0$，$\Gamma_{ki}>0$，而后者等价于$k\overset{\phi(\lambda)}{\Longrightarrow}i$。所以(1)成立。反之，由(1)得(3)式右方为无穷。

设(1)成立．由引理2.1和本定理的注，从(10.4.5)，

$$\lim_{\lambda\downarrow 0}\lambda\psi_{ii}(\lambda)=\lim_{\lambda\downarrow 0}\lambda\phi_{ii}(\lambda)+\lim_{\lambda\downarrow 0}\xi_i(\lambda)\frac{\sum\limits_k \pi_k\phi_{ki}(\lambda)}{\sum\limits_k \pi_k\dfrac{1-\xi_k(\lambda)}{\lambda}}$$

$$=\frac{\sum_k \pi_k \Gamma_{ki}}{\sum_k \pi_k N_k}.$$

已指出，上式右方分母有穷且为正，故i的$\psi(\lambda)$遍历性等价于(2)，证毕。

系1 设对任意i, j, $i \overset{\phi(\lambda)}{\Longleftrightarrow} j$, 且$\phi(\lambda)$非常返。则$(Q,\ \pi)$杜勃过程常返当且只当$\sum_k a_k \xi_k = 1$。如果过程常返，则遍历的充要条件是(2)。

证 关于常返的必要性明显。充分性则由于$\sum_k \pi_k \xi_k = 1$，因而必有某i使$\pi_i > 0$，$\xi_i = 1$，从而对此i，(1)满足，故i为$\psi(\lambda)$常返。又显然$i \overset{\phi(\lambda)}{\Longleftrightarrow} j$，故$\psi(\lambda)$常返

系2 对任i, j, $i \overset{\phi(\lambda)}{\Longleftrightarrow} j$, Q是单流出的。则$(Q,\ \pi)$杜勃过程常返的充要条件是它不中断，即Q保守且$\sum_k \pi_k = 1$。

证 对单流出的Q，$P_i\{\tau < \infty \mid x(\tau - 0) \in B_e\} = 1$。故由定理6.12.8，$\xi_i = P_i\{\tau < \infty\} = 1$。条件$\sum_k \pi_k \xi_k = 1$化为$\sum_k \pi_k = 1$。

§4. 单流出过程

引理1 对于引理1.11.4中的$\eta(\lambda)$和ξ，

$$\lambda[\eta(\lambda),\ \xi] \downarrow 0,\ (\lambda \downarrow 0). \tag{1}$$

证 由(1.11.40)和$\xi(\lambda) \uparrow \xi (\lambda \downarrow 0)$得出。

定理2 设$\psi(\lambda)$为定理2.2.1中的过程，i为$\phi(\lambda)$非常返。则i为$\psi(\lambda)$常返的充要条件是

$$\bar{X}_i > 0,\ \sum_k \alpha_k \Gamma_{kj} + \bar{\eta}_j > 0,\ c = [\alpha,\ X^0] + \overline{\sigma^0} = 0 \tag{2}$$

如果i为$\psi(\lambda)$常返，则i为$\psi(\lambda)$遍历的充要条件是

$$\sum_k \{\alpha_k \sum_j \Gamma_{kj}\overline{X}_j + \overline{\eta_k}\overline{X}_k\} < \infty. \tag{3}$$

其中 $\overline{\eta_j}(\lambda)\uparrow\overline{\eta_j}$，$(\lambda\downarrow 0)$。

证 仿定理3.1进行，只需注意，由（1.11.7）有

$$\frac{\overline{X}_i - \overline{X}_i(\lambda)}{\lambda} = \sum_j \phi_{ij}(\lambda)\overline{X}_j \uparrow \sum_j \Gamma_{ij}\overline{X}_j,\ (\lambda\downarrow 0),$$

并应用引理1即可。

系 设Q保守，并且是单流出的，对任意 i，$j\in E$，$i \overset{\phi(\lambda)}{\Longleftrightarrow} j$，$\phi(\lambda)$非常返。则$\psi(\lambda)$常返当且只当$c = [a, X^0] + \overline{\sigma^0} = 0$。

§5. 一阶过程

我们考虑一阶过程 $X = \{x(t),\ t<\sigma\} \in \mathscr{X}_1(Q)$。一阶过程$X$是最小$Q$过程$X^0 = \{x(t),\ t<\tau\}$的$\{\Pi(a, \cdot),\ a\in B_e\}$广义$D^*$型延拓过程，其中$\Pi(a, \cdot)$ 满足（10.6.2）。因此我们称X为 $\{Q, \Pi(a, \cdot),\ a\in B_e\}$一阶过程。

为了简单，本节中我们假定由Q导出的非原子流出边界B_{e2}为空集．即流出边界B_e全是原子的边界点。简记$\mathscr{A} = B_e$。

根据（10.5.26），$\{Q, \Pi(a, \cdot),\ a\in\mathscr{A}\}$ 一阶过程的预解算子$\psi(\lambda)$有表现：

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \sum_{a\in\mathscr{A}}\sum_{b\in\mathscr{A}} X_i^a(\lambda)G_{ab}(\lambda)A_j^b(\lambda), \tag{1}$$

其中

$$\left.\begin{aligned}
&X_i^a(\lambda) = E_i\{e^{-\lambda\tau},\ x(\tau-0)=a\},\quad A_j^b(\lambda) = \sum_k \Pi(b,k)\phi_{kj}(\lambda).\\
&V_{ab}(\lambda) = \sum_k \Pi(a, k)X_k^b(\lambda),\\
&\mathscr{V}(\lambda) = \{V_{ab}(\lambda)\},\quad \{\mathscr{V}(\lambda)\}^l = \{V^l_{ab}(\lambda)\}\text{是 } \mathscr{A}\times\mathscr{A} \text{ 方阵}.\\
&G_{ab}(\lambda) = \sum_{l=0}^{\infty} V^l_{ab}(\lambda).
\end{aligned}\right\}$$

(2)

如果令

$$A=\{i \mid i\in E,\ 存在a\in \mathscr{A}使\Pi(a,\ i)>0\}. \tag{3}$$

则显然有

$$\Pi(a,\ E-A)=0,\ a\in \mathscr{A}. \tag{4}$$

而(1) 可以改写成

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+\sum_{r\in A}\sum_{q\in A}Z_{ir}(\lambda)D_{rq}(\lambda)\phi_j(\lambda). \tag{5}$$

其中

$$\left.\begin{aligned}
&Z_{iq}(\lambda)=\sum_{a\in\mathscr{A}}X_i^a(\lambda)\Pi(a,\ q),\ i\in E,\ q\in A.\\
&\mathscr{Z}(\lambda)=\{Z_{rq}(\lambda)\},\{\mathscr{Z}(\lambda)\}^l=\{Z^l{}_{rq}(\lambda)\}是A\times A方阵,\\
&D_{rq}(\lambda)=\sum_{l=0}^{\infty}Z^l{}_{rq}(\lambda).
\end{aligned}\right\}$$

(6)

显然，当$\lambda\downarrow 0$时

$$\left.\begin{aligned}
&X_i^a(\lambda)\uparrow X_i^a=P_i\{x(\tau-0)=a\},\ A_j^a(\lambda)\uparrow A_j^a=\sum_k\Pi(a,k)\Gamma_{kj}.\\
&\mathscr{Y}(\lambda)\uparrow\mathscr{Y}=\{\sum_k\Pi(a,\ k)X_k^b\},\ \{\mathscr{Y}(\lambda)\}^l\uparrow\mathscr{Y}^l,\\
&Z_{iq}(\lambda)\uparrow Z_{iq}=\sum_{a\in A}X_i^a\Pi(a,\ q),\ \{\mathscr{Z}(\lambda)\}^l\uparrow\mathscr{Z}^l.\\
&\mathscr{G}(\lambda)\uparrow\mathscr{G}=\sum_{l=0}^{\infty}\mathscr{Y}^l,\ \mathscr{D}(\lambda)\uparrow\mathscr{D}=\sum_{l=0}^{\infty}\mathscr{Z}^l.
\end{aligned}\right\}$$

(7)

又显然$\mathscr{Y}(\lambda)$、$\mathscr{Y}$和$\mathscr{Z}(\lambda)$，$\mathscr{Z}$可以作为状态空间为$\mathscr{A}$和A的马氏链的一步转移矩阵，故可说a为 $\mathscr{Y}(\lambda)$ 常返等等。因而由（1.2.10），

$$\left.\begin{aligned}
&G_{ab}(\lambda)\leqslant G_{bb}(\lambda),\ G_{ab}\leqslant G_{bb},\\
&D_{rq}(\lambda)\leqslant D_{qq}(\lambda),\ D_{rq}\leqslant D_{qq}.
\end{aligned}\right\} \tag{8}$$

引理1 (i) 设对某a，$b\in\mathscr{A}$有$\lim\limits_{\lambda\downarrow 0}\lambda G_{bb}(\lambda)=0$，且$a\overset{\mathscr{Y}}{\Longrightarrow}b$，则$\lim\limits_{\lambda\downarrow 0}\lambda G_{ab}(\lambda)=\lim\limits_{\lambda\downarrow 0}\lambda G_{ba}(\lambda)=\lim\limits_{\lambda\downarrow 0}\lambda G_{aa}(\lambda)=0$。

(ii) 设对某r，$q\in A$有$\lim\limits_{\lambda\downarrow 0}\lambda D_{qq}(\lambda)=0$，且$r\overset{\mathscr{Z}}{\Longrightarrow}q$，则$\lim\limits_{\lambda\downarrow 0}\lambda D_{rq}(\lambda)=\lim\limits_{\lambda\downarrow 0}\lambda D_{qr}(\lambda)=\lim\limits_{\lambda\downarrow 0}\lambda D_{rr}(\lambda)=0$。

证 只证(i)。由假设，存在α，β使$V^{\alpha}_{ab}V^{\beta}_{ba}>0$。而由Chung [1，第22页注]，

$$V^{\alpha+\beta+l}_{bb}(\lambda)\geqslant V^{\beta}_{ba}(\lambda)V^{l}_{aa}(\lambda)V^{\alpha}_{ab}(\lambda).$$

于是

$$\lambda\{G_{bb}(\lambda)-\sum_{l=0}^{\alpha+\beta}V^{l}_{bb}(\lambda)\}\geqslant V^{\beta}_{ba}(\lambda)\lambda G_{aa}(\lambda)V^{\alpha}_{ab}(\lambda).$$

注意(7)，有$V^{\beta}_{ba}(\lambda)V^{\alpha}_{ab}(\lambda)\uparrow V^{\beta}_{ba}V^{\alpha}_{ab}$，$(\lambda\downarrow 0)$。从而

$$\lim_{\lambda\downarrow 0}\lambda G_{bb}(\lambda)\geqslant V^{\beta}_{ba}\lim_{\lambda\downarrow 0}\lambda G_{aa}(\lambda)V^{\alpha}_{ab}.$$

故$\lim\limits_{\lambda\downarrow 0}\lambda G_{aa}(\lambda)=0$。再由(8)得证引理。

引理2 设$\psi(\lambda)$为$\{Q,\ \Pi(a,\ \cdot),\ a\in\mathscr{A}\}$一阶过程。$i$为$\phi(\lambda)$非常返而为$\psi(\lambda)$常返。

(i) 如果$X^{a}_{i}>0$，则a为$\mathscr{Y}$常返。

(ii) 如果$Z_{ir}>0$，则r为$\mathscr{Z}$常返。

证 (i) 设X为$\psi(\lambda)$过程，τ^{n}为第n个飞跃点。由一阶过程的结构，$x(\tau^{n}-0)(n\geqslant 1)$和$x(\tau^{n})$分别为$\mathscr{Y}$链和$\mathscr{Z}$链。由假设及引理2.2，

$$P_i\{\tau<\infty\}=P_i\{\sigma=\infty\}=1. \tag{9}$$

令 $\delta_0=0$，$\beta_0=\tau$，δ_1为β_0后X首回i的时刻，β_1为X在δ_1后的第一个飞跃点。由(9)及i的$\psi(\lambda)$常返性，$P_i\{\delta_1<\infty\}=P_i\{\beta_1<\infty\}=1$。又令$\delta_n$为$\beta_{n-1}$后首回$i$的时刻，$\beta_n$为$\delta_n$后的第一个飞跃点，则可证$P_i\{\delta_n<\infty一切n\}=1$。

易知$\{x(\beta_n-0)=a\}\in\mathscr{F}'_{\delta_n}\cap\mathscr{F}_{\delta_{n+1}}$。由定理6.7.3，它们彼

此独立。由强马氏性，

$$P_i\{x(\beta_l-0)=a\}=P_i\{\delta_l<\infty,\ \theta_{\delta_l}[x(\tau-0)=a]\}=X_i^a>0. \tag{10}$$

故按Borel-Cantalli引理，

$$P_i\{x(\beta_l-0)=a\text{对无穷多个}l\}=1,$$

从而更有

$$P_i\{x(\tau^l-0)=a\text{对无穷多个}l\}$$

$$=\sum_{b\in\mathscr{A}}X_i^bP\{x(\tau^l-0)=a\text{对无穷多个}l\mid x(\tau^1-0)=b\}=1.$$

因$X_i^a>0$，故$P\{x(\tau^l-0)=a$对无穷多个$l\mid x(\tau^l-0)=a\}=1$。即a为$\mathscr{Y}$常返。

(ii) 同(i)，代替(10)的是$P_i\{x(\beta_l)=r\}=Z_{ir}>0$。故$P_i\{x(\beta_l)=r$对无穷多个$l\}=1$，更有

$$P_i\{x(\tau^l)=r\text{对无穷多个}l\}$$

$$=\sum_{q\in A}Z_{iq}P_i\{x(\tau^l)=r\text{对无穷多个}l\mid x(\tau^1)=q\}=1.$$

因$Z_{ir}>0$，故$P_i\{x(\tau^l)=r$对无穷多个$l\mid x(\tau^1)=r\}=1$。即r为$\mathscr{Z}$常返，证毕。

定理3 设$\psi(\lambda)$为$\{Q,\ \Pi(a,\ \cdot),\ a\in\mathscr{A}\}$一阶过程，$i$为$\phi(\lambda)$非常返。

(i) i为$\psi(\lambda)$常返的充要条件是：存在$a,\ b\in\mathscr{A}$使$X_i^a>0$，$A_i^b>0$，a为$\mathscr{Y}$常返，且$a\overset{\mathscr{Y}}{\Longrightarrow}b$。[1]

(ii) i为$\psi(\lambda)$常返的充要条件是：存在$r,\ q\in A$使$Z_{ir}>0$ $r\overset{\mathscr{Z}}{\Longrightarrow}q\overset{\phi(\lambda)}{\Longrightarrow}i$，$r$为$\mathscr{Z}$常返。

证 只证(i)。在(1)中令$\lambda\downarrow0$得

$$\lim_{\lambda\downarrow0}\psi_{ii}(\lambda)=\Gamma_{ii}+\sum_{a\in\mathscr{A}}\sum_{b\in\mathscr{A}}X_i^aG_{ab}A_i^b$$

---

1) a，b可以相同，$a\overset{\mathscr{Y}}{\Rightarrow}a$

$$=\Gamma_{ii}+\sum_{a;b\in\mathscr{A}}{}^{+}X_i^a g_{ab}G_{bb}A_i^b. \tag{11}$$

其中Σ^+表示对被加项为正的项进行求和，g_{ab}为 $\mathscr{V}$ 链自a经有穷($\geqslant 0$) 步到b的概率。

如果i为$\psi(\lambda)$常返，(11) 左方为无穷，故必存在a, $b\in\mathscr{A}$，使$X_i^a>0$, $g_{ab}>0$, $A_i^b>0$。由引理2，a为$\mathscr{V}$常返。而$g_{ab}>0$ 正好表示$a\overset{\mathscr{V}}{\Longrightarrow}b$。

设充分性条件满足。如果其中的$a=b$，则 $g_{ab}=1$，$G_{aa}=\infty$。(11) 式右方为无穷。如果$a\neq b$，则由于a为$\mathscr{V}$常返且$a\overset{\mathscr{V}}{\Longrightarrow}b$，即$g_{ab}>0$，故$b$也$\mathscr{V}$常返，即$G_{bb}=\infty$。于是(11) 右方也为无穷，证毕。

定理4 设对任意i,j，$i\overset{\phi(\lambda)}{\Longleftrightarrow}j$且$\phi(\lambda)$非常返。设$\psi(\lambda)$为 $\{Q,\Pi(a,\ \cdot\ ),\ a\in\mathscr{A}\}$一阶过程。

(i) 如$\psi(\lambda)$常返，则必定

Q保守，$\xi_i=1(i\in E)$，$\Pi(a,\ E)=1$，$a\in\mathscr{A}$。 (12)

其中$\xi_i=P_i\{\tau<\infty\}$如(2.3)。

(ii) 如果(12) 成立，而且下面情况之一成立。①$\mathscr{A}$有限；②A有限；③$\mathscr{A}$中有一个$\mathscr{V}$常返状态；④A中有一个$\mathscr{Z}$常返状态。则$\psi(\lambda)$常返。

证 (i) 由引理2.2，$\xi_i=P_i\{\tau<\infty\}=1(i\in E)$ 且 $P_i\{\sigma=\infty\}=1$。由此得Q必定保守。于是对$a\in\mathscr{A}$，由 (10.6.2)，

$$\sum_k\Pi(a,\ k)=\sum_k P\{x(\tau)=k|x(\tau-0)=a\}$$

$$=P\{\tau<\sigma|x(\tau-0)=a\}=P\{\tau<\infty|x(\tau-0)=a\}=1.$$

(ii) 设(12) 成立，并在定理前提假设下，必定 $X_i^a>0$，$A_i^a>0$对一切$a\in\mathscr{A}$及一切$i\in E$成立，也必定$Z_{ir}>0$对一切$i\in E$及$r\in A$成立。又

$$\sum_{b\in\mathscr{A}} V_{ab}=\sum_k \Pi(a,\ k)\sum_{b\in\mathscr{A}} X_k^b=\sum_k \Pi(a,\ k)\xi_k=1,$$

$$\sum_{q\in A} Z_{rq}=\sum_{a\in\mathscr{A}} X_r^a\sum_{q\in A}\Pi(a,\ q)=\sum_{a\in\mathscr{A}} X_r^a=\xi_r=1.$$

所以如果①—④中之一成立，都可推出有$a\in\mathscr{A}$为$\mathscr{Y}$常返和有$r\in A$为$\mathscr{Z}$常返。因此定理3的充分性条件满足，从而$\psi(\lambda)$常返，证毕。

定理5 设i为$\phi(\lambda)$非常返，$\psi(\lambda)$为$\{Q,\ \Pi(a,\cdot),\ a\in\mathscr{A}\}$一阶过程。设$\delta\equiv\sup\limits_{a\in\mathscr{A}}\Pi(a,\ E)<1$，则$i$为$\psi(\lambda)$非常返。

证 由假设$\Gamma_{ki}\leqslant\Gamma_{ii}<\infty$。因

$$\sum_{a\in\mathscr{A}} X_i^a(\lambda)\leqslant\sum_{a\in\mathscr{A}} X_i^a\leqslant 1,$$

故由(7)易得

$$A_i^b(\lambda)\leqslant\sum_k \Pi(b,\ k)\Gamma_{ki}\leqslant\delta\Gamma_{ii},$$

$$\sum_{b\in\mathscr{A}} T_{ab}^l(\lambda)\leqslant\delta^l,\quad \sum_{b\in\mathscr{A}} G_{ab}(\lambda)\leqslant\sum_{l=0}^{\infty}\delta^l=\frac{1}{1-\delta}<\infty.$$

由(1)得

$$\psi_{ii}(\lambda)\leqslant\Gamma_{ii}+\sum_{a\in\mathscr{A}} X_i^a(\lambda)\frac{1}{1-\delta}\delta\Gamma\leqslant\frac{\Gamma_{ii}}{1-\sigma}<\infty.$$

由此得i为$\psi(\lambda)$非常返，证毕。

引理6 设$\psi(\lambda)$为$\{Q,\ \Pi(a,\ \cdot),\ a\in\mathscr{A}\}$一阶过程。对任意$j$，$i\overset{\psi(\lambda)}{\Longleftrightarrow}j$，则

(i) 如果$\Pi(a,\ E)>0$，则$a\overset{\mathscr{Y}}{\Longrightarrow}b$，$(b\in\mathscr{A})$。

(ii) 对任意r，$q\in A$，$r\overset{\mathscr{Z}}{\Longrightarrow}q$。

证 设X为$\psi(\lambda)$过程。η_i^*如(6.7.7)，因i和j关于$\psi(\lambda)$互通，故

$$u_i = P_i\{\eta_j^* < \sigma\} > 0. \tag{13}$$

(i) 由假设，存在i使$\Pi(a, i) > 0$，对$b \in \mathscr{A}$存在j使$X_j^b > 0$。于是

$$P\{存在 l \geqslant 1 使 x(\tau^l - 0) = b \mid x(\tau^1 - 0) = a\}$$
$$\geqslant \Pi(a, i) u_{ij} X_j^b > 0.$$

即$a \stackrel{\mathscr{T}}{\Longrightarrow} b$。对(ii) 可类似证明

定理7 设$\psi(\lambda)$是 $\{Q, \Pi(a, \cdot), a \in \mathscr{A}\}$ 一阶过程。对任意i, j, $i \stackrel{\phi(\lambda)}{\Longleftrightarrow} j$，$\phi(\lambda)$非常返，而$\psi(\lambda)$常返。

(i) 如A有穷，则$\psi(\lambda)$遍历的充要条件是

$$N_k = E_k \tau < \infty, \quad k \in A. \tag{14}$$

(ii) 如果$\mathscr{A}$有穷，则$\psi(\lambda)$遍历的充要条件是

$$\sum_k \Pi(a, k) N_k < \infty, \quad a \in \mathscr{A}. \tag{15}$$

证 只证 (i)，(ii) 可类似证明。因$\psi(\lambda)$常返，由引理2.2，$\psi(\lambda)$不中断，即

$$\lambda \sum_j \psi_{ij}(\lambda) = 1, \quad i \in E. \tag{16}$$

因$\psi(\lambda)$满足向后方程组，故Q必保守。由定理3，存在$r \in A$，r为$\mathscr{Z}$常返。由引理6,A为$\mathscr{Z}$常返类。将(5) 代入(16)并注意 (1.5)，以及Q保守时，$\xi_i(\lambda) = \sum_{a \in \mathscr{A}} X_i^a(\lambda)$，得

$$\sum_{a \in \mathscr{A}} X_i^a(\lambda) = \sum_{a \in \mathscr{A}} X_i^a(\lambda) \sum_{r, q \in \mathscr{A}} \Pi(a, r) \lambda D_{rq}(\lambda) \sum_j \phi_{qj}(\lambda).$$

由$X^a(\lambda)$的线性独立性，

$$1 = \sum_{r, q \in \mathscr{A}} \Pi(a, r) \lambda D_{rq}(\lambda) \sum_j \phi_{qj}(\lambda), \quad a \in \mathscr{A}. \tag{17}$$

(一) 如果存在$q \in A$使$N_q = \infty$。对此q，存在$a \in \mathscr{A}$使$\Pi(a, q) > 0$。由(17)，

$$\Pi(a, q)\lambda D_{qq}(\lambda)\sum_j \phi_{qj}(\lambda)\leqslant 1,$$

$$\lambda D_{qq}(\lambda)\leqslant\frac{1}{\Pi(a, q)\sum_j \phi_{qj}(\lambda)}.$$

当$\lambda\downarrow 0$时得$\lim\limits_{\lambda\downarrow 0}\lambda D_{qq}(\lambda)=0$，由引理1,对任意$r,q\in A$，$\lim\limits_{\lambda\downarrow 0}\lambda D_{rq}(\lambda)=0$。于是由(5) 及$A$的有限性，得 $\lim\limits_{\lambda\downarrow 0}\lambda\psi_{ii}(\lambda)=0$。即$\psi(\lambda)$不遍历。

(二) 设对一切 k，$N_k<\infty$。不妨设对一切 r，$q\in A$，$\lim\limits_{\lambda\downarrow 0}\lambda D_{rq}(\lambda)$存在，否则可用子列代替之。由(17) 及$A$的有限性

$$\begin{aligned}1&=\sum_{r,q\in A}\Pi(a, r)\lim_{\lambda\downarrow 0}\lambda D_{rq}(\lambda)\sum_j \Gamma_{qj}\\&=\sum_j\sum_{r,q\in A}\Pi(a, r)\lim_{\lambda\downarrow 0}\lambda D_{rq}(\lambda)\Gamma_{qj},\ a\in\mathscr{A}.\end{aligned}$$

故对固定的$a\in\mathscr{A}$，存在$j\in E$，r，$q\in A$使

$$\Pi(a, r)\lim_{\lambda\downarrow 0}\lambda D_{rq}(\lambda)\Gamma_{qj}>0.$$

而对$a\in\mathscr{A}$，必存在i使$X_i^a>0$。于是由(5)，

$$\begin{aligned}\pi_j&=\lim_{\lambda\downarrow 0}\lambda\psi_{ij}(\lambda)\geqslant\lim_{\lambda\downarrow 0}z_{ir}(\lambda)\lambda D_{rq}(\lambda)\phi_{qj}(\lambda)\\&\geqslant X_i^a\Pi(a, r)(\lim_{\lambda\downarrow 0}\lambda D_{rq}(\lambda))\Gamma_{qj}>0.\end{aligned}$$

即$\psi(\lambda)$遍历，证毕。

定理8 设Q导出的非原子流出边界B_{e2}为空集，$\mathscr{A}=B_e$为有限集。则任何D型Q过程$X\in\mathscr{X}_D(Q)$都是 $\{Q, \Pi(a, \cdot), a\in\mathscr{A}\}$一阶过程，其中

$$P\{x(\tau)=j|x(\tau-0)=a\}=\Pi(a, j).\tag{18}$$

τ为第一个飞跃点。

证 因$X\in\mathscr{X}_D$，故

$$\begin{aligned}p_{ij}(t)&=P_{ij}\{x(t)=j,\ t<\tau\}+P_i\{\tau\leqslant t,\ x(t)=j\}\\&=f_{ij}(t)+\sum_{a\in\mathscr{A}}\sum_k P_i\{x(\tau-0)=a,\ \tau\leqslant t,\ x(\tau)=k,\end{aligned}$$

$x(t)=j\}$

利用强马氏性及定理6.17.1，易证上式被加项等于

$$\int_{(x(\tau-0)=a,x(\tau)=k,\tau\leqslant t)} p_{kj}(t-\tau)dP_i=\int_0^t p_{kj}(t-s)dP_i\{x(\tau-0)=a,\ x(\tau)=k,\ \tau\leqslant s\}.$$

$$P_i\{x(\tau-0)=a,\ x(\tau)=k,\tau\leqslant s\}=P_i\{x(\tau-0)=a,\tau\leqslant s\}\cdot P\{x(\tau)=k\,|\,x(\tau-0)=a\}=L_i^a(s)\Pi(a,\ k).$$

故(19)成为

$$p_{ij}(t)=f_{ij}(t)+\sum_{a\in\mathscr{A}}\sum_k \Pi(a,\ k)\int_0^t p_{kj}(t-s)dL_i^a(s),$$

取拉氏变换得

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+\sum_{a\in\mathscr{A}}X_i^a(\lambda)\sum_k \Pi(a,k)\psi_{kj}(\lambda). \quad (20)$$

或者 $\psi(\lambda)=\phi(\lambda)+\{X^a(\lambda)\}'\{B^a(\lambda)\}$. (21)

其中 $B_j^a(\lambda)=\sum\limits_k \Pi(a,\ k)\psi_{kj}(\lambda)$。左乘上式$\Pi(a,\cdot)$得

$$\{B^a(\lambda)\}=\{A^a(\lambda)\}+\mathscr{V}(\lambda)\{B^a(\lambda)\},$$

$$\{I-\mathscr{V}(\lambda)\}\{B^a(\lambda)\}=\{A^a(\lambda)\}. \quad (22)$$

其中$A^a(\lambda)$如(7)。$\mathscr{A}\times\mathscr{A}$矩阵$\mathscr{V}(\lambda)$如(4.2)。因为矩阵$\mathscr{V}(\lambda)$有行和$\sum\limits_{b\in\mathscr{A}}\sum\limits_k \Pi(a,\ k)X_k^b(\lambda)<1$，且$\mathscr{A}$有限，故$\{I-\mathscr{V}(\lambda)\}^{-1}$存在而且等于$\mathscr{G}(\lambda)=\sum\limits_{l=0}^{\infty}(\mathscr{V}(\lambda))^l$，故由(22)，

$$\{B^a(\lambda)\}=\{I-\mathscr{V}(\lambda)\}^{-1}\{A^a(\lambda)\}=\mathscr{G}(\lambda)\{A^a(\lambda)\}.$$

代入(21)得$\psi(\lambda)$有形式(1)，即$\psi(\lambda)$是$\{Q,\ \Pi(a,\ \cdot),\ a\in\mathscr{A}\}$一阶过程，证毕。

注 本节许多结果可推广到k阶瞬返过程，但从略。

§6. 双边生灭过程

这里将采用第四章的记号。

设X为双边生灭过程，τ为第一个飞跃点。按（11.2.6）定义首达i的时刻ξ_i。

引理1 设$i\leqslant k\leqslant n$,

$$P_k\{\xi_i<\xi_n\}=\frac{z_n-z_k}{z_n-z_i},\quad P_k\{\xi_n<\xi_i\}=\frac{z_k-z_i}{z_n-z_i}. \tag{1}$$

证 仿定理11.2.2证明。

定理2 用C_{kn}表示X从k出发经有穷（$\geqslant 0$）步跳跃到达n的概率，即$C_{kn}=P_k\{\xi_n<\tau\}$，则

$$C_{kn}=\begin{cases}\dfrac{r_2-z_k}{r_2-z_n}, & \text{如} n\leqslant k,\\[2ex] \dfrac{z_k-r_1}{z_n-r_1}, & \text{如} n>k.\end{cases} \tag{2}$$

证 在(1)中分别令$n\to+\infty$或$i\to-\infty$而得。

定理3 最小Q过程常返的充要条件是r_1和r_2均无穷。

证 因为从0出发离开0后又回到0的概率$f_0^*=\dfrac{a_0}{a_0+b_0}C_{-10}+\dfrac{b_0}{a_0+b_0}C_{10}$。由(2)可见$f_0^*=1$当且只当$r_1$，$r_2$均无穷。

定理4 设最小Q过程常返，则遍历的充要条件是

$$\sum_k \mu_k<\infty. \tag{3}$$

而且此时

$$m_{ii}=\frac{\sum\limits_k \mu_k}{q_i\mu_i}. \tag{4}$$

$$m_{ij}=(z_j-z_i)\sum_{s\leqslant i}\mu_s+\sum_{i<s<j}(z_j-z_s)\mu_s,\quad \text{如} j>i. \tag{5}$$

$$m_{ij}=\sum_{j<s<i}(z_s-z_j)\mu_s+(z_i-z_j)\sum_{s\geqslant i}\mu_s,\quad \text{如} j<i. \tag{6}$$

其中$m_{ij}=E_i\eta_j^*$，η_j^*按(6.7.7)确定。

证 考虑最小 Q 过程 $X=\{x(t),\ t<\infty\}$。注意由常返性有 $P_i\{\eta_i^*<\infty\}=P_i\{\xi_n<\infty\}=1$。易知，对 $i<k<n$，$u_k=\min(\xi_i,\xi_n)$ 满足 $f_i=f_n=0$，$f_k=1(i<k<n)$ 的方程(4.4.11)，故由(4.4.12)得

$$E_k\min(\xi_i,\ \xi_n)=\sum_{i<s<k}\frac{(z_s-z_i)(z_n-z_k)}{z_n-z_i}\mu_s+\sum_{k<i<n}\frac{(z_k-z_i)(z_n-z_s)}{z_n-z_i}\mu_s. \tag{7}$$

在上式中令 $i\to-\infty$，$n\to+\infty$ 并注意 r_1，r_2 无穷得(5)(6)，由强马氏性，

$$m_{ii}=\frac{1}{q_i}+\frac{a_i}{q_i}m_{i-1\,i}+\frac{b_i}{q_i}m_{i+1\,i}$$

注意 $a_i(z_i-z_{i-1})=b_i(z_{i+1}-z_i)=\mu_i^{-1}$，将(5)(6)代入上式得(4)。

引理5 当最小 Q 过程非常返时，

$$N_k=E_k\tau=\sum_{S<k}\frac{(z_s-r_1)(r_2-z_k)}{r_2-r_1}\mu_s+\sum_{k<S}\frac{(z_k-r_1)(r_2-z_s)}{r_2-r_1}\mu_s. \tag{8}$$

证 在(7)中令 $i\to-\infty$，$n\to+\infty$ 而得。

定理6 设 r_1 流入或自然，r_2 流出或正则，则 Q 过程 $\psi(\lambda)$ 常返的充要条件是 r_1 为无穷且 $\psi(\lambda)$ 不中断。

证 本定理是定理4.2.的特款。在本定理假设下，$\overline{X_i}=\overline{X_i}^2$，$X_i^0=X_i^1$，(4.2)前面二式是满足的。故(4.2)等价于 $c=0$，$X^0=X^1=0$。注意(4.7.2)即得证定理。

定理7 设 r_1 流入或自然，r_2 流出或正则，$\psi(\lambda)$ 常返。则

(i) 设 r_1 流入，则 $\psi(\lambda)$ 遍历。

(ii) 设 r_1 自然。

① 如 $\sum_{S<0}\mu_s=\infty$，则 $\psi(s)$ 不遍历。

② 如 $\sum_{S<0}\mu_s<\infty$，且 $\psi(\lambda)$ 有表现(4.8.3)，则 $\psi(\lambda)$ 遍历

的充要条件是

$$\sum_{S<0} a_s N_s < \infty. \tag{9}$$

特别地，如果$\sum_{S<0} \mu_s < \infty$，$\sum_{S<0} a_s(r_2 - z_s) < \infty$，则$\psi(\lambda)$遍历。

证 在(4.8.12)中令$\lambda \downarrow 0$得

$\overline{\eta_i} = P_1(r_2 - z_j)\mu_j + P_2\mu_j$，$r_1$自然时$P_1 = 0$，$r_2$流出时$P_2 = 0$。 (10)

但是r_2正则或流出，而且

$$\sum_{j<0} (r_2 - z_j)\mu_j < \infty, \ (r_1\text{流入}), \ \sum_{j>0} \mu_j < \infty (r_2\text{正则}). \tag{11}$$

故$\sum_i \overline{\eta_i} < \infty$等价于$P_2 \sum_{S<0} \mu_s < \infty$。注意$\overline{X} = X^2 = 1$，故遍历性条件(4.3)成为

$$\sum_s a_s N_s + P_2 \sum_{s<0} \mu_s < \infty. \tag{12}$$

因为由定理6有r_1无穷，故由(8)，

$$N_h = (r_2 - z_h) \sum_{h \leq s} \mu_s + \sum_{s<h} (r_2 - z_s)\mu_s. \tag{13}$$

往证：如果$\sum_{s \leq 0} \mu_s < \infty$，则

$$\sum_{s \geq 0} a_s N_s < \infty. \tag{14}$$

实际上，当$h \geq 0$时，

$$N_h = (r_2 - z_h) \sum_{s<0} \mu_s + M_h. \tag{15}$$

这里的M_h即(4.11.11)中的N_h。由定理4.11.3和4.11.4，

$\sum_{h \geq 0} a_h N_h < \infty$。

(i) 设r_1流入，则必定$\sum\limits_{s\leqslant 0}\mu_s<\infty$。由（14），遍历性等价于(9)。但当$i\leqslant 0$时，由(11)，

$$N_i\leqslant\sum_{s\leqslant h}(r_2-z_s)\mu_s+\sum_{h<s}(r_2-z_s)\mu_s=\sum_{s}(r_2-z_s)\mu_s<\infty.$$

故由定理4.11.5，

$$\sum_{i\leqslant 0}a_iN_i\leqslant\Big(\sum_{i\leqslant 0}a_i\Big)\Big(\sum_{s}(r_2-z_s)\mu_s\Big)<\infty.$$

所以$\psi(\lambda)$遍历。

(ii) 设r_1自然。

① 设$\sum\limits_{s\leqslant 0}\mu_s=\infty$。由(13)有$N_i=\infty$。如果$a\neq 0$，则(12)不满足。如果$a=0$，则必定$P_2>0$。因而(12)也不成立，故$\psi(\lambda)$不遍历。

② 设$\sum\limits_{s\leqslant 0}\mu_s<\infty$。由于(14)，遍历性条件(12)成为(9)。更设$\sum\limits_{s\leqslant 0}a_s(r_2-z_s)<\infty$。由(13)，对$i\leqslant s$，

$$\begin{aligned}N_i\leqslant&(r_2-z_i)\sum_{s\leqslant i}\mu_s+\sum_{i<s\leqslant 0}(r_2-z_s)\mu_s\\&+\sum_{s>0}(r_2-z_s)\mu_s\leqslant(r_2-z_i)\sum_{s\leqslant 0}\mu_s\\&+\sum_{s>0}(r_2-z_s)\mu_s.\end{aligned}$$

由定理4.11.5有$\sum\limits_{h\leqslant 0}a_h<\infty$。故有$\sum\limits_{s\leqslant 0}a_sN_s<\infty$。从而$\psi(\lambda)$遍历，证毕。

定理8 设r_1，r_2正则或流出。

(i) Q过程$\psi(\lambda)$常返当且只当它不中断。

(ii) 不中断Q过程必遍历。

证 由引理2.2，$\psi(\lambda)$常返必不中断。

首先证：如$a\geqslant 0$使$a\phi(\lambda)\in l$，则

$$\sum_k a_k N_k < \infty. \tag{16}$$

其中N_k根据(8)确定。当$k\geqslant 0$时，由(8)，

$$N_k \leqslant (r_2 - z_k)\sum_{i<0}\frac{z_i - r_1}{r_2 - r_1}\mu_i + M_k. \tag{17}$$

其中的M_k即(4.11.11)中的N_k。由定理4.11.3和4.11.4，$\sum\limits_{k\geqslant 0} a_k N_k$ $<\infty$。类似证$\sum\limits_{k<0} a_k N_k < \infty$。

其次，对于(4.7.12)中的$\overline{\eta_j}(\lambda)$。令$\lambda\downarrow 0$得

$$\overline{\eta} = P_1 X^1\mu + P_2 X^2\mu，\ r_a\text{流出时}P_a = 0. \tag{18}$$

而r_2正则时，$\sum\limits_j X_j^2\mu_j \leqslant \dfrac{1}{r_2 - r_1}\sum\limits_{j<0}(z_j - r_1)\mu_j + \sum\limits_{j\geqslant 0}\mu_j < \infty$。同样，$r_1$正则时，$\sum\limits_j X_j^1\mu_j < \infty$。因此恒有

$$\sum_j \overline{\eta_j} < \infty. \tag{19}$$

再次，由引理4.1，

$$[\alpha^a,\ X^a - X^a(\lambda)]\downarrow 0, \lambda[\overline{\eta}(\lambda),\ X^a]\downarrow 0,\ (\lambda\downarrow 0). \tag{20}$$

如果$\psi(\lambda)$有表现(4.10.16)($c=0$，$d_1 = d_2$)。由(16)(19)，仿照定理4.2可得$\psi(\lambda)$常返而且遍历，并且

$$\pi_j = \lim_{\lambda\downarrow 0}\lambda\psi_{ij}(\lambda) = \frac{\sum\limits_k a_k\Gamma_{kj} + P_1 X_j^1\mu_j + P_2 X_j^2\mu_j}{\sum\limits_k a_k N_k + \sum\limits_k (P_1 X_k^1 + P_2 X_k^2)\mu_k}. \tag{21}$$

设$\psi(\lambda)$有表现 (4.10.1)(4.10.4)(4.10.33)，并且$\overline{S}^{12} = \overline{S}^{21} = 1$。

记

$$\mathscr{G}(\lambda)=\mathscr{Z}_{\lambda}^{-1}=\{I-\overline{\mathscr{S}}+\overline{\mathscr{H}}_{\lambda}+\overline{\mathscr{M}}\mathscr{U}_{\lambda}\}^{-1}$$

$$=\sum_{l=0}^{\infty}\{\mathscr{F}(\lambda)\}^{l}\mathrm{diag}\left(\frac{1}{1-e_{\lambda}^{aa}}\right). \tag{22}$$

其中e_{λ}^{ab}为矩阵$\overline{\mathscr{S}}-\overline{\mathscr{H}}_{\lambda}-\overline{\mathscr{M}}\mathscr{U}_{\lambda}$的元素，$\mathscr{F}(\lambda)=\{f^{ab}(\lambda)\}$。

$$f^{aa}(\lambda)=0,\quad f^{ab}(\lambda)=\frac{e_{\lambda}^{ab}}{1-e_{\lambda}^{aa}},(b\neq a). \tag{23}$$

由(20)及$\overline{S}^{aa}=0$得

$$1-e_{\lambda}^{aa}\uparrow 1,\quad f^{ab}(\lambda)\uparrow\overline{S^{ab}},\quad(\lambda\downarrow 0). \tag{24}$$

故$\mathscr{F}(\lambda)\uparrow\overline{\mathscr{S}}$，$(\lambda\downarrow 0)$。$\mathscr{G}(\lambda)\to\mathscr{G}=\sum_{l=0}^{\infty}\overline{\mathscr{S}}^{l}$。

$\psi(\lambda)$可以写成下式：

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+\sum_{a=1}^{2}X_{i}^{a}(\lambda)G_{ab}(\lambda)\eta_{j}^{b}(\lambda). \tag{25}$$

其中$\eta^{b}(\lambda)=\overline{\alpha}^{b}\phi(\lambda)+\overline{M}^{bb}X^{b}(\lambda)\mu$，$r_b$流出时$\overline{M}^{bb}=0$。 (26)

因$\overline{\mathscr{S}}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$故在(25)中令$\lambda\downarrow 0$知$\psi(\lambda)$常返，由于$\psi(\lambda)$不中断，故

$$\lambda G_{ab}(\lambda)\sum_{j}\eta_{j}^{b}(\lambda)=1,\quad a=1,2. \tag{27}$$

注意(16)(18)(19)，故$\lambda\downarrow 0$时。

$$\sum_{j}\eta_{j}^{b}(\lambda)\uparrow\sum_{k}(\overline{\alpha}_{k}^{b}N_{k}+\overline{M}^{bb}X_{k}^{b}\mu_{k})<\infty.$$

从(27)得$\lim\limits_{\lambda\downarrow 0}\lambda G_{ab}(\lambda)\sum\limits_{j}\eta_{j}^{b}=1$，故有$j$使$\lim\limits_{\lambda\downarrow 0}\lambda G_{ab}(\lambda)\eta_{j}^{b}>0$。从(25)得

$$\lim_{\lambda\downarrow 0}\lambda\psi_{ij}(\lambda)\geqslant X_{i}^{a}(\lim_{\lambda\downarrow 0}\lambda G_{ab}(\lambda))\eta_{j}^{b}>0.$$

从而$\psi(\lambda)$遍历，证毕。

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索　　引

一至三划

1) 这表示此名词首次出现或其定义在第一章第8节。

四至五划

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The Construction Theory of Denumerable Markov Processes

XIANG-QUN YANG
Xiangtan University, Hunan,
People's Republic of China

With a Foreword by
D. G. KENDALL

HUNAN SCIENCE AND TECHNOLOGY
PUBLISHING HOUSE
Changsha

JOHN WILEY & SONS
Chichester · New York · Brisbane · Toronto · Singapore

First published as Keliemaerkefuguochenggouzaolun
by the Hunan Science and Technology Publishing House, Changsha.

Other Wiley Editorial Offices

John Wiley & Sons, Inc., 605 Third Avenue, New York, NY 10158-0012, USA

Jacaranda Wiley Ltd, G.P.O. Box 859, Brisbane, Queensland 4001, Australia

John Wiley & Sons (Canada) Ltd, 22 Worcester Road, Rexdale, Ontario M9W 1L1, Canada

John Wiley & Sons (SEA) Pte Ltd, 37 Jalan Pemimpin 05-04, Block B, Union Industrial Building, Singapore 2057

Library of Congress Cataloging-in-Publication Data:

Yang, Hsiang-ch' ün.
[K' o lieh Ma-erh-k' o-fu kuo ch' eng kou tsao lun. English]
The construction theory of denumerable Markov processes / Xiang-qun Yang ; with a foreword by D. G. Kendall.
p. cm.
Translation of: K' o lieh Ma-erh-k' o-fu kuo ch' eng kou tsao lun.
Includes bibliograhical references.
ISBN 0 471 92490 3
1. Markov processes. I. Title.
QA274.7.Y3613 1990
519.2' 33--dc20 89-22671

British Library Cataloguing in Publication Data:

Yang, Xiang-qun
The construction theory of denumerable Markov processes.
1. Markov processes
I. Title
519.2'33

ISBN 0 471 92490 3

Typeset by Thomson Press (India) Ltd, New Delhi
Printed and bound in Great Britain by
Biddles Ltd, Guildford and King's Lynn

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Foreword to the English Edition

In this book my friend Xiang-qun Yang presents the theory of temporally-homogeneous countable-state Markov chains developing in continuous time. Systems of this kind with a finite set of states have been studied for many years. Indeed, the second volume of Fréchet's book on probability, published in 1938, gives a detailed treatment. The interest and difficulty both increase dramatically when the set of states becomes countably infinite. The first detailed theoretical treatments of this 'infinite' case were given in two papers published in 1951 and written by Kolmogorov and Lévy respectively, and these have subsequently generated an enormous literature. The book by Kai-lai Chung (1967) at once became the standard reference for the subject, but now it by no means covers all that has been done—even by Chung himself.

In recent years mathematicians in the People's Republic of China have taken up the subject with enthusiasm, and there are two especially strong centres of Markovian studies in the Province of Hunan in which Professor Yang lives—Xiangtan University (of which he is the President), and the Chinese Institute of Railways in Changsha (of which Professor Zhen-ting Hou is the Vice-President).

David Williams and I, accompanied by Sheila Williams, spent two weeks at Xiangtan University in 1983. A few lines from my diary attempt to sketch the exotic environment.

'Xiangtan University is set quite by itself on a plateau of red clay standing out in a vast landscape of rice fields. These are arranged in tiers so that the water flowing out of one field forms a source for the next, and if one follows the flow backwards then the system forks as the valleys fork in an complicated network—almost a branching process.... This was well shown when we visited and climbed (in a typhoon) the holy mountain Nan Yue. There the "branching process" ascends the mountain in a succession of ever-decreasing rice-fields (ultimately just of handkerchief size) until the forest takes over as a sort of Martin boundary.'

Our task was to give two weeks of advanced lectures to a group of some forty Chinese probabilists drawn from widely different parts of the People's Republic. I lectured on archaeological seriation, and on the statistical theory

of shape, both novel topics for that audience. David Williams was not so fortunate. He set out armed with his recently published and now famous book *Diffusions, Markov Processes, and Martingales* as a text, but was astonished to see the members of the audience pour into the lecture hall, each clutching a well-thumbed xerox copy of it. 'What we want to know', they said, 'is *what happens next*'. So they received an impromptu and highly original set of lectures from him.

That incident impressed in our minds how seriously mathematical probability is taken in China—dramatic advances in Markov chain theory are exactly what one would expect. And dramatic advances there have been.

The present book first sets out the basic framework of the theory of such Markov chains, and then in the latter chapters develops more recent advanced work by Professor Yang himself and by some other Chinese writers. The author made his own translation into English, and in the earlier chapters this has been carefully checked by Professor G. E. H. Reuter whose own work forms a basis for much of the subsequent developments. But the latter chapters of the book are essentially as the author himself wrote them.

The reader should be alerted to a special feature of Chinese mathematical writing in the English language. This arises from the fact that in different traditions there are different conventions about what must be mentioned, and what need not be mentioned. Anyone who has tried translating Chinese mathematics into English will have noticed this. Sometimes, for example, algebraic manipulations will be completely omitted when we would expect them to be included. Conversely, Chinese readers must find our own mathematical writing to be extraordinarily prolix, and packed with references to apparently irrelevant matters, such as cricket. Readers from both language traditions have to learn to expect and to be patient with this. It is tempting to enlarge on this theme, but I will stop here, and wish the reader an interesting journey.

DAVID KENDALL

Foreword to the Original Edition

Markov processes occupy an extremely important position in random processes. The denumerable Markov processes form a very active and theoretically fairly complete branch of knowledge of Markov processes, which is universally applicable in many fields of science and technology. Quite a few famous probability scholars, such as A. N. Kolmogorov, J. L. Doob, W. Feller, K. L. Chung, etc., have done a lot of work in this area of knowledge for quite some time and have made important contributions. For more than two decades, Chinese workers engaged in probability theory have conducted extensive and profound studies on this subject. So far there have appeared three monographs published respectively under the following titles: *Birth–Death Processes and Markov Chains, Homogeneous and Denumerable Markov Processes* and *Reversible Markov Processes.*

In 1958 Zi-kun Wang, Professor of Nankai University (now President of Beijing Normal University), began to publish his research results in this field. Then he, together with his students and colleagues, further probed and developed his studies on the subject. Because of their long, unremitting and painstaking efforts, they have achieved excellent results on several principal problems closely related to this subject. This book is a treatise by Professor Xiang-qun Yang. It gives a comprehensive summary of his research results on the construction theory of denumerable Markov processes, which he has studied for the past 20 years. Some of the achievements contained in this book are made known for the first time.

The construction problem is a central one in denumerable Markov processes. Xiang-qun Yang's researches are mainly concentrated on the construction of birth–death processes in construction theory, and of the Q processes with finite boundaries, on the probability method in the construction theory, and on the relationship between the probability method and the analytical method. For these aspects he has obtained very good results, which come up to the most advanced level in the world.

Quite a number of people in the world have studied birth–death processes. S. Karlin gave the integral representation of the minimal solution. W. Feller constructed all birth–death processes by the analytical method, which satisfy

simultaneously the systems of both backward and forward equations. Zi-kun Wang succeeded in constructing all honest birth-death processes by means of the probability method. Xiang-qun Yang has constructed all the birth–death processes by using these two methods and, moreover, has meticulously investigated the properties of birth–death processes. Therefore, the construction problem of birth–death processes has been completely solved.

The general construction problem of denumerable Markov processes is a very difficult one. In 1957 Feller constructed all Q processes satisfying the system of forward equations under the conditions that Q is conservative, finite exit and finite entrance. Latter, Xiang-qun Yang constructed all Q processes under the same conditions. D. Williams and Kai-Lai Chung derived all Q processes respectively in the case that Q is conservative and finite exit. Xiang-qun Yang has recently made further advances on this problem. He has derived all Q processes under the more extensive conditions that the non-conservative state and exit boundary of Q are both finite.

The two methods in the construction theory, i.e. the probability method and the analytical method, each have their own merits and demerits. By applying either of the two methods we should achieve certain results. But the results generated by these two methods are quite different in form. Xiang-qun Yang has found the relatonship between the two methods with regard to birth–death processes and combined these two kinds of results. Hence, he succeeds not only in giving a clear, definitive probability meaning to the results of analysis but also in expressing the probability results in equally concise forms. The work in this aspect pioneered by Xiang-qun Yang is to be further studied because it is a branch of knowledge with a broad, bright future for development.

This book is not a simple collection of the author's research achievements, but a treatise in which his research results are elaborately put into good order and carefully compiled. I believe that the reader can read this book through without much difficulty and reach the forefront of research in this field if he or she has finished reading the first three chapters of the treatise entitled *Birth–Death Processes and Markov Chains* by Zi-kun Wang. Consequently, the publication of this work will certainly give an impetus to the development of probability theory in our country.

At a time when this work is about to be published, I have expressed my views briefly on it. On account of my limited knowledge, my personal remarks cannot be without something inappropriate. Colleagues are sincerely requested to be kind enough to make comments or criticisms.

ZHEN-TING HOU
Changsha Railway Institute
3 March, 1980

Preface to the English Edition

I feel greatly honoured and very happy that my treatise *The Construction Theory of Denumerable Markov Processes* is to be available outside China in the form of this English edition. Readers abroad will be able to read the book and my research results on denumerable Markov processes will be introduced to the UK and the world.

Comprehension of this book presupposes one being familiar with the basic knowledge of random processes and the fundamental theory of Markov chains, which can be found in the book *Markov Chains with Stationary Transition Probabilities* by K. L. Chung and in the treatise *The Birth–Death Processes and Markov Chains* by Zi-kun Wang and Xiang-qun Yang. These two books were published by Springer-Verlag. For the readers' convenience, to this English edition I have added a chapter entitled 'Theoretical background', most of which is devoted to an introduction to the basic concepts and fundamental conclusions of random processes and, especially, to these of Markov chains, which are necessary for reading this book.

I sincerely thank Professor D. G. Kendall, who recommended my book to John Wiley & Sons Ltd. I am most grateful to John Wiley & Sons Ltd and to the Hunan Science and Technology Publishing House for having decided to publish the English edition of this book; likewise I would like to extend my heartfelt thanks to the Editor, Mrs Charlotte Farmer, and Associate Editor, Mr Hai-qing Hu, for their kind cooperation and great efforts. In particular, I feel very much indebted to Professor David Kendall, who has been kind enough to write a foreword to the English edition of this book. The first draft of the English version of this book was mainly done by Dr Shou-jun Luo and Mr Ying-qiu Li, and Madam Dong-ya Zou and Mr Wei-guo Tan translated the forward and introduction and some difficult paragraphs. Finally, the entire English manuscript was examined and approved by myself. Mrs Jun-fang Peng typed out the approved manuscript. Therefore, the English version of this book was born of joint efforts. To all those mentioned above, I would like to give my hearty thanks.

XIANG-QUN YANG
Xiangtan University
February 1989

Preface to the Second Edition

The second edition differs from the first one in that some major revisions and additions have been made in the following aspects: first, a concise proof of the uniqueness criterion is given; secondly, the theory of the Martin boundary is elaborated and verified, and therefore considerably enriched; thirdly, those sections about the construction of the bifinite (finite non-conservative and finite exit) Q processes are rewritten; fourthly, the approximating Markov chains and approximating minimal processes are supplemented, and DV-type and (DV)*-type extension processes and generalized DV-type and (DV)*-type extension processes are also supplemented. In addition, a large number of minor changes have been made.

The book has been enthusiastically received and heartily supported by a good many colleagues, who have expressed their interest in it since its first edition came out in 1981. Thus, the author has been greatly encouraged. What is particularly exciting is that quite a number of colleagues have suggested valuable improvements. To all these colleagues I extend my hearty thanks. My heartfelt thanks also go to the Hunan Science and Technology Publishing House for offering me the opportunity of having the second edition of this book published.

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5 April 1984

Introduction

This book is a summary of the research results of the author on the construction theory of denumerable Markov processes. With an introduction to the analytical basis of the construction theory in the first chapter, special topics are presented progressively.

Construction theory is a central subject in the theory of Markov processes. It is aimed at constructing Markov processes on the basis of some known conditions. In other words, we describe the Markov processes one by one. Thus we can study the properties of each Markov process according to its general and specific characters. For example, by basing our approach on the construction theory, we may select with relative ease those processes possessing inversibility from the different types of constructed Markov processes just as does the author of the treatise entitled *Inversible Markov Processes*.

Up to now there have been two methods for dealing with the construction theory. One is the analytical method, and the other is the probability method (both of which have their merits and demerits). Certain results have been achieved by means of one or other of these two methods. We can find a brief account of this aspect in this book.

The present work has six parts, basically dealing with three major aspects. In Parts I–III, the analytical method and results of the construction theory are discussed. In Parts IV and are expounded the probability method and results of the construction theory, and the relationship between these two methods. In Part VI a discussion is given on some properties of Markov processes related to the construction theory. Considerable importance is attached to the birth–death processes, not only because they have their own important theoretical significance and application value but also because they are often the source that gives the ideas and methods for solving general problems.

The analytical basis of the construction theory is discussed in Part I. First, a study is made of the analytical propertis of the Q process as a transition probability, of the construction and properties of the minimal solution, and of the general form of the Q process; then Q processes in simple cases are directly constructed; finally, the uniqueness problem for the Q processes is discussed.

Part II is devoted to an exposition of the construction theory of birth–death processes. We have succeeded in constructing all bilateral birth–death processes and all unilateral birth–death processes, the results being complete and

enlightening. A good grasp of the construction theory of birth–death processes is extremely conductive to a thorough understanding of the general construction of the Q processes.

Part III is concentrated on a study of the Martin boundary of Q processes and of its application in the construction theory. First of all, we apply the Martin boundary theory of discrete parameter Markov chains to the Q processes, conducting an extensive and profound discussion. Next we introduce the Martin exit boundary of the minimal Q process and further describe the general form of Q processes in the light of the exit boundary. Finally, we construct all the finite non-conservative and finite exit Q processes. That is to say, all the Q processes in the 'double finite' case are constructed.

In Part IV emphasis is laid upon analysing the path structures of probabilistic Q processes. First the concepts of W transformation and strong limit are introduced. General Q processes can be changed into various processes with simpler path structures by W transformation. Thus it is convenient to study the paths of processes from various aspects. The strong limit theorems demonstrate that the paths of relatively complicated Q processes can be approached by those of Q processes with simpler structures. Secondly, the concept of leaping intervals is introduced to study the exit and entrance of Q processes. A discussion is given on how leaping intervals and leaping points are related to the system of Kolmogorov equations, and the entrance decomposition theorem is derived.

Finally, the extension of processes is studied and the sample paths of processes with simpler structures are directly constructed. We mainly consider the D-type extension. In order that the Q matrix of the extension process remains unchanged, we also take into account the D^*-type extension.

Part V exclusively treats construction of the probability method for birth–death processes. As these processes are particular, the results obtained are relatively profound. Each birth–death process is not only the strong limit of a sequence of Doob processes but also corresponds to a characteristic sequence. We have established the relations between the birth–death processes constructed by the analytical method and those constructed by the probability method, so that the processes constructed by the analytical method have clear probability structure and the processes constructed by the probability method have succinct analytical expression. In this way the virtues of each method can be brought out. With simultaneous use of the two methods, the results are more remarkable.

In Part VI some properties of Markov processes, mainly recurrence and ergodic properties, are investigated. These properties depend closely on the construction theory.

I extend my hearty thanks to my teacher, Professor Zi-kun Wang, for his patient guidance, without which it would have been impossible to attain these research results. I have also greatly benefited by numerous discussions with

Professor Zhen-ting Hou. Moreover, he has read through the manuscript of this book very carefully and suggested many valuable improvements. Among those who had discussions with me and rendered me much support and encouragement are Associate Professor Qing-feng Guo and colleagues Rong Wu, Wen-chuan Mo and Mu-fa Chen. To all these I express my sincere thanks.

CHAPTER 1

Theoretical Background

1.1 INTRODUCTION

A number of basic concepts and major conclusions of random processes and Markov chains are given briefly in this chapter, but the proofs of conclusions are omitted. These concepts and conclusions will be used often in this book.[1]

1.2 RANDOM PROCESSES

1.2.1 Definitions

One or finitely many random trials are usually considered in elementary probability theory. These random trials can be described by one or finitely many random variables. In the law of large numbers and the central limit theorems, a sequence of random variables is involved, but it is assumed that the variables are independent. However, in practice, we have to study the development and process of a random phenomenon, and the events considered must be concerned with infinitely many (not necessarily denumerable) random variables. Consequently, we need to study all the random variables involved in the development and process of the random phenomenon. Thus we can depict the whole statistical law and the development of a random phenomenon. Hence we usually call a family of random variables a random process.

Example 1

Let $T=\{1,2,\ldots,N\}$ or $T=\{0,1,2,\ldots\}$ and $X(i)$ be a random variable for each $i\in T$. Then $X=\{X(i), i\in T\}$ is just a random process. But at this time we call it a random vector or a random sequence.

Example 2

In order to investigate how much service is performed at a telephone service counter, we calculate it by starting from a certain time $t=0$ and by letting $X(t)$

[1] *A note on notation.* 'Equation (1)' and 'Theorem 1', 'equation (2.1)' and 'Theorem 2.1', 'equation (3.2.1)' and 'Theorem 3.2.1', etc., refer respectively to equation (1) and Theorem 1 in the same section, in section 2 of the same chapter, and in section 2 of Chapter 3.

denote the number of telephone calls received at the service counter up to time t. Thus $X = \{X(t), t \in [0, \infty)\}$ is a random process.

Example 3

When considering the storing capacity of a warehouse, we denote by $X(t)$ the amount or quantity of some material stored in the warehouse at time t. Then $\{X(t), t \in [0, \infty)\}$ is a random process.

Example 4

When considering the problem of how far and wide an infectious disease has been spread, we use $X(t)$ to denote the number of people suffering from the disease in some region at time t. Then $\{X(t), t \in [0, \infty)\}$ is a random process.

Definition 1. Let a set T of parameters, a measurable space $(E, \mathscr{B})$ and a probability space $(\Omega, \mathscr{F}, P)$ be given. For each $t \in T$, there exists a measurable mapping from $(\Omega, \mathscr{F})$ into $(E, \mathscr{B})$ (the image of ω for the mapping $X(t)$ is denoted by $X(t, \omega)$). Then $\{X(t), t \in T\}$ is called a random process defined on the probability space $(\Omega, \mathscr{F}, P)$ and taking values in $(E, \mathscr{B})$.

An element ω of Ω is called a sample point while an element of $\mathscr{F}$ is called an event. The E or $(E, \mathscr{B})$ is called the state space; an element of E is called a state.

The parameter t is usually understood to stand for time; of course, the parameter t can also be understood to refer to something else, for instance the points in a plane or in a space. The state space $(E, \mathscr{B})$ is in general an abstract space. The one often used is $(E_0, \mathscr{B}_0)$ or $(R_1, \mathscr{B}_1)$, where E_0 is a set of all non-negative integers, $\mathscr{B}_0$ is the Borel field composed of all subsets of E_0, $R_1 = (-\infty, +\infty)$ and $\mathscr{B}_1$ is the Borel field composed of all Borel sets in R_1.

According to the definition of a random process, $X(t, \omega)$ $(t \in T, \omega \in \Omega)$ is a bivariate function taking values in E. When t is fixed, $X(t, \cdot)$ is a random variable; when ω is fixed, $X(\cdot, \omega)$ is a function defined on T, and is called the sample function or the path corresponding to the sample point ω.

From now on, if we make no special statement, $(\Omega, \mathscr{F}, P)$ always refers to a complete probability space, i.e. a probability space with the property that any subset of a null-probability event is also an event. T always refers to $\{0, 1, 2, \dots\}$ or $[0, \infty)$; $(E, \mathscr{B})$ always refers to $(E_0, \mathscr{B}_0)$ or $(R_1, \mathscr{B}_1)$.

1.2.2 Family of finite-dimensional distribution functions and existence theorem

Just as the probabilistic feature of a random variable is represented by the distributions of the random variable in elementary probability theory, so we use the family of finite-dimensional distribution functions in the case of random processes.

Definition 2. Let $X = \{X(t), t \in T\}$ be a random process. For any positive integer n and $t_i \in T$, $1 \leqslant i \leqslant n$, denote the n-dimensional joint distribution function of $X(t_1), X(t_2), \dots, X(t_n)$ by $F_{t_1, t_2, \dots, t_n}$, i.e.

$$F_{t_1, t_2, \dots, t_n}(\lambda_1, \lambda_2, \dots, \lambda_n) = P\{X(t_i) \leqslant \lambda_i, 1 \leqslant i \leqslant n\} \qquad \lambda_i \in R_1, \quad 1 \leqslant i \leqslant n \quad (1)$$

The family of distribution functions

$$F = \{F_{t_1, t_2, \dots, t_n} : t_i \in T, 1 \leqslant i \leqslant n, n = 1, 2, \dots\} \quad (2)$$

is called a family of finite-dimensional distribution functions of the random process X.

Obviously, the family F of finite-dimensional distribution functions of a random process satisfies the following consistency conditions (a) and (b):

(a) Assume that $(\alpha_1, \alpha_2, \dots, \alpha_n)$ is any permutation of $(1, 2, \dots, n)$. Then

$$F_{t_1, t_2, \dots, t_n}(\lambda_1, \dots, \lambda_n) = F_{t_{\alpha_1}, t_{\alpha_2}, \dots, t_{\alpha_n}}(\lambda_{\alpha_1}, \lambda_{\alpha_2}, \dots, \lambda_{\alpha_n})$$

(b) Assume $m < n$. Then

$$F_{t_1, \dots, t_m}(\lambda_1, \dots, \lambda_m) = F_{t_1, \dots, t_m, t_{m+1}, \dots, t_n}(\lambda_1, \dots, \lambda_m, \infty, \dots, \infty)$$

Theorem 1. Given a set T of parameters and a family (2) of finite-dimensional distribution functions satisfying the consistency conditions (a) and (b), then there exists a probability space $(\Omega, \mathscr{F}, P)$ on which is defined a random process $X = \{X(t), t \in T\}$ whose family of finite-dimensional distribution functions coincides with the given F.

1.2.3 Separability

In researching into random processes, non-denumerable union ($\cup$) and intersection ($\cap$) of events are often involved. For instance, sometimes we have to consider the set A of those sample points for which the corresponding sample functions of the random process $X = \{X(t), t \in T\}$ are bounded by C in the interval $[0, 1]$, i.e.

$$A = \{\omega : X(\cdot, \omega) \text{ is bounded by } C \text{ in } [0, 1]\} = \bigcap_{0 \leqslant t \leqslant 1} \{\omega : |X(t, \omega)| \leqslant C\} \quad (3)$$

Although each $\{\omega : |X(t, \omega)| \leqslant C\} \in \mathscr{F}$, the set A, as a non-denumerable intersection of sets of $\mathscr{F}$, does not necessarily belong to $\mathscr{F}$. Consequently we have to introduce the concept of separability.

In the discussion of separability, we allow the random process to take the

value ∞, but for each $t \in T$,

$$P\{X(t) = \infty\} = 0 \tag{4}$$

Definition 3. A random process $X = \{X(t), t \in T\}$ is called separable if there exist a denumerable dense subset R of T and a null-probability event N such that, for every $\omega \notin N$, the sample function $X(\cdot, \omega)$ possesses the following property: for any $t \in T$ there exists a sequence of points $r_n \in R$ such that

$$\lim_{n \to \infty} r_n = t \qquad \lim_{n \to \infty} X(r_n, \omega) = X(t, \omega) \tag{5}$$

The set R is said to be a separability set and the set N to be an exceptional set (relative to R).

If the process is separable, then for the set A in (3), the following holds:

$$A_R \supset A \supset A_R \cap N^c$$

where $N^c = \Omega - N$ and

$$\begin{aligned} A_R &= \{\omega: X(\cdot, \omega) \text{ is bounded by } C \text{ in } [0, 1] \cap R\} \\ &= \bigcap_{\substack{0 \leqslant r \leqslant 1 \\ r \in R}} \{\omega: |X(r, \omega)| \leqslant C\} \in \mathscr{F} \end{aligned}$$

Since $(\Omega, \mathscr{F}, P)$ is complete, $A \in \mathscr{F}$.

Definition 4. Let $X = \{X(t), t \in T\}$ be a random process. If there exist a denumerable dense subset R of T and a null-probability ω set N, such that for any closed set Λ in $[-\infty, +\infty]$ and any open interval I, the following holds:

$$\{\omega: X(r, \omega) \in \Lambda, r \in IR\} - \{\omega: X(t, \omega) \in \Lambda, t \in IT\} \subset N \tag{6}$$

then the process X is called separable.

Theorem 2. The two definitions in Definitions 3 and 4 are equivalent.

In order to emphasize the separability set, we say that the separable process X is separable relative to R. If X is separable relative to any denumerable dense set R of T, then X is called well separable.

Definition 5. Two random processes $X = \{X(t), t \in T\}$ and $Y = \{Y(t), t \in T\}$ defined on the same probability space $(\Omega, \mathscr{F}, P)$ are said to be stochastically equivalent if, for any $t \in T$, it is true that

$$P\{X(t) = Y(t)\} = 1$$

In that case, X is called a modification or a version of Y.

Theorem 3. Let $X = \{X(t), t \in T\}$ be a real-valued random process. Then there always exists a separable process $Y = \{Y(t), t \in T\}$ stochastically equivalent to X.

Y is called a separable version of X. Notice that Y take the value ∞ but $P\{Y(t) = \infty\} = 0$, $t \in T$.

Definition 6. A random process $X = \{X(t), t \in T\}$ is called stochastically continuous if, for any $t_0 \in T$, we have

$$P \lim_{t \to t_0} X(t) = X(t_0) \tag{7}$$

Here the limit is the limit in probability.

Theorem 4. Assume that a separable process X is stochastically continuous; then X is well separable.

1.2.4 Measurability

In order to study the measurability of a sample function $X(\cdot, \omega)$, we need to define measurable random processes.

Let $T = [0, \infty)$, $\mathscr{B}_T$ be the Borel field composed of all Borel sets in T, L be the Lebesgue measure on $\mathscr{B}_T$, $\mu = L \times P$ be the product measure defined on $\mathscr{B}_T \times \mathscr{F}$ and $\overline{\mathscr{B}_T \times \mathscr{F}}$ be the completion of $\mathscr{B} \times \mathscr{F}$ relative to μ.

Definition 7. A random process $X = \{X(t), t \in T\}$ is called Borel-measurable if, for any $c \in R_1$, we have

$$\{(t, \omega): X(t, \omega) \leqslant c\} \in \mathscr{B}_T \times \mathscr{F} \tag{8}$$

X is called measurable if, for any $c \in R_1$, we have

$$\{(t, \omega): X(t, \omega) \leqslant c\} \in \overline{\mathscr{B}_T \times \mathscr{F}}. \tag{9}$$

Theorem 5. A right-continuous process (i.e. a process whose sample functions are all right-continuous) is Borel-measurable. A process almost all of whose sample functions are right-continuous is measurable.

Theorem 6. A stochastically continuous process must have a measurable version.

1.2.5 Sample functions: continuity and step functions

Theorem 7. Let $X = \{X(t), t \in [a, b]\}$ be a separable random process. Assume that there exist three real numbers $\alpha > 0$, $\varepsilon > 0$ and $c \geqslant 0$ such that for arbitrary

$t \in [a, b]$, $t + \Delta \in [a, b]$, we have

$$E\{|X(t+\Delta) - X(t)|^{\alpha}\} \leqslant c|\Delta|^{1+\varepsilon} \tag{10}$$

Then almost all sample functions of $\{X(t), t \in [a, b]\}$ are continuous.

Definition 8. Let $\{X(t), t \in [a, b]\}$ be a real-valued function, and suppose that there is a finite subdivision $a = t_0 < t_1 < \cdots < t_n = b$ such that

(i) $X(t) = c_i$ for $t_i < t < t_{i+1}$, $i = 0, 1, \ldots, n-1$, so that the limits $X(t_i - 0) = c_{i-1}$ $(0 < i \leqslant n)$ and $X(t_i + 0) = c_i$ $(0 \leqslant i < n)$ exist.

(ii) Either $X(t_i) = X(t_i - 0)$ $(0 < i \leqslant n)$, in which case $X(\cdot)$ is left-continuous on $(a, b]$, or $X(t_i) = X(t_i + 0)$ $(0 \leqslant i < n)$, in which case $X(\cdot)$ is right-continuous on $[a, b)$.

Then $\{X(t), t \in [a, b]\}$ is said to be a step function defined on $[a, b]$; if $X(t_i - 0) \neq X(t_i + 0)$, we call t_i a jumping point.

A function $\{X(t), t \in [0, \infty)\}$ is called a step function if $\{X(t), t \in [0, b_n]\}$ is step function for a sequence $b_n \uparrow \infty$.

Theorem 8. Let $\{X(t), t \in [a, b]\}$ be a separable random process. If there exists a constant $c \geqslant 0$ such that, for arbitrary $t \in [a, b]$, $t + \Delta \in [a, b]$, the following holds:

$$P\{X_{t+\Delta} \neq X_t\} \leqslant c|\Delta| \tag{11}$$

then almost all sample functions of $\{X(t), t \in [a, b]\}$ are step functions.

1.2.6 Classification

According to whether the set T of parameters and the state space E are denumerable or not, we can simply classify random processes into four classes:

(i) T and E are denumerable;

(ii) T is denumerable whereas E is not;

(iii) T and E are not denumerable at all;

(iv) T is not denumerable, but E is.

A random process with a denumerable set of parameters is called a random sequence or a time series. A random process with a denumerable state space is called a denumerable process.

The classifications above are only in form. They are not concerned with the inherent probability relation of a random process. If we classify random processes according to the inherent probability relation of a process, then a lot of important classes of processes can be obtained. For example, consider the following.

Definition 9. Let $X(t), t \in T$, be independent random variables. Then $X = \{X(t), t \in T\}$ is said to be an independent process.

Definition 10. Suppose that, for arbitrary $t_i \in T$, $t + t_i \in T$, $1 \leqslant i \leqslant n$,

$$X(t_1), X(t_2), \ldots, X(t_n)$$

and

$$X(t_1 + t), X(t_2 + t), \ldots, X(t_n + t)$$

have the same joint distribution function. Then the random process $X = \{X(t), t \in T\}$ is called stationary.

An extremely important class of random processes is that of Markov processes, which will be introduced below.

1.3 MARKOV PROCESSES

1.3.1 Definitions

The intrinsic probability structure of an independent random process is simple. Let $T = [0, \infty)$; the parameter is understood as time, and $X(t)$ is understood as the position of a particle making stochastic movement at time t. We consider the time s as the 'present'; then the time interval $[0, s)$ is the 'past', and the time interval (s, ∞) is the 'future'. Suppose that we want to predict the 'future' of the random process if we have information about the 'past' and the 'present', or about the 'present', or know nothing about the 'past' and 'present'. For an independent process, the results predictable are the same in the three cases above. But for a Markov process, the result predictable are the same in the first and second cases above; roughly speaking, under the condition that the 'present' is known, the 'future' and the 'past' are conditionally independent. This property is said to be the Markov property.

Examples in reality possessing the Markov property are numerous. For instance, if we leave out secondary factors, population growth is characterized by the Markov property. The number of a population in future is relative only to the present population base and not relative to the number of people in the past. The process in example 2 in section 1.2 is also characterized by the Markov property.

Definition 1. A random process $X = \{X(t), t \in T\}$ is called a Markov process if, for arbitrary $0 \leqslant t_1 < t_2 < \cdots < t_n$ and $\Gamma \in \mathscr{B}$, we have

$$P\{X(t_n) \in \Gamma | X(t_1), \ldots, X(t_{n-1})\} = P\{X(t_n) \in \Gamma | X(t_{n-1})\} \quad \text{a.e.} \tag{1}$$

The property (1) is called the Markov property. The Markov property has

many equivalent forms. For instance:

(a) For arbitrary $0 \leqslant s \leqslant t$, $\Gamma \in \mathscr{B}$,

$$P\{X(t)\in\Gamma \mid X(u), 0 \leqslant u \leqslant s\} = P\{(t)\in\Gamma \mid X(s)\} \qquad \text{a.e.} \tag{2}$$

(b) For arbitrary $0 \leqslant s \leqslant t$, and N^s-measurable random variable η, $E|\eta| < \infty$,

$$E\{\eta \mid X(u), 0 \leqslant u \leqslant s\} = E\{\eta \mid X(s)\} \qquad \text{a.e.} \tag{3}$$

where $N^s = \sigma\{X(u), u \geqslant s\}$ is the Borel field generated by $\{X(u), u \geqslant s\}$.

1.3.2 Transition functions and homogeneity

Definition 5. A Markov process $\{X(t), t\in T\}$ is called homogeneous if there exists a regular transition probability function (simply, transition function) if the following four conditions are satisfied:

(i) for fixed s, x, t, $P(s, x; t, \cdot)$ is a probability measure on $\mathscr{B}$;

(ii) for fixed s, t, Γ, $P(s, \cdot; t, \Gamma)$ is a $\mathscr{B}$-measurable function;

(iii) for any $s \in T$

$$P(s, x; s, \Gamma) = I_\Gamma(x) \tag{4}$$

where I_Γ is the indicator of Γ;

(iv) the Chapman–Kolmogorov equation holds: for arbitrary $0 \leqslant s \leqslant t \leqslant u$, there is

$$P(s, x; u, \Gamma) = \int_E P(s, x; t, \mathrm{d}y) P(t, y; u, \Gamma) \tag{5}$$

The condition (iii) is called the regularity condition.

Definition 3. A transition function $P(s, x; t, \Gamma)$ is said to be homogeneous if for fixed x and Γ the values of the function only depend on $t - s$. That is, there exists a three-variate function $P(t, x, \Gamma)$ such that

$$P(s, x; t, \Gamma) = P(t - s, x, \Gamma) \tag{6}$$

In that case (i)–(iv) in Definition 2 become:

(a) for fixed t and x, $P(t, x, \cdot)$ is a probability measure on $\mathscr{B}$;

(b) for fixed t and Γ, $P(t, \cdot, \Gamma)$ is a $\mathscr{B}$-measurable function;

(c) $P(0, x, \Gamma) = I_\Gamma(x)$;

(d) for arbitrary $s, t \geqslant 0$, we have

$$P(s + t, x, \Gamma) = \int_E P(s, x, \mathrm{d}y) P(t, y, \Gamma) \tag{7}$$

The three-variate function $P(t, x, \Gamma)$ satisfying (a)–(d) above is said to be a homogeneous transition functions.

Example 1

(Wiener transition function).

$$P(t, x, \Gamma) = \begin{cases} \dfrac{1}{\sigma(2\pi t)^{1/2}} \displaystyle\int_\Gamma \exp\left(-\frac{(y - x)^2}{2\sigma^2 t}\right) \mathrm{d}y & t > 0 \\ I_\Gamma(x) & t = 0 \end{cases}$$

Here $\sigma > 0$ is a constant.

Example 2

(Poisson transition function). Let $E = \{0, 1, 2, \ldots\}$. Put

$$p_{ij}(t) = \begin{cases} \dfrac{(\lambda t)^{j-i}}{(j - i)!} \mathrm{e}^{-\lambda t} & \text{if } t > 0, j \geqslant i \\ 0 & \text{if } t > 0, j < i \\ \delta_{ij} & \text{if } t = 0 \end{cases}$$

Here $\lambda > 0$ is a constant. And then set

$$P(t, i, \Gamma) = \sum_{j\in\Gamma} p_{ij}(t) \qquad t \geqslant 0$$

Definition 4. A Markov process is said to have a transition function if there exists a transition function $P(s, x; t, \Gamma)$ such that

$$P\{X(t)\in\Gamma \mid X(s)\} = P(s, X(s); t, \Gamma) \qquad \text{a.e.} \tag{8}$$

Definition 5. A Markov process $\{X(t), t\in T\}$ is called homogeneous if there exists a homogeneous transition function $P(t, x, \Gamma)$ such that

$$P\{X(s + t)\in\Gamma \mid X(s)\} = P\{t, X(s), \Gamma\} \qquad \text{a.e.} \tag{9}$$

1.3.3 Family of finite-dimensional distributions and existence theorem

Theorem 1. Let $\{X(t), t\in T\}$ be a random process. In order that $\{X(t), t\in T\}$ is a homogeneous Markov process, it is necessary and sufficient that there exists a homogeneous transition function $P(t, x, \Gamma)$ such that for any $0 \leqslant t_1 < t_2 < \cdots < t_n$ and $\Gamma_i \in \mathscr{B}$, $1 \leqslant i \leqslant n$, the finite-dimensional distributions are given by

$$P(X(t_i)\in\Gamma, 1 \leqslant i \leqslant n)$$

$$= \int_E \mu(\mathrm{d}y_0) \int_{\Gamma_1} P(t_1, y_0, \mathrm{d}y_1) \int_{\Gamma_2} P(t_2 - t_1, y_1, \mathrm{d}y_1) \cdots \int_{\Gamma_n} P(t_n - t_{n-1}, y_{n-1}, \mathrm{d}y_n) \tag{10}$$

where $\mu(\Gamma) = P(X(0)\in\Gamma)$ is the initial distribution.

Theorem 2 (Existence theorem). Let a homogeneous transition function $P(t, x, \Gamma)$, $t \geqslant 0$, $x \in R_1$, $\Gamma \in \mathscr{B}_1$, and a probability measure $\mu(\Gamma), \Gamma \in \mathscr{B}$, be given. Then there exists a probability space $(\Omega, \mathscr{F}, P)$ on which is defined a homogeneous Markov process $\{X(t), t \geqslant 0\}$ with the given transition function $P(t, x, \Gamma)$ and the given initial distribution $\mu(\Gamma)$.

1.3.4 Semigroup property and Feller transition functions

Let B be the collection of all bounded $\mathscr{B}$-measurable functions defined on E. For $f \in B$ we define a norm $\|f\| = \sup_{x \in E} |f(x)|$. Then B is a Banach space. For $f \in B$ we define a function by

$$T_t f(x) = \int_E f(y) P(t, x, \mathrm{d}y) \tag{11}$$

Obviously, $T_t f \in B$, and moreover, from (a) and (b) for a homogeneous transition function (Definition 3) we have

$$\| T_t \| \leqslant 1 \qquad T_{s+t} = T_s T_t \tag{12}$$

That is, $\{T_t, t \geqslant 0\}$ is a family of contraction operators, forming a semigroup. Whence we can determine its strong (or weak) infinitesimal operator A (or $\tilde{A}$) and strong (or weak) resolvent operators R_λ (or $\tilde{R}_\lambda$), $\lambda > 0$. Therefore we may research into the tansition function of Markov processes by means of the theory of semigroups in functional analysis.

Assume that the state space E is endowed with a topology and C denotes the collection of all bounded continuous functions defined on E. If $T_t f \in C$ for any $f \in C$, the transition function is called the Feller transition function.

1.3.5 Strong Markov property

Definition 6. Let $X = \{X(t), t \in T\}$ be a random process. We say that X is progressively measurable if, for any $t \in T$ and $\Gamma \in \mathscr{B}$, it is valid that

$$\{(s, \omega): 0 \leqslant s \leqslant t, X(s, \omega) \in \Gamma\} \in \mathscr{B}_{[0,t]} \times N_t \tag{13}$$

where $\mathscr{B}_{[0,t]}$ is the Borel field composed of all Borel sets in $[0, t] \cap T$, and N_t is the Borel field generated by $\{X(s), 0 \leqslant s \leqslant t, s \in T\}$, which is called the pre-t field.

Theorem 3. A random process whose sample functions all are right- (or left-) continuous is progressively measurable.

Definition 7. Let τ be a non-negative (including $+\infty$) $\mathscr{F}$-measurable function defined on Ω. If for any $t \in T$ the following holds:

$$(\tau \leqslant t) \in N_t \tag{14}$$

τ is called a stopping time of the process $\{X(t), t \in T\}$. For a stopping time τ, the Borel field

$$N_\tau = \{A: A \in \mathscr{F}, \text{ and moreover, } A \cap (\tau \leqslant t) \in N_t \text{ for each } t \geqslant 0\} \tag{15}$$

is called the pre-τ field.

Definition 8. Let $X = \{X(t), t \in T\}$ be a homogeneous Markov process. Assume that X is progressively measurable, and moreover, for any stopping time τ, it is valid that

$$P\{X(\tau + t) \in \Gamma \mid N_\tau\} = P(t, X(\tau), \Gamma) \qquad \text{a.e. } \Omega_\tau \tag{16}$$

where a.e. Ω_τ indicates that it is valid for almost all $\omega \in \Omega_\tau = (\tau < \infty)$. X is called a strong Markov process and (16) is said to be the strong Markov property.

Theorem 4. A homogeneous Markov process $X = \{X(n), \; n = 0, 1, 2, \ldots\}$ possesses the strong Markov property.

Theorem 5. A right-continuous Feller process is a strong Markov process.

1.4 MARKOV CHAINS

1.4.1 Definitions and transition matrices

A homogeneous Markov chain is a special homogeneous Markov process. When $T = \{0, 1, 2, \ldots\}$ and E is denumerable, we might as well set $E = \{0, 1, 2, \ldots\}$, a homogeneous Markov process is said to be a homogeneous Markov chain, which is called a Markov chain for short. Of course, we can define a Markov chain directly.

Definition 1. $\{X(n), n = 0, 1, 2, \ldots\}$ is called a Markov chain if, for arbitrary $m \geqslant 0$, $n \geqslant 0$, $i_0, i_1, \ldots, i_{m-1}, i, j \in E$, the following equality holds:

$$\begin{aligned} &P\{X(m+n) = j \mid X(0) = i_0, X(1) = i_1, \ldots, X(m-1) = i_{m-1}, X(m) = i\} \\ &\quad = P\{X(m+n) = j \mid X(m) = i\} \end{aligned} \tag{1}$$

and moreover, the right side is independent of m, provided the conditional probability is defined.

We may as well assume that for each $i \in E$, there exists a parameter $m \in T$ such that $P\{X(m) = i\} > 0$. Otherwise, i is a non-essential state and i may be combed out of E. We can consider the state space $E - \{i\}$ instead of E.

Denote the right in (1) by $p_{ij}^{(n)}$, which is called the n-step transition probability from i to j. The matrix $P^{(n)} = (p_{ij}^{(n)}, i, j \in E)$ is called the n-step transition matrix. When $n = 1$, we write $P = P^{(1)}$, $p_{ij} = p_{ij}^{(1)}$. The matrix P is called the transition matrix of the chain. Owing to the Chapman–Kolmogorov equations, it follows

that

$$P^{(n)} = P^n \tag{2}$$

Consequently, the transition law of the Markov chain is determined by its one-step transition matrix. Therefore, sometimes we call P the Markov chain.

For a Markov chain $\{X(n),\ n=0,1,2,\ldots\}$, the distribution $\gamma=(\gamma_i)$, where $\gamma_i = P(X(0)=i)$, is called the initial distribution of the chain. Moreover, the family of distributions

$$P(X(0)=i_0, X(1)=i_1,\ldots,X(n)=i_n) = \gamma_{i_0} P_{i_0 i_1} \cdots P_{i_{n-1} i_n}, \qquad i_0, i_1, \ldots, i_n \in E$$

and the family of finite-dimensional distributions

$$P(X(n_1)=i_1, X(n_2)=i_2, \ldots, X(n_k)=i_k), \qquad n_1 < n_2 < \cdots < n_k,$$

are determined by the initial distribution and the transition matrix P. Conversely, given a distribution $\gamma=(\gamma_i)$ and a nonnegative matrix $P=(P_{ij})$ with $\sum_j P_{ij}=1$ for every i, by the existence theorem, there exists a probability space $(\Omega, \mathscr{F}, P)$, on which is defined a Markov chain $\{X(n),\ n \geqslant 0\}$. This chain has the initial distribution $\gamma=(\gamma_i)$ and transition matrix $P=(P_{ij})$. The probability measure P depends on the initial distribution $\gamma=(\gamma_i)$, and we also denote it by P_γ instead of the probability measure P. The measure P_γ is said to be the measure generated by the initial distribution $\gamma=(\gamma_i)$ and the matrix $P=(P_{ij})$. When $\gamma_i=1$, $\gamma_j=0$ $(j\neq i)$, we denote it by $p_\gamma=p_i$. For general γ we have $p_\gamma = \sum_i \gamma_i p_i$.

Example 1

(Random Walk). Assume that $Z(0)=0$ is a constant, $Z(i)$ $(i=1,2,\ldots)$ are independent and of identical distribution, and, moreover,

$$P\{Z(i)=1\}=p \qquad P\{Z(i)=-1\}=q \qquad P\{Z(i)=0\}=r$$

put

$$p+q+r=1 \qquad 0<p, \qquad 0<q, \qquad 0\leqslant r<1.$$

$$X(n)=Z(0)+Z(1)+\cdots+Z(n)$$

Then $\{X(n),\ n=0,1,2,\ldots\}$ is a Markov chain, its state space is $\{\cdots,-2,-1,0,1,2,\ldots\}$, and its transition probabilities are

$$P_{i,i+1}=p \qquad P_{i,i-1}=q \qquad P_{ii}=r \tag{3}$$

$\{X(n), n=0,1,2,\ldots\}$ is called a simple random walk, and a symmetric random walk when $r=0$ and $p=q=\frac{1}{2}$.

The example above can be explained intuitively as follows. A particle moves stochastically in a number axis. It starts from the origin. Then it moves once per unit time. It shifts a unit distance either towards the left with probability q or towards the right with probability p; it may stand still with probability r. $X(n)$ denotes the position of the particle after it has moved n times.

Example 2

(Random walk with barriers). Let the state space be $\{0,1,2,\ldots,b\}$. Where the particle is at i $(0<i<b)$, it makes the simple random walk. Let the particle be at 0. If the particle goes to 1 after unit time with probability 1, i.e. $p_{01}=1$, the barrier 0 is called a reflecting barrier; if the particle stays at 0 after unit time with probability 1, then 0 is called an absorbing barrier. If the particle stays at 0 with probability r_0 $(0<r_0<1)$ and goes to 1 with probability $p_0=1-r_0$, the barrier 0 is called an elastic barrier. If we place variously different barriers on 0 and b, we will obtain variously different random walks. For instance, the matrix

$$P=\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ q & r & p & \cdots & 0 & 0 & 0 \\ 0 & q & r & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & q & r & p \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

represents a random walk with an absorbing barrier 0 and a reflecting barrier b.

Example 3

(Biological chains). Let the state space be $\{0,1,2,\ldots\}$. If the particle is at i, when the particle walks stochastically to the left or to the right, or stands motionless its probabilities are related to the position i. For example, the probabilities are $q_i, p_i, r_i, q_i+p_i+r_i=1$, $q_0=0$, respectively, that is,

$$P=\begin{pmatrix} r_0 & 1-r_0 & 0 & 0 & \cdots \\ q_1 & 1-(p_1+q_1) & p_1 & 0 & \cdots \\ 0 & q_2 & 1-(p_2+q_2) & p_2 & \cdots \\ 0 & 0 & q_3 & 1-(p_3+q_3) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

A chain with the transition matrix P above is called a birth–death chain. The intuitive meaning goes as follows:

Assume that there is a biological group, the number of whose individuals is changeable. Suppose that at time n the number of individuals of the group is denoted by i. Then at time $n+1$, the biological group increases to $i+1$ individuals with probability p_i, reduces to $i-1$ individuals with probability q_i and keeps i individuals with probability r_i.

1.4.2 Classification of states

Let $P=(p_{ij}, i,j\in E)$ be the transition matrix of a Markov chain $\{X(n),\ n=0, 1,2,\dots\}$.

Defintion 2. Assume that the state i is fixed and the set $\{n:n\geqslant 1,\ p_{ii}^{(n)}>0\}$ is non-empty. Let $d=d(i)$ be the greatest common divisor of the set. The number d is called the period of the state i. The state i is called periodic when $d>1$ and is non-periodic when $d=1$.

Let $\tau=\tau(j)$ denote the first time of reaching the state j:

$$\tau=\begin{cases}\min\{n\geqslant 1:X(n)=j\} & \text{if the set is non-empty}\\ +\infty & \text{otherwise}\end{cases}$$

Then

$$f_{ij}^{(n)}=P\{\tau=n\mid X(0)=i\}\qquad n\geqslant 1$$

is the probability that the Markov chain will first visit the state j at time n, given that it starts from state i. Furthermore,

$$f_{ij}^{*}=\sum_{n=1}^{\infty} f_{ij}^{(n)}$$

is the probability that the Markov chain will first visit the state j after finitely many steps, given that it starts from the state i, and

$$\mu_{ij}=\sum_{n=1}^{\infty} n f_{ij}^{(n)}$$

is the mean number of steps that the Markov chain visits j first, given that it starts from state i. Write $f_i^*=f_{ii}^*$, $\mu_i=\mu_{ii}$.

Definition 3. (a) The state i is called recurrent if $f_i^*=1$ and transient if $f_i^*<1$. (b) Let the state i be recurrent. The state i is called positive-recurrent if $\mu_i<\infty$ and null-recurrent if $\mu_i=\infty$. (c) The state i is said to be ergodic if it is non-periodic and positive-recurrent.

Theorem 1. For arbitrary $i,j\in E$ and $n\geqslant 1$, it is true that

$$p_{ij}^{(n)}=\sum_{k=1}^{n} f_{ij}^{(k)}p_{jj}^{(n-k)}=\sum_{k=0}^{n-1} p_{ij}^{(k)}f_{jj}^{(n-k)}$$

where $p_{ij}^{(0)}=\delta_{ij}$, $f_{jj}^{(0)}\doteq 1$.

Set

$$G_{ij}=\sum_{n=0}^{\infty} p_{ij}^{(n)}$$

Theorem 2. It is true that

$$G_{ii}=1/(1-f_i^*)$$
$$G_{ij}=f_{ij}^*G_{jj}\qquad (i\neq j)$$

Theorem 3. For the state i to be recurrent it is necessary and sufficient that $G_{ii}=\infty$.

Theorem 4. Let the state i be recurrent.

(a) i is null-recurrent if and only if $\lim_{n\to\infty} p_{ii}^{(n)}=0$.
(b) i is ergodic if and only if $\lim_{n\to\infty} p_{ii}^{(n)}=1/\mu_i>0$.

1.4.3 Decomposition of the state space

Definition 4. We say that the state i leads to the state j if there exists an integer $n\geqslant 1$ such that $p_{ij}^{(n)}>0$. In this case we write $i\to j$. Also we write $i\leftrightarrow j$ to mean that i and j communicate, i.e. $i\to j$ and $j\to i$.

Definition 5. A subset C of E is called closed if for any $i\in C$ and $k\bar{\in} C$ it is true that $p_{ik}=0$. If two arbitrary states in a closed set C communicate, the set C is called an irreducible closed set.

Theorem 5 (Decomposition theorem). The state space has a unique decomposition

$$E=E_0\cup\bigcup_{a\in\mathscr{A}} E_a$$

where $\mathscr{A}$ is empty, or the finite set $\{1,2,\dots,b\}$, or the denumerable set $\{1,2,3,\dots\}$; E_0 is the set If all transient states, which may be empty; for each $a\in\mathscr{A}$, E_a is a irreducible and recurrent class; the states in E_a are all null, or all positive, or all ergodic; finally, all E_a, $a\in\{0\}\cup\mathscr{A}$, are mutually disjoint.

1.5 DENUMERABLE MARKOV PROCESSES

1.5.1 Definitions

A denumerable Markov process is a Markov process with a continuous time and a discrete state space. We usually take $T=[0,\infty)$, $E=\{0,1,2,\dots\}$.

Definition 1. Let the state space E of the random process $X=\{X(t),\ t\geqslant 0\}$ be denumerable. X is said to be a Markov process if for arbitrary $n\geqslant 1$,

$0 \leqslant t_1 < \cdots < t_n < s < t$, $i_1, \ldots, i_n$, $i, j \in E$, we have

$$P\{X(s+t)=j \mid X(t_1)=i_1, \ldots, X(t_n)=i_n, X(s)=i\} = P\{X(s+t)=j \mid X(s)=i\} \quad (1)$$

provided that

$$P\{X(t_1)=i_1, \ldots, X(t_n)=i_n, X(s)=i\} > 0$$

If the right side of (1) depends on only $t-s$ and does not depend on s, the Markov process X is called homogeneous, or a denumerable Markov process.

Denoting the right side of (1) by $p_{ij}(s,t)$, then the matrices $P(s,t)=\{p_{ij}(s,t), i,j \in E\}$, $0 \leqslant s \leqslant t$, are called the transition matrices of X. Write $E_+(s) = \{k: P\{X(s)=k\} > 0\}$, $E_0(s) = E - E_+(s)$. If $E_+(s) = E$ for each $s \geqslant 0$, then the transition matrices $\{P(s,t), 0 \leqslant s \leqslant t\}$ of X exist, and satisfy the following conditions:

(a) $p_{ij}(s,t) \geqslant 0 \qquad \sum_j p_{ij}(s,t) = 1$

(b) $p_{ij}(s,t) = \sum_k p_{ik}(s,u) p_{kj}(u,t) \qquad 0 \leqslant s \leqslant u \leqslant t$

(c) $p_{ij}(s,s) = \delta_{ij}$

But if $E_+(s) \neq E$ for some s, $p_{ij}(s,t)$ is defined for $i \in E_+(s)$ and undefined for $i \in E_0(s)$. Does X possess the transition matrices satisfying the above condition (a), (b), (c) in the general case? The books that I have read affirm the truth of the conclusion but the proof is not given. My postgraduate student Yu-Quan Xie gives a proof of this below.

Theorem 1. Each Markov process X possesses the transition matrices $P(s,t)$ satisfying the above conditions (a), (b), (c), such that when $i \in E_+(s)$, the right side of (1) is $p_{ij}(s,t)$.

Proof. For $i \in E_+(s)$, we define $p_{ij}(s,t)$ by the right side of (1).
Let $i \in E_0(s)$. Take arbitrary non-negative numbers $u_{ij}(s)(j \in E)$ such that $u_{ij}(s) = 0$ for $j \in E_0(s)$ and $\sum_{j \in E_+(s)} u_{ij}(s) = 1$. Set

$$p_{ij}(s,t) = \sum_{k \in E_+(s)} u_{ik}(s) p_{kj}(s,t) \qquad 0 \leqslant s \leqslant t \quad (2)$$

By a direct verification, it easily follows that $P(s,t) = \{p_{ij}(s,t), i,j \in E\}$, $0 \leqslant s \leqslant t$, are just what we want. QED

In the following X is always assumed to be a homogeneous denumerable Markov process. Then $p_{ij}(s,t) = p_{ij}(t-s)$ is independent of s, and moreover, (a), (b), (c) become

(α) $p_{ij}(t) \geqslant 0 \qquad \sum_j p_{ij}(t) = 1$

(β) $p_{ij}(s+t) = \sum_k p_{ik}(s) p_{kj}(t)$

(γ) $p_{ij}(0) = \delta_{ij}$

Furthermore, we assume the condition of standardness:

(δ) $\lim_{t \downarrow 0} p_{ij}(t) = p_{ij}(0)$

1.5.2 Properties of sample functions

A family of matrices $P(t) = \{p_{ij}(t), i,j \in E\}$, $t \geqslant 0$, satisfying the above conditions (α)–(δ) are called standard transition matrices.

Theorem 2. For standard transition matrices there exist right derivatives at $t=0$,

$$q_{ij} = p'_{ij}(0) = \lim_{t \downarrow 0} \frac{p_{ij}(t) - p_{ij}(0)}{t} \quad (3)$$

and moreover,

$$0 \leqslant q_{ij} < \infty \qquad (i \neq j) \qquad 0 \leqslant -q_{ii} \leqslant \infty$$

$$\sum_{j \neq i} q_{ij} \leqslant -q_{ii} \quad (4)$$

From (3) it follows that

$$p_{ij}(t) = q_{ij}t + o(t) \qquad i \neq j \quad (5)$$

if $q_i < \infty$ then

$$p_{ii}(t) = 1 - q_i t + o(t) \quad (6)$$

The matrix $Q = (q_{ij}, i,j \in E)$ is called the Q matrix of the process X or of the transition matrix. It can be seen from (3) that Q is determined by the values of $P(t)$ in a small interval $[0. \varepsilon)$. In practice Q is easier to calculate than $P(t)$. Therefore how to calculate from Q constitutes the central problem discussed in this book, that is, the construction problem.

Definition 2. Write $q_i = -q_{ii}$. The state i is said to be stable if $q_i < \infty$, to be absorbing if $q_i = 0$, and to be instaneous if $q_i = +\infty$.

Notice that $X(t, \omega)$ may be $+\infty$ if we consider a separable process $X = \{X(t), t \geqslant 0\}$, but $P\{X(t) = \infty\} = 0$ for fixed $t \geqslant 0$.

Theorem 3. Let X be separable and measurable, and i be an instantaneous state.

Then for almost all $\omega\in\Omega$, the set

$$S_i(\omega)=\{t:t\geqslant 0, X(t,\omega)=i\}$$

is nowhere dense in $[0,\infty)$.

Theorem 4. Let X be separable and measurable, and $t\geqslant 0$ be fixed. Then for allmost all $\omega\in\Omega$, the following conclusions hold:

(i) $\lim_{s\to t}X(s,\omega)=i$ if $X(t,\omega)=i$ and i is stable.

(ii) $X(s,\omega)$ has exactly two limiting values i and ∞ as $s\downarrow t$ (or $s\uparrow t$) if $X(t,\omega)=i$ and i is instantaneous.

(iii) $X(t,\omega)\neq\infty$.

Theorem 5. Assume that X is separable and measurable. Then for almost all $\omega\in\Omega$, the sample functions $X(\cdot,\omega)$ have the following property.

Property (A): for any generic $t\geqslant 0$, as $s\downarrow t$ (or $s\uparrow t$) $X(\cdot,\omega)$ has at most one finite limiting value. There are three posibilities:

(a) $X(s,\omega)\to i$, where i is stable;

(b) $X(s,\omega)$ has exactly two limiting values i and ∞, where i is instantaneous;

(c) $X(s,\omega)\to\infty$.

Furthermore

(α) if $X(t,\omega)=i$ where i is stable, then (a) is true (with the same i) as $s\to t$ from at least one side;

(β) if $X(t,\omega)=i$ where i is instantaneous, then (b) is true (with the same i) as $s\to t$ from at least one side.

1.5.3 Canonical processes

Definition 3. A process $X=\{X(t),\ t\geqslant 0\}$ is called right lower semicontinuous if, for all $\omega\in\Omega$, we have

$$\varliminf_{s\downarrow t}X(s,\omega)=X(t,\omega)\qquad \text{all } t\geqslant 0 \tag{4}$$

A separable and measurable process that is right lower semicontinuous is said to be a canonical process.

Theorem 6. Given a denumerable Markov process X, there always exists a canonical process stochastically equivalent to X.

Theorem 7. A canonical process possesses the strong Markov property. That

is, for any stopping time τ and $\Lambda\in N_\tau$, arbitrary $0\leqslant t_0<t_1<\cdots<t_N$, j_0, $j_1,\ldots,j_N\in E$, the following equality holds:

$$P\{\Lambda, X(\tau+t_\nu)=j_\nu, 0\leqslant\nu\leqslant N\}=P\{\Lambda, X(\tau+t_0)=j_0\}\prod_{\nu=0}^{N-1}p_{i_\nu i_{\nu+1}}(t_{\nu+1}-t_\nu) \tag{7}$$

Moreover,

$$P\{X(\tau+t)=\infty\mid\Delta\}=0\qquad t>0 \tag{8}$$

where $\Delta=\{\tau<\infty\}$ and $P\{X(\tau)=\infty\mid\Delta\}$ may be positive.

1.5.4 Special examples

(i) Poisson process. Let $\lambda>0$ be a constant. The process $X=\{X(t), t\geqslant 0\}$ with the Q matrix

$$Q=\begin{pmatrix}-\lambda & \lambda & 0 & 0 & \cdots\\ 0 & -\lambda & \lambda & 0 & \cdots\\ 0 & 0 & -\lambda & \lambda & \cdots\\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

is called a Poisson process.

The Poisson process is a special birth process. Assume that there is a biological group, whose individuals are deathless and may give birth to new individuals. Let $X(t)$ denote the content of the biological group (i.e. the number of all individuals) at time t. Assume that Δt is very small, and the probability that the content increases by one in the interval $[t,t+\Delta t]$ is $\lambda\Delta t+o(\Delta t)$, the probability that the content increases by two and over in the interval $[t,t+\Delta t]$ is $o(\Delta t)$, and the probability that the content is invariable in $[t,t+\Delta t]$ is $1-\lambda\Delta t+o(t)$. Such a process $\{X(t), t\geqslant 0\}$ is just a Poisson process.

The transition probabilities of the Poisson process must be

$$p_{ij}(t)=\begin{cases}0 & \text{if } j<i\\ e^{-\lambda t}\dfrac{(\lambda t)^{j-i}}{(j-i)!} & \text{if } j\geqslant i\end{cases} \tag{9}$$

(ii) Birth processes. A denumerable Markov process $X=\{X(t),\ t\geqslant 0\}$ with the Q matrix

$$Q=\begin{pmatrix}-\lambda_0 & \lambda_0 & 0 & 0 & \cdots\\ 0 & -\lambda_1 & \lambda_1 & 0 & \cdots\\ 0 & 0 & -\lambda_2 & \lambda_2 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \tag{10}$$

is called a birth process. Here $\lambda_i>0\,(i=0,1,2,\ldots)$ also. Let $X(t)$ be the content

of the biological group at time t, and the individuals be deathless and give birth to new individuals. Suppose $X(t)=i$. The probability that $X(t+\Delta t)=i+1$ is $\lambda_i\Delta t+o(\Delta t)$, the probability that $X(t+\Delta t)>i+1$ is $o(\Delta t)$, and the probability that $X(t+\Delta t)=i$ is $1-\lambda_i\Delta t+o(\Delta t)$. Such a denumerable Markov process $\{X(t), t\geqslant 0\}$ is a birth process. The transition matrix $P(t)$ with its Q matrix (10) may not be unique.

(iii) Birth–death processes. A denumerable Markov process $X=\{X(t), t\geqslant 0\}$ with Q matrix

$$Q=\begin{pmatrix} -(a_0+b_0) & b_0 & 0 & 0 & \cdots \\ a_1 & -(a_1+b_1) & b_1 & 0 & \cdots \\ 0 & a_2 & -(a_2+b_2) & b_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

is called a birth–death process. Here $a_0\geqslant 0$, $b_0>0$, $a_i>0$, $b_i>0\,(i=1,2,\ldots)$.

Let $X(t)$ be the content of the biological group whose individuals may give birth to new individuals and may die. Assume $X(t)=i$. For $i>0$ the probability that $X(t+\Delta t)=i-1$ is $a_i\Delta t+o(\Delta t)$. For $i\geqslant 0$ the probability that $X(t)=i+1$ is $b_i\Delta t+o(\Delta t)$, and the probability that $X(t)=i$ is $1-(a_i+b_i)\Delta t+o(\Delta t)$. The Markov process $X=\{X(t), t\geqslant 0\}$ is a birth–death process.

PART I GENERAL CONSTRUCTION THEORY

CHAPTER 2

Introduction to Construction Theory

2.1 INTRODUCTION

In this chapter the fundamental results of the construction theory are introduced. They are summarized as follows: the analytical properties of processes, such as continuity, existence of Q matrices, differentiability; the conditions under which the Kolmogorov equations are satisfied; the construction and properties of the minimal solution and the general form of the Q processes. The greater part of this chapter is derived from Reuter (1957). Sections 2.4 and 2.5 and Theorem 6.2 are obtained from Chung (1967) and the conclusions in section 2.8 from Reuter (1959, 1962) and Feller (1957a). Sections 2.10–2.12 are derived from Xiang-qun Yang (1981a).

2.2 NOTATION AND DEFINITIONS

Let E be a denumerable set of indices. We call it the state space. We denote by m the Banach space that is composed of the bounded column vectors (or bounded functions) on E. Let the norm of $f \in m$ be defined by $\|f\| = \sup_{i\in E}|f_i|$. We denote by l the Banach space that is composed of summable row vectors on E and let the norm of $g \in l$ be defined by $\|g\| = \sum_{j\in E}|g_i|$. If $f\in m, g\in l$, their inner product will be defined by

$$[g, f] = \sum_{i\in E} g_i f_i \tag{1}$$

Matrix notation will be used and the limit of matrices will be defined element by element. From the analytical point of view, a Markov process (abbreviated to process) is a family of real matrices $P(t) = \{p_{ij}(t)\}$ $(i, j\in E, t \geqslant 0)$ which satisfy the following conditions:

(A) $P(t) \geqslant 0 \qquad P(t)\mathbf{1} \leqslant \mathbf{1}$

(B) $P(t+s) = P(t)P(s)$

(C) $\lim\limits_{t\downarrow 0} P(t) = P(0) = \boldsymbol{I}$

where 0 represents the zero matrix and sometimes the zero column (or zero row) vector; **1** represents the unit column vector whose components are 1. $\boldsymbol{I}$ represents the unit matrix.

In terms of the elements of the matrix $P(t)$, conditions (A), (B) and (C) become: for arbitrary $i, j \in E$, $s, t \geqslant 0$, we have

(A) $p_{ij}(t) \geqslant 0 \qquad \sum_j p_{ij}(t) \leqslant 1$

(B) $p_{ij}(t+s) = \sum_k p_{ik}(t) p_{kj}(s)$

(C) $\lim_{t \downarrow 0} p_{ij}(t) = p_{ij}(0) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

Here the sums are taken over E. (B) is usually called the Chapman–Kolmogorov equation.

We call

$$d_i(t) = 1 - \sum_j p_{ij}(t) \tag{2}$$

the stopping function of the process $P(t)$.

Lemma 1. $d_i(t)$ is a non-decreasing function of t. If $d_i(t) = 0$ holds for some $t > 0$ and all $i \in E$, then $d_i(t) = 0$ holds for all $t \geqslant 0$ and $i \in E$.

Proof. For $s, t > 0$, by (B), (C),

$$\begin{aligned} d_i(s+t) &= 1 - \sum_j p_{ij}(s+t) \\ &= 1 - \sum_j \sum_k p_{ik}(s) p_{kj}(t) \\ &= 1 - \sum_k p_{ik}(s) \sum_j p_{kj}(t) \geqslant 1 - \sum_k p_{ik}(s) \\ &= d_i(s) \end{aligned}$$

From this we see that $d_i(t)$ is non-decreasing. From the above expression we find out that if $\sum_j p_{kj}(t) = 1$ for some $t > 0$ and all k, then $d_i(s+t) = 0$ $(s > 0)$. Since $d_i(t) = 0$, an easy induction gives $d_i(nt) = 0$ for all position integers n. For any $u \geqslant 0$, choose n so that $u \leqslant nt$, then $d_i(u) \leqslant d_i(nt) = 0$ because $d_i(\cdot)$ is non-decreasing, and so $d_i(u) = 0$. QED

When $d_i(t) = 0$ $(i \in E)$ for some (hence all) $t > 0$, namely,

(D) $P(t)\mathbf{1} = \mathbf{1} \qquad t > 0$

or equivalently

(D) $\sum_j p_{ij}(t) = 1 \qquad i \in E, t > 0$

then the process $P(t)$ is said to be an honest process; otherwise it is called a stopping process.

For a stopping process $P(t)$ we can get an honest process $\tilde{P}(t)$ by enlarging the state space as follows: take an arbitrary index $\Delta \notin E$, and write $\tilde{E} = E \cup \{\Delta\}$; set

$$\begin{gathered} \tilde{p}_{ij}(t) = p_{ij}(t) \qquad \tilde{p}_{i\Delta}(t) = d_i(t) \qquad i, j \in E \\ \tilde{p}_{\Delta j}(t) = \begin{cases} 1 & \text{if } j = \Delta \\ 0 & \text{if } j \in E \end{cases} \end{gathered} \tag{3}$$

We then have the following lemma.

Lemma 2. Let $P(t) = \{p_{ij}(t)\}$ $(i, j \in E, t \geqslant 0)$ be a stopping process. Then $\tilde{P}(t) = \{\tilde{p}_{ij}(t)\}$ $(i, j \in \tilde{E}, t \geqslant 0)$ will be an honest process.

The class of all processes $P(t)$ is denoted by $\mathscr{P}$.

It will be proved latter (Theorems 5.1 and 5.2) that for each $P(t) \in \mathscr{P}$, its right derivative $P'(0)$ at $t = 0$ exists, that is, the limits

$$q_{ij} = p'_{ij}(0) = \lim_{t \downarrow 0} \frac{p_{ij}(t) - p_{ij}(0)}{t} \tag{4}$$

exist, and

$$\begin{gathered} 0 \leqslant q_{ij} < \infty \qquad (i \neq j) \\ \sum_{j \neq i} q_{ij} \leqslant -q_{ii} \equiv q_i \leqslant \infty \end{gathered} \tag{5}$$

We call the matrix $Q = (q_{ij})$ $(i, j \in E)$ the Q matrix of the process $P(t)$.

When $q_i < \infty$, we say that the state i is stable. From now on we only study the process whose states are all stable, that is, the process has a finite Q matrix. We denote by $\mathscr{P}_s$ the class of such processes, that is, their Q matrices satisfy the following conditions:

$$\begin{gathered} 0 \leqslant q_{ij} < \infty \qquad (i \neq j) \\ q_i \equiv -q_{ii} < \infty \qquad d_i \equiv q_i - \sum_{j \neq i} q_{ij} \geqslant 0 \end{gathered} \tag{6}$$

In order to emphasize the relation (4) between $P(t) \in \mathscr{P}_s$ and the finite matrix Q, that is,

$$P'(0) = Q \tag{7}$$

we say that $P(t)$ is a Q process. It should be emphatically pointed out that in this book all the states of the Q process $P(t)$ are supposed to be stable. The class of all Q processes with the same Q matrix will be denoted by $\mathscr{P}_s(Q)$.

2.3 THE PROBLEM OF CONSTRUCTION

The problem of construction is a converse one. Except for Williams (1976), so far consideration of the problem of construction has been limited almost entirely to Q matrices satisfying equation (2.6).

Let Q be a fixed matrix satisfying (2.6). We call $d = (d_i, i \in E)$ a non-conservative column vector of Q. If $d_i = 0$, we call i a conservative state. We call

$$H = \{i \mid d_i > 0\} \tag{1}$$

the set of non-conservative states. If H is empty, we call Q itself conservative.

The formulation of the problem of construction is as follows. Suppose a fixed Q matrix satisfying (2.6) is given. Problem 1: is there a process $P(t)$ satisfying (2.7)? In other words, does there exist such a Q process? Problem 2: if there exists such a Q process, is it unique? Problem 3: if such a Q process is not unique, how can we construct all Q processes?

The problem of construction was first proposed by Kolmogorov (1931), and he was the first to write down the system of backward differential equations

$$\text{(KB)}\quad p'_{ij}(t) = \sum_k q_{ik} p_{kj}(t) \qquad i, j \in E, t \geqslant 0$$

and the system of forward differential equations

$$\text{(KF)}\quad p'_{ij}(t) = \sum_k p_{ik}(t) q_{kj} \qquad i, j \in E, t \geqslant 0$$

Feller (1940) proved that the Q process always exists. Moreover, he constructed the minimal Q process. So problem 1 is settled.

Doob (1945) proved that for a conservative Q either there is only one Q process, i.e. the minimal Q process, or there exist infinitely many Q processes. For a conservative Q Reuter (1957) found out the necessary and sufficient condition under which the Q process is unique. Therefore for a conservative Q matrix, problem 2 was solved. For a general Q, Zhen-ting Hou (1974) obtained the uniqueness criterion for Q processes, and thus problem 2 is solved completely.

As far as problem 3 is concerned, a complete solution seems still far away. There are two approaches to this problem at present. One is the analytical approach, as used in Reuter (1957, 1959, 1962, 1976), Feller (1940, 1945, 1956, 1957a, b, 1958, 1971), Williams (1964, 1966, 1976), Zhen-Zu Sun (1962), Xiang-qun Yang (1964b, 1965a, 1966a) and Di-he Hu (1965, 1966, 1983). They mainly make use of analytical tools and methods to find solutions satisfying Kolmogorov's backward or forward equations, which are Q processes, or to get the infinitesimal operators of the contraction semigroups derived from the Q processes, or to find the resolvent operators of the Q processes. The other approach is the probability method, i.e. the limit transition method. This method was given by Zi-kun Wang (1958), resulting in a very successful solution to the construction problem for birth–death processes (Zi-kun Wang, 1962; Zi-kun Wang and Xiang-qun Yang 1978, 1979). The basic idea of this method is to approach the sample paths of a Q process by those of Doob processes which have simpler structures. That is, the sample paths of a Q process are the strong limits of the sample paths of a sequence of Doob processes. This method was latter improved by Zhen-ting Hou (1975) and Xiang-qun Yang (1978, 1979, 1980a, b, c).

The solution to problem 3 by means of the analytical approach goes as follows. Suppose that Q is conservative. Reuter (1959) and Zhen-zu Sun (1962) constructed all Q processes in the case that Q is single exit. Feller (1957a), in the case of finite exit and finite entrance, derived all Q processes that satisfy simultaneously Kolmogorov's backward and forward equations. Xiang-qun Yang (1966a), with the same hypotheses as Feller's, obtained all Q processes. Williams (1964, 1966) and Chung (1963, 1966), with the finite exit hypothesis, found all Q processes.

Since each Q process must satisfy Kolmogorov's backward equations when Q is conservative but need not satisfy the backward equations when Q is non-conservative, it follows that few books or articles are concerned with the problem of the construction of a general Q.

As for the birth–death matrix Q, which is conservative except perhaps at the state 0, Feller (1957b, 1971) got all birth–death processes that satisfy both Kolmogorov's backward equations and forward equations. Xiang-qun Yang (1965a) has constructed all birth–death processes, that is, the birth–death processes that satisfy one of the two systems of equations, or niether. In Chapter 4, for general Q in the case when Q is single exit, we have constructed all the Q processes that satisfy the system of backward equations, and in the case when Q is single entrance we have constructed all the Q processes that satisfy the forward equations. Especially, we have derived all Q processes in the case of null exit when there exists only one non-conservative state. In Chapter 8, we find all the Q processes when Q is finite exit and finite non-conservative.

It can be seen in the bibliography of this work that much work on the subject of the construction theory has been carried out by Chinese writers.

2.4 CONTINUITY

Theorem 1. If $P(t) \in \mathscr{P}$, then for arbitrary $i, j \in E$ and $h > 0$,

$$|p_{ij}(t \pm h) - p_{ij}(t)| \leqslant 1 - p_{ii}(h) \tag{1}$$

$$|d_i(t \pm h) - d_i(t)| \leqslant 1 - p_{ii}(h) \tag{2}$$

and, furthermore, the stopping function $d_i(t)$ and $p_{ij}(t)$ are uniformly continuous in $[0, \infty)$.

Proof. It suffices to prove the theorem in the case of the plus sign. By condition

(B) in section 2.2[1],

$$p_{ij}(t+h)-p_{ij}(t)=\sum_k p_{ik}(h)p_{kj}(t)-p_{ij}(t)$$
$$=[p_{ii}(h)-1]p_{ij}(t)+\sum_{k\neq i} p_{ik}(h)p_{kj}(t) \tag{3}$$

The first term is non-positive and $\geqslant -(1-p_{ii}(h))$; the second term is non-negative, and $\leqslant \sum_{k\neq i}p_{ik}(h)\leqslant 1-p_{ii}(h)$; hence (1) follows. Extend $P(t)$ to $\tilde{P}(t)$ according to (2.3) and apply (1) to $\tilde{p}_{i\Delta}(t)$, and we obtain (2). The proof is terminated. QED

Theorem 2. The series $\sum_j p_{ij}(t)$ is uniformly convergent in any finite interval $[0,b]$.

Proof. By Theorem 1, both $p_{ij}(t)$ and $1-d_i(t)$ are continuous functions; hence it suffices to quote the Dini theorem (Titchmarsh, 1939). QED

Theorem 3. Let $h>0$. The sum

$$\sum_j |p_{ij}(t+h)-p_{ij}(t)| \tag{4}$$

is non-increasing when t increases and is uniformly convergent to zero for $t\geqslant\delta>0$ as $h\to 0$. In particular, for every $\delta>0$, $p_{ij}(t)$ is uniformly continuous in $[\delta,\infty]$.

Proof. Let $0\leqslant s<t$. By conditions (A) and (B) is section 2.2[1],

$$\sum_j |p_{ij}(t+h)-p_{ij}(t)|=\sum_j\left|\sum_k[p_{ik}(s+h)-p_{ik}(s)]p_{kj}(t-s)\right|$$
$$\leqslant\sum_k|p_{ik}(s+h)-p_{ik}(s)|\sum_j p_{kj}(t-s)$$
$$\leqslant\sum_k|p_{ik}(s+h)-p_{ik}(s)| \tag{5}$$

Thus, we have proved the first conclusion. Then on account of Theorem 1, $p_{ij}(t)$ is continuous; hence the above expression can be integrated for $s\in[0,\delta]$. So if $t\geqslant\delta>0$,

$$\sum_j|p_{ij}(t+h)-p_{ij}(t)|\leqslant\sum_k(1/\delta)\int_0^\delta|p_{ik}(s+h)-p_{ik}(s)|\,ds$$

But if $0\leqslant h\leqslant\delta$, the second series is dominated by the series

$$\sum_k(2/\delta)\int_0^{2\delta}p_{ik}(s)\,ds$$

[1]Shortened in the following to (2.A) and (2.B) etc.

and, therefore, uniformly continuous in $h\in[0,\delta]$. But according to a theorem in Titchmarsh (1939), for every k

$$\lim_{h\to 0}\int_0^\delta|p_{ik}(s+h)-p_{ik}(s)|\,ds=0 \tag{6}$$

It follows that uniformly for $t\geqslant\delta$,

$$\overline{\lim_{h\to 0}}\sum_j|p_{ij}(t+h)-p_{ij}(t)|\leqslant\sum_k(1/\delta)\lim_{h\to 0}\int_0^\delta|p_{ik}(s+h)-p_{ik}(s)|\,ds$$
$$=\sum_k(1/\delta)0=0$$

The proof is complete. QED

Theorem 4. We have $p_{ii}(t)>0$ in $(0,\infty)$; for $i\neq j$, we have $p_{ij}(t)=0$ or $p_{ij}(t)>0$ in $(0,\infty)$.

Proof. (i) By (2.B), for arbitrary $t>0$, we have

$$p_{ii}(t)\geqslant[p_{ii}(t/n)]^n \qquad \text{for all } n \tag{7}$$

From (2.C) it follows that $p_{ii}(t)>0$.

(ii) Suppose $i\neq j$. By (2.B)

$$p_{ij}(t+s)\geqslant p_{ii}(s)p_{ij}(t) \tag{8}$$

Hence if we have $p_{ij}(t_1)>0$ for some $t_1>0$, then $p_{ij}(t)>0$ for all $t\geqslant t_1$.

(iii) Suppose that $t_0>0$ exists so that

$$p_{il}(t)=0 \qquad (0<t\leqslant t_0)$$
$$p_{il}(2t_0)=c>0 \tag{9}$$

To simplify things, we may suppose that E is the set of all non-negative integers. By Theorem 2 there exists a positive integer N so that

$$\sum_{j>N}p_{ij}(t)<c/4 \qquad 0<t\leqslant 2t_0 \tag{10}$$

Set $s=t_0/(2N)$, and define

$$A_m=\{k\,|\,p_{ik}(ms)>0\} \qquad m\geqslant 1 \tag{11}$$

By (ii), $A_m\subset A_{m+1}$. Write $B_1=A_1$, $B_m=A_m-A_{m-1}$ $(m\geqslant 2)$. If $k\notin A_n$, then

$$0=p_{ik}(ms)=\sum_j p_{ij}[(m-1)s]p_{jk}(s)$$
$$=\sum_{j\in A_{m-1}}p_{ij}[(m-1)s]p_{jk}(s) \tag{12}$$

Therefore,

$$p_{jk}(s)=0 \qquad j\in A_{m-1},k\notin A_m \tag{13}$$

If for some m, $1 < m \leqslant 2N$, we have $A_m = A_{m-1}$, then by the above expression we obtain

$$p_{ik}[(m+1)s] = \sum_{j \in A_m} p_{ij}(ms) p_{jk}(s) = 0 \qquad k \notin A_m$$

Hence $A_{m+1} = A_m$. Repeating this proof we get $A_n = A_m$ $(n > m)$ and in particular we have $A_{2N} = A_{4N}$. But by (9), $l \notin A_{2N}$ and $l \in A_{4N}$. This contradiction shows that all B_m $(1 \leqslant m \leqslant 2N)$ are non-empty and mutually disjoint.

Let $1 \leqslant m \leqslant 2N$, then $A_m \subset A_{2N}$. If $k \notin A_m$, then by (13), for every $n \geqslant 1$, we have

$$p_{ik}[(n+1)s] = \left(\sum_{j \notin A_m} + \sum_{j \in B_m} + \sum_{j \in A_{m-1}} \right) p_{ij}(ns) p_{jk}(s)$$

By (13) the third sum is zero, so

$$\sum_{j \notin A_m} p_{ik}[(n+1)s] \leqslant \sum_{j \notin A_m} p_{ij}(ns) + \sum_{j \in B_m} p_{ij}(ns)$$

Summing n from 1 to $4N-1$, we have

$$\sum_{k \notin A_m} p_{ik}(4Ns) \leqslant \sum_{n=1}^{4N} \sum_{j \in B_m} p_{ij}(ns)$$

Because $l \notin A_{2N}$ the left-hand side at least equals $p_{il}(4NS) = c$, and therefore

$$c \leqslant \sum_{n=1}^{4N} \sum_{j \in B_m} p_{ij}(ns) \qquad 1 \leqslant m \leqslant 2N \tag{14}$$

Since $B_1, B_2, \ldots, B_{2N}$ are non-empty and mutually disjoint, there exists at least N of B_N $(1 \leqslant n \leqslant 2N)$, whose union is denoted by B, which is disjoint from the set $(1, 2, \ldots, N)$. Thus

$$Nc \leqslant \sum_{n=1}^{4N} \sum_{j \in B} p_{ij}(ns) \tag{15}$$

On the other hand, we have by (10)

$$\sum_{j \in B} p_{ij}(ns) \leqslant \sum_{j > N} p_{ij}(ns) < c/4$$

Consequently the right-hand side of (15) is strictly less than $4N(c/4) = Nc$. This contradiction implies that (9) cannot hold. The proof is concluded. QED

Corollary

For any fixed i, $d_i(t)$ is identically zero in $(0, \infty)$ or never zero.

Proof. If suffices to combine Lemma 2.2 with Theorem 4.4

2.5 EXISTENCE OF Q MATRICES

Theorem 1. Let $P(t) \in \mathscr{P}$, then for each $i \in E$

$$-p'_{ii}(0) = \lim_{t \downarrow 0} \frac{1 - p_{ii}(t)}{t} \tag{1}$$

exists, but may be infinite.

Proof. Put

$$\Phi(t) = -\ln p_{ii}(t) \tag{2}$$

Then $\Phi(t)$ is non-negative and finite by Theorem 5.4. By (2.A) and (2.B) we have

$$p_{ii}(s+t) \geqslant p_{ii}(s) p_{ii}(t) \tag{3}$$

From the foregoing expression it follows that

$$\Phi(s+t) \leqslant \Phi(s) + \Phi(t) \tag{4}$$

Denote

$$q_i \equiv \sup_{t > 0} [\Phi(t)/t] \tag{5}$$

If $q_i < \infty$, then for arbitrary $\varepsilon > 0$, there exists $t_0 > 0$ so that $\Phi(t_0)/t_0 > q_i - \varepsilon$. But for every $t > 0$, we can write $t_0 = nt + \delta$ $(0 \leqslant \delta < t)$, so

$$q_i - \varepsilon \leqslant \frac{\Phi(t_0)}{t_0} \leqslant \frac{n\Phi(t) + \Phi(\delta)}{t_0} = \frac{nt}{t_0} \frac{\Phi(t)}{t} + \frac{\Phi(\delta)}{t_0}$$

$nt/t_0 \to 1$ when $t \to 0$. By (2.C) we have $\Phi(\delta) \to 0$. It follows that

$$q_i - \varepsilon \leqslant \varliminf_{t \downarrow 0} \frac{\Phi(t)}{t} \leqslant \varlimsup_{t \downarrow 0} \frac{\Phi(t)}{t} \leqslant q_i$$

Since ε is arbitrary, $\lim_{t \downarrow 0} \Phi(t)/t = q_i$. If $q_i = \infty$, upon replacing $q_i - \varepsilon$ by the arbitrarily large positive number M, we still get $\lim_{t \downarrow 0} \Phi(t)/t = \infty$. However,

$$\lim_{t \downarrow 0} \frac{\Phi(t)}{t} = \lim_{t \downarrow 0} \frac{-\ln\{1 - [1 - p_{ii}(t)]\}}{t}$$

$$= \lim_{t \downarrow 0} \frac{1 - p_{ii}(t)}{t}$$

Thus, the theorem is proved. QED

Theorem 2. For arbitrary $i \neq j$,

$$p'_{ij}(0) = \lim_{t \downarrow 0} \frac{p_{ij}(t)}{t} \tag{6}$$

exists and is non-negative but finite.

Proof. Let $i \neq j$ and $h > 0$. Define ${}_jp_{ii}^0(h) = 1$

$$ {}_jp_{ii}^n(h) = \sum p_{ik_1}(h)p_{k_1k_2}(h)\cdots p_{k_{n-1}i}(h) \tag{7}$$

$$ f_{ij}^n(h) = \sum p_{ik_1}(h)p_{k_1k_2}(h)\cdots p_{k_{n-1}j}(h) \tag{8}$$

where the sum is taken over those $k_1 \neq j, k_2 \neq j, \ldots, k_{n-1} \neq j$. Although we define ${}_jp_{ii}^n(h)$ and $f_{ij}^n(h)$ by analytical expressions, in actual fact they are respectively the probability of going from i to i in n steps without visiting j and the probability of the first visit to j from i in n steps for a Markov chain whose one-step transition probability matrix is $\{p_{ij}(h)\}$.

From (2.A) and (2.B), we can deduce that

$$ p_{ij}(nh) \geqslant \sum_{m=0}^{n-1} {}_jp_{ii}^m(h)p_{ij}(h)p_{jj}[(n-m-1)h] \tag{9}$$

$$ {}_jp_{ii}(mh) = {}_jp_{ii}^m(h) + \sum_{a=1}^{m-1} f_{ij}^a(h)p_{ji}[(m-a)h] \tag{10}$$

$$ p_{ij}(mh) = \sum_{a=1}^{m} f_{ij}^a(h)p_{jj}[(m-a)h] \tag{11}$$

From the probability point of view, the above relations are even clearer. By (2.C), for $\varepsilon < \frac{1}{2}$, there exists $t_0 > 0$ so that

$$ \max_{0 \leqslant t \leqslant t_0} p_{ij}(t) < \varepsilon \qquad \max_{0 \leqslant t \leqslant t_0} p_{ji}(t) < \varepsilon $$

$$ \min_{0 \leqslant t \leqslant t_0} p_{ii}(t) > 1 - \varepsilon \qquad \min_{0 \leqslant t \leqslant t_0} p_{jj}(t) > 1 - \varepsilon \tag{12}$$

From this and (11), we obtain

$$ \sum_{a=1}^{m} f_{ij}^a(h)(1-\varepsilon) < \varepsilon $$

thus

$$ \sum_{a=1}^{m} f_{ij}^a(h) \leqslant 1 \tag{13}$$

Hence, by (10), we get

$$ {}_jp_{ii}^m(h) \geqslant p_{ii}(mh) - \max_{1 \leqslant a \leqslant m} p_{ji}[(m-a)h] $$

From this, if $nh < t_0$, then ${}_jp_{ii}^m(h) > 1 - 2\varepsilon$. So by (9), we find

$$ p_{ij}(nh) > (1-2\varepsilon)\sum_{m=0}^{n-1} p_{ij}(h)(1-\varepsilon) \geqslant (1-3\varepsilon)np_{ij}(h) $$

$$ \frac{p_{ij}(nh)}{nh} > (1-3\varepsilon)\frac{p_{ij}(h)}{h} \qquad \text{if } nh < t_0 \tag{14}$$

Set

$$ q_{ij} = \lim_{t \downarrow 0} \frac{p_{ij}(t)}{t} $$

Then we have $q_{ij} < \infty$ by (14), and moreover, there exists $t_1, 0 < t_1 < t_0/2$, so that

$$ p_{ij}(t_1)/t_1 < q_{ij} + \varepsilon $$

The left-hand side is continuous in t_1; hence, there exists $h_0 > 0$ so that

$$ p_{ij}(t)/t < q_{ij} + 2\varepsilon \qquad |t - t_1| < h_0 \tag{15}$$

For arbitrary $h \in (0, \min(h_0, t_1))$, there exists n so that $t_1 \leqslant nh < t_1 + h < t_0$. By (14) and (15), we get

$$ (1-3\varepsilon)\frac{p_{ij}(h)}{h} < \frac{p_{ij}(nh)}{nh} < q_{ij} + 2\varepsilon $$

Since ε is arbitrary, we obtain $\lim_{t \downarrow 0} p_{ij}(t)/t \leqslant q_{ij}$. QED

Corollary

For all $i \in E$, the right derivative

$$ D_i \equiv d_i'(0) = \lim_{t \downarrow 0} \frac{d_i(t)}{t} \geqslant 0 \tag{16}$$

exists and is finite.

Proof. By Lemma 2.2, to get (16) it suffices to apply Theorem 2 to $\tilde{p}_{i\Delta}(t) = d_i(t)$. QED

Theorem 3. For arbitrary $i \in E$, we have

$$ \sum_{j \neq i} p_{ij}'(0) + d_i'(0) \leqslant -p_{ii}'(0) \tag{17}$$

Proof. Because

$$ \sum_{j \neq i} \frac{p_{ij}(t)}{t} + \frac{d_i(t)}{t} = \frac{1 - p_{ii}(t)}{t} $$

(17) follows by using the Fatou lemma. QED

2.6 DIFFERENTIABILITY

Let $P(t) \in \mathscr{P}$. $Q = (q_{ij}) = \{p_{ij}'(0)\}$ is its Q matrix, and $q_i = -q_{ii}$, $D_i = d_i'(0)$.

Theorem 1. If $q_i < \infty$[1] for fixed i, then $p_{ij}(t)$ (for all $j \in E$) and $d_i(t)$ have finite and continuous derivatives in $[0, \infty)$. Moreover

$$|p_{ij}(t+h) - p_{ij}(t)| \leqslant q_i h \qquad t \geqslant 0, h \geqslant 0 \tag{1}$$

$$|d_i(t+h) - d_i(t)| \leqslant q_i h \qquad t \geqslant 0, h \geqslant 0 \tag{2}$$

$$\sum_j |p'_{ij}(t)| + d'_i(t) \leqslant 2q_i \qquad t > 0 \tag{3}$$

$$\sum_j p'_{ij}(t) + d'_i(t) = 0 \qquad t > 0 \tag{4}$$

$$p'_{ij}(t_1 + t_2) = \sum_k p'_{ik}(t_1) p_{kj}(t_2) \qquad t_1 > 0, t_2 \geqslant 0 \tag{5}$$

Proof. (i) By (4.7) and the definition of q_i, we have

$$p_{ii}(t) \geqslant e^{-q_i t} \geqslant 1 - q_i t \tag{6}$$

From this, (4.1) and (4.2), (1) and (2) follow.

Inequalities (1) and (2) show that $p_{ij}(t)$ and $d_i(t)$ satisfy Lipschitz's condition; hence, they are absolutely continuous. It follows that $p'_{ij}(t)$ and $d'_i(t)$ exist for almost all $t \geqslant 0$. Furthermore,

$$p_{ij}(t) = \delta_{ij} + \int_0^t p'_{ij}(u)\,du \tag{7}$$

$$d_i(t) = \int_0^t d'_i(u)\,du \tag{8}$$

(ii) Set

$$\Delta_{ij}(t, t+s) = \frac{p_{ij}(t+s) - p_{ij}(t)}{s} \qquad t \geqslant 0, s > 0 \tag{9}$$

$$\Delta_i(t, t+s) = \frac{d_i(t+s) - d_i(t)}{s} \qquad t \geqslant 0, s > 0 \tag{10}$$

Since $\sum_j p_{ij}(t) + d_i(t) = 1$ and $d_i(t)$ are non-decreasing, we have

$$\sum_j \Delta_{ij}(t, t+s) + \Delta_i(t, t+s) = 0 \tag{11}$$

$$\Delta_i(t, t+s) \geqslant 0 \tag{12}$$

[1] When $q_i = \infty$, it follows also that $p_{ij}(t)$ and $d_i(t)$ have finite and continuous derivatives in $(0, \infty)$, and (5) holds for $t_1 > 0$, $t_2 \geqslant 0$. See Zi-kun Wang (1980, §2.2, Theorem 2) or Chung (1967a, II.12, Theorem 8).

By (4.8) and (6), it follows that

$$\Delta_{ij}(t, t+s) \geqslant -\frac{1 - p_{ii}(s)}{s} p_{ij}(t) \geqslant -q_i p_{ij}(t) \tag{13}$$

Hence for arbitrary set $A \subset E$, we have

$$\sum_{j \in A} \Delta_{ij}(t, t+s) \geqslant -q_i \sum_{j \in A} p_{ij}(t) \geqslant -q_i \tag{14}$$

Starting from this and noticing (12), by (11), we get

$$\sum_{j \in A} \Delta_{ij}(t, t+s) = -\sum_{j \in E-A} \Delta_{ij}(t, t+s) - \Delta_i(t, t+s) \leqslant q_i \tag{15}$$

So for arbitrary $A \subset E$,

$$\left| \sum_{j \in A} \Delta_{ij}(t, t+s) \right| \leqslant q_i \tag{16}$$

Moreover

$$0 \leqslant \Delta_i(t, t+s) = -\sum_j \Delta_{ij}(t, t+s) \leqslant q_i \tag{17}$$

Take $A = \{j | \Delta_{ij} \geqslant 0\}$ in (14) and (15). We obtain

$$\sum_j |\Delta_{ij}| = \sum_{j \in A} \Delta_{ij} + \sum_{j \in E-A} (-\Delta_{ij}) \leqslant 2q_i \tag{18}$$

From this it follows that

$$\sum_j |p'_{ij}(t)| \leqslant 2q_i \tag{19}$$

so long as the derivatives involved exist. In particular the foregoing equation holds for almost all $t \geqslant 0$.

Sum (7) over j and add it to (8); it follows by (19) that the order of summation and integration can be interchanged, therefore

$$\sum_j p_{ij}(t) + d_i(t) = 1 + \sum_j \int_0^t p'_{ij}(u)\,du + \int_0^t d'_i(u)\,du$$
$$= 1 + \int_0^t \left(\sum_j p'_{ij}(u) + d'_i(u) \right) du$$

and it follows that

$$\int_0^t \left(\sum_j p'_{ij}(u) + d'_i(u) \right) du = 0 \qquad t \geqslant 0$$

Hence for almost all $t \geqslant 0$, we have

$$\sum_j p'_{ij}(t) + d'_i(t) = 0 \tag{20}$$

(iii) Suppose that $p'_{ij}(t)$ and $d'_i(t)$ exist for some t and (20) holds. We prove

$$\Sigma(s) \equiv \sum_j |\Delta_{ij}(t, t \pm s) - p'_{ij}(t)| \to 0 \qquad s \downarrow 0 \tag{21}$$

First we prove the case for the plus sign. Given arbitrary $\varepsilon > 0$, by (19) we can choose a finite set $A \subset E$ such that

$$q_i \sum_{j \notin A} p_{ij}(t) + \sum_{j \notin A} |p'_{ij}(t)| < \varepsilon \tag{22}$$

Hence,

$$\Sigma(s) \leqslant \sum_{j \in A} |\Delta_{ij}(t, t+s) - p'_{ij}(t)| + \sum_{j \notin A} |\Delta_{ij}(t, t+s)| + \varepsilon$$

Use $\sum'$ to denote the summation of index j for $\Delta_{ij} < 0$. Then by (14) and (22), we have

$$\begin{aligned}\sum_{j \notin A} |\Delta_{ij}| &= \sum_{j \notin A} \Delta_{ij} - 2 \sum_{j \notin A}{}' \Delta_{ij} \\ &\leqslant \sum_{j \notin A} \Delta_{ij} + 2q_i \sum_{j \notin A} p_{ij}(t) \\ &\leqslant \sum_{j \notin A} \Delta_{ij} + 2\varepsilon \end{aligned} \tag{23}$$

Hence by (11) we have

$$\Sigma(s) \leqslant \sum_{j \notin A} |\Delta_{ij}(t, t+s) - p'_{ij}(t)| - \sum_{j \in A} \Delta_{ij} - \Delta_i + 3\varepsilon$$

By (20) and (22), we obtain

$$\begin{aligned}\overline{\lim_{s \downarrow 0}}\, \Sigma(s) &\leqslant 0 - \sum_{j \in A} p'_{ij}(t) - d'_i(t) + 3\varepsilon \\ &= \sum_{j \notin A} p'_{ij}(t) + 3\varepsilon < 4\varepsilon \end{aligned}$$

Thus we have proved (21) where the plus sign appears.

In the case of the minus sign the above proof is still valid after a few revisions. Inequality (22) will be replaced by

$$q_i \sum_{j \notin A} p_{ij}(t-s) + \sum_{j \notin A} |p'_{ij}(t)| < \varepsilon \qquad (s \leqslant t) \tag{24}$$

The above expression holds by Theorem 4.2. Inequality (23) still holds after replacing $p_{ij}(t)$ by $p_{ij}(t-s)$.

(iv) Suppose that $p'_{ij}(t)$ and $d'_i(t)$ exist for some $t > 0$ and (20) holds. Then for arbitrary $u > t$, by (21)

$$\begin{aligned}\sum_j \left| \Delta_{ij}(u, u+s) - \sum_k p'_{ik}(t) p_{kj}(u-t) \right| &= \sum_j \left| \sum_k [\Delta_{ik}(t, t+s) - p'_{ik}(t)] p_{kj}(u-t) \right| \\ &\leqslant \sum_k |\Delta_{ik}(t, t+s) - p'_{ik}(t)| \to 0 \qquad s \downarrow 0 \end{aligned} \tag{25}$$

From this, it follows that

$$\left| \Delta_i(u, u+s) - \sum_j \left(-\sum_k p'_{ik}(t) p_{kj}(u-t) \right) \right| \leqslant \sum_k |\Delta_{ik}(t, t+s) - p'_{ik}(t)| \to 0 \qquad s \downarrow 0$$

Hence the right derivatives of $p_{ij}(u)$ and $d_i(u)$ exist. It is analogous to prove that the left derivatives also exist. Hence $p'_{ij}(u)$ and $d'_i(u)$ exist in (t, ∞). Moreover

$$p'_{ij}(u) = \sum_k p'_{ik}(t) p_{kj}(u-t) \tag{26}$$

$$d'_i(u) = -\sum_k p'_{ik}(u) \tag{27}$$

From (19) and the two above equations it follows that $p'_{ij}(u)$ and $d'_i(u)$ are continuous functions for $u \in (t, \infty)$. Since by (i) it follows that the above t can be arbitrarily small, hence $p'_{ij}(t)$ and $d'_i(t)$ exist and are continuous functions in $(0, \infty)$; moreover, (4) and (5) hold. Because (19) holds for all $t > 0$, according to Lemma 2.2 and by applying conclusion (19) to $\tilde{P}(t)$, we obtain (3).

(v) We are going to prove

$$\lim_{t \downarrow 0} p'_{ij}(t) = p'_{ij}(0) \qquad \lim_{t \downarrow 0} d'_i(t) = d'_i(0) \tag{28}$$

We only need to prove the first statement, because according to Lemma 2.2 we can get the second statement by applying the first statement to $\tilde{P}(t)$.

By (6)

$$p_{ij}(t+h) - p_{ij}(t) \geqslant [p_{ii}(h) - 1] p_{ij}(t) \geqslant -q_i h p_{ij}(t) \tag{29}$$

Therefore

$$R_{ij}(t) \equiv p'_{ij}(t) + q_i p_{ij}(t) \geqslant 0 \tag{30}$$

Moreover, by (5) and (2.B), we have

$$R_{ij}(t+s) = \sum_k R_{ik}(t) p_{kj}(s) \qquad s > 0, t > 0 \tag{31}$$

Hence

$$R_{ij}(t+s) \geqslant R_{ij}(t) p_{jj}(s)$$

Since $p'_{ij}(t)$ is continuous in $(0, \infty)$, we obtain

$$R_{ij}(s) \geqslant \overline{\lim_{t \downarrow 0}}\, R_{ij}(t) p_{jj}(s) \qquad \underline{\lim_{t \downarrow 0}}\, R_{ij}(s) \geqslant \overline{\lim_{t \downarrow 0}}\, R_{ij}(t)$$

Hence $\lim_{t \downarrow 0} R_{ij}(t)$ exists, i.e. $\lim_{t \downarrow 0} p'_{ij}(t)$ exists. And by the mean value theorem of differential calculus, we have

$$\lim_{t \downarrow 0} p'_{ij}(t) = p'_{ij}(0)$$

and the proof is complete. QED

Corollary

If $q_i < \infty$, then for arbitrary $\delta > 0$

$$\lim_{s \to 0} \sum_j \left| \frac{p_{ij}(u+s) - p_{ij}(u)}{s} - p'_{ij}(u) \right| = 0 \tag{32}$$

holds uniformly for $u \geqslant \delta$.

Proof. The conclusion follows from (25) and (5). QED

Theorem 2. Suppose $q_j < \infty$ for fixed j, then for all $i \in E$ there exist finite and continuous derivatives of $p_{ij}(t)$ in $[0, \infty)$. Moreover

$$p'_{ij}(s+t) = \sum_k p_{ik}(t) p'_{kj}(s) \qquad t \geqslant 0, s > 0 \tag{33}$$

Proof. By (4.8) and (6)

$$p_{ij}(t+h) - p_{ij}(t) \geqslant p_{ij}(t)[p_{jj}(h) - 1] \geqslant -p_{ij}(t) q_j h$$

Thus

$$D[p_{ij}(t) e^{q_j t}] = [D p_{ij}(t) + p_{ij}(t) q_j] e^{q_j t} \geqslant 0$$

where D represents the right lower derivative. Therefore $p_{ij}(t) e^{q_j t}$ is non-decreasing when t increases and $D p_{ij}(t)$ are almost everywhere finite.

Set

$$v_{ij}(t) = D p_{ij}(t) + p_{ij}(t) q_j \geqslant 0 \tag{34}$$

Rewriting (2.B) we have

$$p_{ij}(s+t) e^{q_j(t+s)} = e^{q_j t} \sum_k p_{ik}(t) p_{kj}(s) e^{q_j s}$$

By differentiating the above equation for s and using Fubini's theorem on differentiation[1] it follows that for every $t \geqslant 0$ and almost all s we have

$$v_{ij}(s+t) = \sum_k p_{ik}(t) v_{kj}(s) \tag{35}$$

If we use Fatou's lemma, then for all $s \geqslant 0$ and $t \geqslant 0$ we have

$$v_{ij}(s+t) \geqslant \sum_k p_{ik}(t) v_{kj}(s) \tag{36}$$

In particular, for almost all t_0 and all $s \leqslant t_0$,

$$\infty > v_{ij}(t_0) \geqslant p_{ii}(t_0 - s) v_{ij}(s)$$

holds. So $v_{ij}(s)$ is bounded in any finite interval. By Fubini's theorem it follows

[1] See Saks (1937; p. 117)

that (35) holds for $s \notin Z$ and $t \notin Z_s$, where the measures of Z and Z_s are zero. For some $s_0 \notin Z$, suppose that

$$v_{ij}(t + s_0) > \sum_k p_{ik}(t) v_{ki}(s_0) \tag{37}$$

holds for some t, then for $t' > t$ we have

$$\begin{aligned} v_{ij}(t' + s_0) &\geqslant \sum_l p_{il}(t' - t) v_{lj}(t + s_0) \\ &> \sum_l p_{il}(t' - t) \sum_k p_{lk}(t) v_{kj}(s_0) \\ &= \sum_k p_{ik}(t') v_{kj}(s_0) \end{aligned}$$

The above inequality cannot hold, because (35) holds for almost all t when $s_0 \notin Z$. Thus Z_s is empty when $s \notin Z$. Also let $s > 0$ be arbitrary, $0 < s' < s$ and $s' \notin Z$. Then

$$\begin{aligned} v_{ij}(t+s) &= v_{ij}(t + s - s' + s') \\ &= \sum_k p_{ik}(t + s - s') v_{kj}(s') \\ &= \sum_k \sum_l p_{il}(t) p_{lk}(s - s') v_{kj}(s') = \sum_l p_{il}(t) v_{lj}(s) \end{aligned}$$

and it follows that Z is also empty. Hence (35) holds for all $t \geqslant 0$ and $s > 0$.

Let $t \geqslant 0$ and moreover let $t_n \downarrow 0$, $t'_n \downarrow 0$ so that $v_{ij}(t + t_n) \to a_{ij}$, $v_{ij}(t + t'_n) \to a'_{ij}$. We might as well suppose $t'_n < t_n$. By (35),

$$v_{ij}(t + t_n) \geqslant p_{ii}(t_n - t'_n) v_{ij}(t + t'_n)$$

Hence $a_{ij} \geqslant a'_{ij}$. By symmetry it follows that $a_{ij} = a'_{ij}$. That is, $v_{ij}(t+0)$ $(t \geqslant 0)$ exists. Similarly, we can prove that $v_{ij}(t-0)$ $(t > 0)$ exists. Moreover, from $v_{ij}(t + t_n) \geqslant p_{ii}(t_n) v_{ij}(t)$ it follows that $v_{ij}(t+0) \geqslant v_{ij}(t)$ $(t \geqslant 0)$. From $v_{ij}(t) \geqslant p_{ii}(t_n) v_{ij}(t - t_n)$ it follows that $v_{ij}(t) \geqslant v_{ij}(t-0)$ $(t > 0)$. Therefore, $v_{ij}(t+0) \geqslant v_{ij}(t) \geqslant v_{ij}(t-0)$. For $t > 0$ we can take $0 < s < t$, since

$$v_{ij}(t - t_n) = \sum_k p_{ik}(t - s - t_n) v_{kj}(s)$$

$$v_{ij}(t - 0) \geqslant \sum_k p_{ik}(t - s) v_{kj}(s) = v_{ij}(t)$$

Thus $v_{ij}(t)$ is continuous in $(0, \infty)$. Hence $p_{ij}(t)$ has the continuous Dini's derivative $D p_{ij}(t)$ in $(0, \infty)$, and in fact there exists the continuous derivative $p'_{ij}(t)$ in $(0, \infty)$ (Saks 1937, p. 204). As just stated, $v_{ij}(0+)$ exists, namely, $p'_{ij}(0+) = \lim_{t \downarrow 0} p'_{ij}(t)$ exists. By the mean value theorem we get $p'_{ij}(0+) = p'_{ij}(0)$. So $p'_{ij}(t)$ is continuous in $[0, \infty)$, and (35), which holds for all $t \geqslant 0$ and $s > 0$, becomes (33). The proof is complete. QED

2.7 THE KOLMOGOROV EQUATIONS

Let $P(t)\in\mathscr{P}$. Notice that (6.5) and (6.33) do not necessarily hold for $t_1=0$ and $s=0$ respectively. That is, when $q_i<\infty$,

$$(\mathrm{KB}_i)\quad p'_{ij}(t)=\sum_k q_{ik}p_{kj}(t)\qquad t\geqslant 0, j\in E$$

does not necessarily hold, and when $q_j<\infty$,

$$(\mathrm{KF}_j)\quad p'_{ij}(t)=\sum_k p_{ik}(t)q_{kj}\qquad t\geqslant 0, i\in E$$

does not necessarily hold. But from

$$\Delta_{ij}(t,t+s)=\sum_k \Delta_{ik}(0,s)p_{kj}(t)=\sum_k p_{ik}(t)\Delta_{kj}(0,s)$$

and Fatou's lemma it follows that when $q_i<\infty$

$$p'_{ij}(t)\geqslant\sum_k q_{ik}p_{kj}(t)\qquad t\geqslant 0, j\in E \tag{1}$$

holds and when $q_j<\infty$

$$p'_{ij}(t)\geqslant\sum_k p_{ik}(t)q_{kj}\qquad t\geqslant 0, i\in E \tag{2}$$

holds. We call (1) and (2) backward inequalities and forward inequalities respectively. We call the system of equations (KB_i) $(i\in E)$, i.e. (3.KB), the system of backward equations; and (KF_j) $(j\in E)$, i.e. (3.KF), is said to be the system of forward equations.

Applying (1) to $\tilde{P}(t)$ of Lemma 2.2, we obtain

$$d'_i(t)\geqslant\sum_k q_{ik}d_k(t)+D_i \tag{3}$$

where $D_i=d'_i(0)\geqslant 0$.

If $P(t)\in\mathscr{P}$, and moreover (KB_i) or (KF_j) holds, then q_i or q_j is finite and also $p'_{ij}(0)=q_{ij}$ $(j\in E$ or $i\in E)$. In particular, if $P(t)$ satisfies the system of backward equations or forward equations, then $P(t)$ is a Q process, i.e. $P(t)\in\mathscr{P}_s(Q)$. In fact, by (KB_i),

$$p'_{ii}(t)=q_{ii}p_{ii}(t)+\sum_{k\neq i}q_{ik}p_{ki}(t)$$

Hence q_{ii} is finite. Similarly, by (KF_j) it follows that q_{jj} is finite. Taking $t=0$ in (KB_i) or (KF_j), we get $p'_{ij}(0)=q_{ij}$.

Naturally we may put forward the question: What are the conditions under which the Q process $P(t)\in\mathscr{P}_s(Q)$ satisfies the backward or forward equations? We shall discuss this problem now.

Lemma 1. Let $P(t)\in\mathscr{P}$. If (KB_i) holds for almost all t, then (KB_i) holds for all $t\geqslant 0$.

Proof. By hypothesis it follows that q_{ii} is finite. Moreover

$$p_{ij}(t)=\delta_{ij}+\int_0^t\left(\sum_k q_{ik}p_{kj}(u)\right)\mathrm{d}u\qquad t\geqslant 0 \tag{4}$$

The integrand expression is continuous by (2.6) and (KB_i) holds for all $t\geqslant 0$ by differentiating with respect to t. QED

Theorem 2. Assume that $P(t)\in\mathscr{P}$. For fixed i, the necessary and sufficient condition under which (KB_i) holds is that q_i is finite and moreover

$$\sum_j q_{ij}+D_i=0 \tag{5}$$

where $D_i=d'_i(0)$.

Proof. Suppose (KB_i) holds, then q_i is finite and

$$\sum_j p'_{ij}(t)=\sum_k q_{ik}\sum_j p_{kj}(t)$$

By (6.4) it follows that the above equation is equivalent to

$$-d'_i(t)=\sum_k q_{ik}[1-d_k(t)]$$

By Theorem 6.1 it follows that, setting $t\to 0$, we obtain

$$-D_i=\sum_k q_{ik}$$

That is (5).

Conversely if q_i is finite and (5) holds, then by (1), (3) and (6.4) we find

$$\begin{aligned}0&=\sum_j p'_{ij}(t)+d'_i(t)\\&\geqslant\sum_k q_{ik}[1-d_k(t)]+\sum_k q_{ik}d_k(t)+D_i\\&=\sum_k q_{ik}+D_i=0\end{aligned}$$

Hence equality must hold in (1), i.e. (KB_i) holds. QED

When i is a conservative state, that is, when

$$q_i=\sum_{j\neq i}q_{ij}<\infty \tag{6}$$

by Theorem 5.3 it necessarily follows that $D_i=0$. Therefore (5) holds. Hence we have as a corollary of Theorem 2, the following.

Theorem 3. If (6) holds, then (KB_i) holds. In particular, when Q is conservative, each Q process satisfies the system of backward equations.

Consequently when Q is conservative, the problem of construction is simpler, and the problem of deriving all the Q processes becomes the problem of deriving all the Q processes that satisfy the system of backward equations.

Theorem 4. Let $P(t)\in\mathscr{P}_s(Q)$. A necessary and sufficient condition under which the system of backward equations holds is that, for arbitrary $t\geqslant 0$,

$$\lim_{h\to 0}\sum_j\left|\frac{p_{ij}(t+h)-q_{ij}(t)}{h}-\sum_k q_{ik}p_{kj}(t)\right|=0 \tag{7}$$

And a necessary and sufficient condition under which the system of forward equations holds is that, for arbitrary $t\geqslant 0$,

$$\lim_{h\to 0}\sum_j\left|\frac{p_{ij}(t+h)-p_{ij}(t)}{h}-\sum_k p_{ik}(t)q_{kj}\right|=0 \tag{8}$$

Proof. The necessity is derived from the corollary of Theorem 6.1. The sufficiency is obvious.

Theorem 5. Let $P(t)\in\mathscr{P}$. If the system of forward equations (3.KF) holds for almost all $t\geqslant 0$, then it must hold for all $t\geqslant 0$.

Proof. Suppose Z is a null set and (3.KF) holds if $t\notin Z$. From this it follows that

$$p_{ij}(t)=\delta_{ij}+\int_0^t\left(\sum_k p_{ik}(u)q_{kj}\right)\mathrm{d}u$$

$$\geqslant\delta_{ij}+q_{ij}\int_0^t p_{ii}(u)\,\mathrm{d}u+(1-\delta_{ij})q_{jj}\int_0^t p_{ij}(u)\,\mathrm{d}u$$

$$\frac{p_{ij}(t)-\delta_{ij}}{t}\geqslant q_{ij}\frac{1}{t}\int_0^t p_{ii}(u)\,\mathrm{d}u+(1-\delta_{ij})q_{jj}\frac{1}{t}\int_0^t p_{ij}(u)\,\mathrm{d}u$$

By (2.C) and Theorem 5.1 and 5.2, we get

$$\bar{q}_{ij}\equiv p'_{ij}(0)\geqslant q_{ij}+(1-\delta_{ij})q_{jj}\delta_{ij}=q_{ij} \tag{9}$$

Therefore $\bar{q}_{ii}\geqslant q_{ii}>-\infty$, and hence all $\bar{q}_{ij}$ are finite; furthermore, by forward inequalities (2) we have

$$p'_{ij}(t)\geqslant\sum_k p_{ik}(t)\bar{q}_{kj}\qquad t\geqslant 0$$

Combining (3.KF) of $t\notin Z$, we obtain

$$\sum_k p_{ik}(t)(\bar{q}_{kj}-q_{kj})\leqslant 0\qquad t\notin Z,\ i,j\in E$$

Since every term is non-negative, in particular, we have $p_{ii}(t)(\bar{q}_{ij}-q_{ij})=0\ (t\notin Z)$. It follows that $\bar{q}_{ij}=q_{ij}$, that is, $p'_{ij}(0)=q_{ij}$. Hence (3.KF) holds for $t=0$.

Now let $u>0$ and moreover $u\in Z$. We can find $t>0$ so that $t\notin Z$, $0<u-t\notin Z$. So by Theorem 6.2 we have

$$\begin{aligned}p'_{ij}(u)&=\sum_l p_{il}(t)p'_{lj}(u-t)\\&=\sum_l p_{il}(t)\sum_k p_{lk}(u-t)q_{kj}\\&=\sum_k\left(\sum_l p_{il}(t)p_{lk}(u-t)\right)q_{kj}\\&=\sum_k p_{ik}(u)q_{kj}\end{aligned}$$

QED

Theorem 6. Let $P(t)\in\mathscr{P}$. If the system of forward equations holds, then $\sum_k p_{ik}(t)q_{kj}$ is uniformly convergent in any finite interval $[0,T]$.

Proof. By Theorems 4.1 and 6.2 it follows that both $p'_{ij}(t)$ and $p_{ik}(t)q_{kj}$ are continuous; then using Dini's theorem we can prove this conclusion. QED

It is to be pointed out that if we only construct the Q processes that satisfy the system of backward equations, we can change the non-conservative case into the conservative case.

Theorem 7. Assume that $P(t)\in\mathscr{P}_s(Q)$ and $P(t)$ satisfies the system of backward equations, and $\tilde{P}(t)=\{\tilde{p}_{ij}(t)\}$ $(i,j\in\tilde{E})$ is determined by (2.3). Then the matrix $\tilde{Q}=\tilde{P}'(0)$ is conservative and $\tilde{P}(t)$ satisfies the system of backward equations.

Proof. Obviously

$$\tilde{Q}=\begin{pmatrix}Q & D\\ 0 & 0\end{pmatrix}$$

where $D=\{D_i\}_{i\in E}$ is a column vector. By Theorem 2 it follows that $\tilde{Q}$ is conservative. By (3.KB) we can see that

$$\tilde{p}_{ij}(t)=\sum_{k\in\tilde{E}}\tilde{q}_{ik}\tilde{p}_{kj}(t)\qquad i,j\in\tilde{E} \tag{10}$$

is valid for $i,j\in E$. For $i=\Delta$ and $j\in\tilde{E}$, the above equation is obviously valid. For $i\in E$, $j=\Delta$, (10) becomes

$$d'_i(t)=\sum_k q_{ik}d_k(t)+D_i \tag{11}$$

When $t=0$, the above equation obviously holds. And when $t>0$, it can be derived from (3.KB) and (6.4), and the proof is complete. QED

2.8 RESOLVENT OPERATORS

Let $P(t)\in\mathscr{P}$. Consider the Laplace transform of $P(t)$, $\psi(\lambda)=\{\psi_{ij}(\lambda)\}$ $(i,j\in E, \lambda>0)$:

$$\psi_{ij}(\lambda)=\int_0^\infty e^{-\lambda t}p_{ij}(t)\,dt \qquad \lambda>0 \tag{1}$$

$\psi(\lambda)(\lambda>0)$ are called resolvent operators of the process $P(t)$.

By (2.A)–(2.C) it follows that for arbitrary $i,j\in E, \lambda,\mu>0$,

$$\psi_{ij}(\lambda)\geqslant 0 \qquad \lambda\sum_j\psi_{ij}(\lambda)\leqslant 1 \tag{2}$$

$$\psi_{ij}(\lambda)-\psi_{ij}(\mu)+(\lambda-\mu)\sum_k\psi_{ik}(\lambda)\psi_{kj}(\mu)=0 \tag{3}$$

$$\lim_{\lambda\to\infty}\lambda\psi_{ij}(\lambda)=\delta_{ij} \tag{4}$$

Here the deduction of (3) from (2.B) requires the calculation of the integral $\int_0^\infty\int_0^\infty p_{ij}(t+s)e^{-\lambda t-\mu s}dt\,ds$, for which see Feller (1971, XIII.8, Ex, (a), p. 452–3). We call (2) the norm condition, (3) the resolvent equation and (4) the continuity condition. The matrix forms of (2), (3) and (4) are as follows:

$$\psi(\lambda)\geqslant 0 \qquad \lambda\psi(\lambda)\mathbf{1}\leqslant\mathbf{1} \tag{5}$$

$$\psi(\lambda)-\psi(\mu)+(\lambda-\mu)\psi(\lambda)\psi(\mu)=0 \tag{6}$$

$$\lim_{\lambda\to\infty}\lambda\psi(\lambda)=\boldsymbol{I} \tag{7}$$

From (2.D) we obtain

$$\lambda\sum_j\psi_{ij}(\lambda)=1 \tag{8}$$

or

$$\lambda\psi(\lambda)\mathbf{1}=\mathbf{1} \tag{9}$$

when $P(t)$ is an honest process.

The systems of backward and forward inequalities (7.1) and (7.2) imply that

$$\lambda\psi_{ij}(\lambda)-\delta_{ij}\geqslant\sum_k q_{ik}\psi_{kj}(\lambda) \tag{10}$$

$$\lambda\psi_{ij}(\lambda)-\delta_{ij}\geqslant\sum_k\psi_{ik}(\lambda)q_{kj} \tag{11}$$

From Theorem 7.3 we have

$$\lambda\psi_{ij}(\lambda)-\sum_k q_{ik}\psi_{kj}(\lambda)=\delta_{ij} \qquad i\in E-H \tag{12}$$

where H is the set of non-conservative states.

Theorem 1. In order that $\psi(\lambda)(\lambda>0)$ are the resolvent operators of a process $P(t)\in\mathscr{P}$, the necessary and sufficient conditions are that the norm condition, resolvent equation and continuity condition hold. The process $P(t)$ is honest if and only if (9) holds.

Proof. The necessity has been proved already. We are now going to prove the sufficiency. Take $\psi(\lambda)$ to be a linear operator that operates on Banach space l described at the beginning of section 2.2: for $g\in l, g\psi(\lambda)\in l$,

$$[g\psi(\lambda)]_j=\sum_i g_i\psi_{ij}(\lambda) \tag{13}$$

By the norm condition it follows that $\psi(\lambda)$ is a non-negative linear operator from l to l, and its norm is bounded by λ^{-1}. By the resolvent equation it follows that in the uniform topology of operators,

$$\left(-\frac{d}{d\lambda}\right)^n\psi(\lambda)=n!\,[\psi(\lambda)]^{(n+1)} \tag{14}$$

Hence

$$0\leqslant\left(-\frac{d}{d\lambda}\right)^n\psi_{ij}(\lambda)\leqslant\frac{n!}{\lambda^{n+1}} \tag{15}$$

$$0\leqslant\left(-\frac{d}{d\lambda}\right)^n\sum_j\psi_{ij}(\lambda)\leqslant\frac{n!}{\lambda^{n+1}} \tag{16}$$

Using the theory of complete monotonic functions (as can be seen in Feller (1971, pp. 415–18)), by (15) it follows that $\psi_{ij}(\lambda)$ is the Laplace transform of some measurable function $f_{ij}(t)$. From (15) and (16) we obtain the following two inequalities:

$$0\leqslant f_{ij}(t)\leqslant 1 \tag{17}$$

$$0\leqslant\sum_j f_{ij}(t)\leqslant 1 \tag{18}$$

for almost all $t\geqslant 0$. Modify the values of $f_{ij}(t)$ on a set t whose Lebesgue measure is zero, so that (17) and (18) hold for all $t>0$. We assume that the values have already been modified.

Then we shall prove that, according to Lebesgue measure in the plane,

$$f_{ij}(s+t)=\sum_k f_{ik}(s)f_{kj}(t) \tag{19}$$

holds for almost all non-negative s and t. By the uniqueness theorem of the double Laplace transform it suffices to prove that the double Laplace transform

$$\int_0^\infty \int_0^\infty \cdots e^{-\lambda s - vt} \mathrm{d}s\,\mathrm{d}t$$

for both sides of (12) are equal. When $\lambda \neq v$, the double Laplace transform for both sides of (19) are

$$(v-\lambda)^{-1}[\psi_{ij}(\lambda) - \psi_{ij}(v)] \qquad \text{and} \qquad \sum_k \psi_{ik}(\lambda)\psi_{kj}(v)$$

According to the resolvent equation, they are equal. Letting $\lambda \to v$, from (14) it follows that the double Laplace transforms for both sides of (19) also are equal in the case $\lambda = v$.

Now by modifying values of $f_{ij}(t)$ in a null measure set we shall obtain $p_{ij}(t)$ such that $p_{ij}(t)$ will become a Markov process.

Define $p_{ij}(0) = \delta_{ij}$ for $t = 0$ and define

$$\begin{aligned} p_{ij}(t) &= t^{-1} \sum_k \int_0^t f_{ik}(u) f_{kj}(t-u)\mathrm{d}u \\ &= t^{-1} \int_0^t \left(\sum_k f_{ik}(u) f_{kj}(t-u) \right) \mathrm{d}u \end{aligned} \tag{20}$$

for $t > 0$.

Since (19) holds for almost all (s, t), for almost all $t > 0$ the integrated expression in (14) is equal to $f_{ij}(t)$ for almost all $u \in (0, t)$. Hence

$$p_{ij}(t) = f_{ij}(t) \qquad \text{for almost all } t > 0$$

and the function

$$g_k(t) = \int_0^t f_{ik}(u) f_{kj}(t-u)\mathrm{d}u \tag{21}$$

as a convolution of two bounded measurable functions f_{ik} and f_{kj}, is continuous. However, the series $\sum_k g_k(t)$ is dominated by the series

$$\sum_k \int_0^t f_{ik}(u)\mathrm{d}u \tag{22}$$

term by term. In addition, each term of series (22) is continuous. Therefore by Dini's theorem it follows that the series (22) is uniformly convergent in any finite interval $[0, T]$ so that series $\sum_k g_k(t)$ is uniformly convergent in $[0, T]$, and hence $p_{ij}(t) = t^{-1}\sum_k g_k(t)$ is continuous in $(0, T)$. It follows that

$$p_{ij}(t) \text{ is continuous in } (0, \infty) \tag{23}$$

By (21) and (23), from (17) we have

$$0 \leqslant p_{ij}(t) \leqslant 1 \qquad \text{for all } t \geqslant 0 \tag{24}$$

By (18) and (21) we obtain that

$$\sum_j p_{ij}(t) \leqslant 1 \tag{25}$$

is valid for almost all $t > 0$. By (23), using Fatou's lemma it follows that the above expression holds for all $t > 0$.

We have already proved that $P(t) = \{p_{ij}(t)\}$ satisfies (2.A) and (2.C). We proceed to prove that $P(t)$ satisfies (2.B). Considering that (21) and (19) hold for almost all (s, t), we have

$$p_{ij}(s+t) = \sum_k p_{ik}(s) p_{kj}(t) \qquad \text{for almost all } (s, t) \tag{26}$$

By (23), the left-hand side of the above expression is continuous for $s > 0, t > 0$. Hence in order to prove that (26) holds for all $s > 0, t > 0$, we only need to prove that for fixed $s > 0$, the right-hand side of (26) is continuous for $t > 0$; and for fixed $t > 0$, the right-hand side of (26) is continuous for $s > 0$. By (23) and (25) it follows that the first conclusion is right. In order to prove the second conclusion, we only need to prove that for fixed $t > 0$, the series

$$\sum_k p_{ik}(s) p_{kj}(t)$$

is uniformly convergent in any interval $[a, b]$ $(0 < a < b)$. By (24) it suffices to prove that the series $\sum_k p_{ik}(s)$ is uniformly convergent in $[a, b]$. In fact, by (20)

$$\begin{aligned} \sum_j p_{ij}(t) &= t^{-1} \int_0^t \left(\sum_k \sum_j f_{ik}(u) f_{kj}(t-u) \right) \mathrm{d}u \\ &= t^{-1} \sum_k \int_0^t f_{ik}(u) \left(\sum_j f_{kj}(t-u) \right) \mathrm{d}u \end{aligned}$$

By (18) it follows that the above series $\sum_k$ is dominated by series (22), and it has been pointed out that series (22) is uniformly convergent in any finite interval $[0, T]$. Hence the series $\sum_j p_{ij}(t)$ converges uniformly in $[a, b]$.

We have already proved that the elements of $P(t) = \{p_{ij}(t)\}$ are measurable functions, and that (2.A) and (2.B) hold, namely, $P(t) = \{p_{ij}(t)\}$ is a measurable generalized transition matrix. Therefore the limit $\lim_{t \to 0+} p_{ij}(t) = u_{ij}$ exists (see Zi-kun Wang 1980, Theorem 1 in section 2.1) or Chung (1967, II.1, Theorem 3). According to Abel's property of Laplace transforms, with which we are familiar, we have

$$\lim_{\lambda \to \infty} \lambda \psi_{ij}(\lambda) = \lim_{t \to 0+} p_{ij}(t)$$

By noticing also the continuity condition (7) of $\psi(\lambda)$ it follows that (2.C) holds. Thus $P(t)=\{p_{ij}(t)\}$ is a Markov process. Its resolvent matrix is the given $\psi(\lambda)$. The sufficiency is proved.

Let $P(t)$ be honest; obviously (9) holds. Conversely, assume that (9) holds. If $\sum_j p_{ij}(t)<1$ for some $t>0$, then by the corollary of Theorem 1.4.4 $\sum_j p_{ij}(t)<1$ for all $t>0$, so that (9) cannot possibly hold. Thus (9) implies that $P(t)$ is honest. The proof is complete. QED

Theorem 2. Let the derivative matrix of $P(t)\in\mathscr{P}$ be $Q=(q_{ij})$, and let its resolvent operators be $\psi(\lambda)$, then

$$q_{ij}=\lim_{\lambda\to\infty}\lambda[\lambda\psi_{ij}(\lambda)-\delta_{ij}] \tag{27}$$

Proof. When q_{ij} is finite, for any arbitrarily given $\varepsilon>0$, there exists $\delta>0$ so that, if $t>\delta$,

$$\left|\frac{p_{ij}(t)-\delta_{ij}}{t}-q_{ij}\right|<\varepsilon$$

Therefore

$$\begin{aligned}|\lambda[\lambda\psi_{ij}(\lambda)-\delta_{ij}]-q_{ij}|&=\left|\lambda^2\int_0^\infty e^{-\lambda t}[p_{ij}(t)-\delta_{ij}-q_{ij}t]\,dt\right|\\&\leqslant\lambda^2\int_0^\delta e^{-\lambda t}\varepsilon t\,dt+\lambda^2\int_\delta^\infty e^{-\lambda t}(2+|q_{ij}|t)\,dt\\&=\varepsilon[-e^{-\lambda\delta}(\lambda\delta+1)+1]+2\lambda e^{-\lambda\delta}+|q_{ij}|e^{-\lambda\delta}(\lambda\delta+1)\end{aligned}$$

When $\lambda\to\infty$ we have

$$\overline{\lim_{\lambda\to\infty}}\,|\lambda[\lambda\psi_{ij}(\lambda)-\delta_{ij}]-q_{ij}|\leqslant\varepsilon$$

Because ε is arbitrary, (27) follows.

When q_{ii} is infinite, for arbitrary $N>0$, there exists $\delta>0$, if $t<\delta$, such that

$$[p_{ii}(t)-1]/t<-N$$

Consequently

$$\begin{aligned}\lambda[\lambda\psi_{ii}(\lambda)-1]&=\lambda^2\int_0^\infty e^{-\lambda t}[p_{ii}(t)-1]\,dt\leqslant\lambda^2\int_0^\delta e^{-\lambda t}(-Nt)\,dt\\&=-N[-e^{-\lambda\delta}(\lambda\delta+1)+1]\to-N\qquad(\lambda\to\infty)\end{aligned}$$

Because N is arbitrary, $\lim_{\lambda\to\infty}\lambda[\lambda\psi_{ij}(\lambda)-1]=-\infty$. Thus the theorem is proved. QED

When Q is finite, we call condition (27) the Q condition.

Theorem 3. In order that $\psi(\lambda)(\lambda>0)$ are the resolvent operators of a Q process $P(t)\in\mathscr{P}_s(Q)$, the necessary and sufficient conditions are that the norm condition, resolvent equation and Q condition are valid.

Proof. This is quite obvious because the Q condition implies the continuity condition. QED

Hereafter, the resolvent operators $\psi(\lambda)(\lambda>0)$ of a process or a Q process will be directly called a process or a Q process, and denoted by $\psi(\lambda)\in\mathscr{P}$ or $\psi(\lambda)\in\mathscr{P}_s(Q)$.

Theorem 4. Assume that Q is finite. In order that $\psi(\lambda)\in\mathscr{P}_s(Q)$ and $\psi(\lambda)$ satisfies the Kolmogorov backward equations (3.KB), the necessary and sufficient conditions are that the norm condition, the resolvent equation and the backward (B) condition

$$(\lambda I-Q)\psi(\lambda)=I\qquad\lambda>0 \tag{28}$$

are valid. The expression of the B condition with elements is

$$\lambda\psi_{ij}(\lambda)-\sum_k q_{ik}\psi_{kj}(\lambda)=\delta_{ij}\qquad\lambda>0,i,j\in E \tag{29}$$

Proof. Taking the Laplace transform of (2.KB) and noticing the initial condition (2.4) we obtain the B condition (29). The necessity is proved.

We now consider sufficiency. By the resolvent equations we get $\psi_{ij}(\lambda)\downarrow(\lambda\uparrow)$. By (27) we have

$$\psi_{ij}(\lambda)\downarrow 0\qquad\lambda\uparrow\infty \tag{30}$$

Also by (29) we obtain

$$(\lambda+q_i)\psi_{ij}(\lambda)\downarrow\delta_{ij}\qquad\lambda\uparrow\infty \tag{31}$$

Hence

$$\lambda\psi_{ij}(\lambda)\to\delta_{ij}\qquad\lambda\to\infty \tag{32}$$

i.e. the continuity condition holds. Again by (29) we have

$$\lambda[\lambda\psi_{ij}(\lambda)-\delta_{ij}]=\sum_k q_{ik}\lambda\psi_{kj}(\lambda) \tag{33}$$

By (32) and the dominated convergence theorem it follows that the Q condition holds. As a result $\psi(\lambda)\in\mathscr{P}_s(Q)$.

According to Theorems 4.1 and 6.1, both $p'_{ij}(t)$ and $\sum_k q_{ik}p_{kj}(t)$ are continuous functions of t, whereas (29) shows that they have the same Laplace transform. Hence they are equal, i.e. the backward equations (3.KB) hold. QED

Theorem 5. Let Q be finite. In order that $\psi(\lambda)\in\mathscr{P}_s(Q)$ and $\psi(\lambda)$ satisfies the system of Kolmogorov forward equations (3.KF) the necessary and sufficient

conditions are that the norm condition, the resolvent equation and the forward (F) condition

$$\psi(\lambda)(\lambda I - Q) = I \qquad \lambda > 0 \tag{34}$$

hold. The expression of the F condition with elements is

$$\lambda\psi_{ij}(\lambda) - \sum_k \psi_{ik}(\lambda)q_{kj} = \delta_{ij} \qquad \lambda > 0, i, j \in E \tag{35}$$

Proof. Taking Laplace transforms of both sides of (3.KF) and noticing the initial condition (2.C) the F condition (35) follows. The necessity is proved.

We prove next the sufficiency. By (35) it follows that (30) still holds. Again by (35) we have

$$\psi_{ij}(\lambda)(\lambda + q_j) \downarrow \delta_{ij} \qquad \lambda \uparrow \infty \tag{36}$$

So we find that (32) still holds, too. By Theorem 1, $\psi(\lambda)(\lambda > 0)$ are the resolvent operators of a process $P(t) \in \mathscr{P}$.

$$\lambda[\lambda\psi_{ij}(\lambda) - \delta_{ij}] = \sum_k \lambda\psi_{ik}(\lambda)q_{kj} \tag{37}$$

Likewise the Q condition holds, i.e. $\psi(\lambda) \in \mathscr{P}_s(Q)$.

By (35) it follows that the functions $p'_{ij}(t)$ and $\sum_k p_{ik}(t)q_{kj}$ have the same Laplace transform. Therefore they are equal for almost all t, i.e. (3.KF) holds for almost all $t \geqslant 0$. According to Theorem 7.5, the system of forward equations (3.KF) holds. The proof is complete. In the proof of Theorem 7.5, it is proved directly that $p'_{ij} = q_{ij}$, i.e. $p(t) \in \mathscr{P}_s(Q), \psi(\lambda) \in \mathscr{P}_s(Q)$. QED

Definition 1. $\psi(\lambda)$, which satisfies (28) or (35), is said to be B-type or F-type respectively.

2.9 FELLER'S EXISTENCE THEOREM

Does a Q process exist for a fixed matrix Q that satisfies (2.6)? In other words, is the $\mathscr{P}_s(Q)$ empty? Feller (1940) solved this problem: he constructed the minimal solution.

Define $f^n_{ij}(t)\,(t \geqslant 0)$ as follows:

$$\begin{aligned} &f^0_{ij}(t) \equiv 0 \\ &f^{n+1}_{ij}(t) = \delta_{ij}e^{-q_i t} + e^{-q_i t}\int_0^t \left(\sum_{k \neq i} q_{ik}f^n_{kj}(u)\right)e^{q_i u}\,du \end{aligned} \tag{1}$$

or equivalently

$$\begin{aligned} &f^0_{ij}(t) \equiv 0 \\ &f^{n+1}_{ij}(t) = \delta_{ij}e^{-q_i t} + e^{-q_j t}\int_0^t \left(\sum_{k \neq j} f^n_{ik}(u)q_{kj}\right)e^{q_j u}\,du \end{aligned} \tag{2}$$

Set

$$f^n_{ij}(t) \uparrow f_{ij}(t) \qquad \text{as } n \uparrow \infty \tag{3}$$

To show that (1) is equivalent to (2), consider the Laplace transforms $\phi^n_{ij}(\lambda)\ (\lambda > 0)$ and $\phi_{ij}(\lambda)$ of $f^n_{ij}(t)$ and $f_{ij}(t)$

$$\begin{aligned} &\phi^0_{ij}(\lambda) = 0 \\ &\phi^{n+1}_{ij}(\lambda) = \frac{1}{\lambda + q_i}\delta_{ij} + \sum_{k \neq i}\frac{q_{ik}}{\lambda + q_i}\phi^n_{kj}(\lambda) \end{aligned} \tag{4}$$

or

$$\begin{aligned} &\phi^0_{ij}(\lambda) = 0 \\ &\phi^{n+1}_{ij}(\lambda) = \frac{1}{\lambda + q_j}\delta_{ij} + \sum_{k \neq j}\phi^n_{ik}(\lambda)\frac{q_{kj}}{\lambda + q_j} \end{aligned} \tag{5}$$

Condition (3) becomes

$$\phi^n_{ij}(\lambda) \uparrow \phi_{ij}(\lambda) \qquad n \uparrow \infty \tag{6}$$

It remains only to prove that (4) is equivalent to (5). Using matrix notation is simpler. Denote

$$\begin{aligned} &\Pi = (\Pi_{ij}) \\ &\Pi_{ij} = \begin{cases} (1 - \delta_{ij}q_{ij})/q_i & \text{as } q_i > 0 \\ \delta_{ij} & \text{as } q_i = 0 \end{cases} \end{aligned} \tag{7}$$

Set

$$\begin{aligned} &\Pi(\lambda) = \{\Pi_{ij}(\lambda)\} \\ &\Pi_{ij}(\lambda) = \frac{q_i}{\lambda + q_i}\Pi_{ij} \end{aligned} \tag{8}$$

Define diagonal matrix

$$\lambda I + q = \mathrm{diag}(\lambda + q_i) \tag{9}$$

Then (4) becomes

$$\begin{aligned} &\phi^0(\lambda) = 0 \\ &\phi^{n+1}(\lambda) = \Pi^0(\lambda)(\lambda I + q)^{-1} + \Pi(\lambda)\phi^n(\lambda) \end{aligned}$$

By induction we have

$$\phi^{n+1}(\lambda) = \sum_{a=0}^{n} \Pi^a(\lambda)(\lambda I + q)^{-1} \qquad n \geqslant 0 \tag{10}$$

From (5) it follows that

$$\phi^0(\lambda) = 0$$

$$\phi^{n+1}(\lambda) = \Pi^0(\lambda)(\lambda I + q)^{-1} + \phi^n(\lambda)(\lambda I + q)\Pi(\lambda)(\lambda I + q)^{-1}$$

By induction we still have (10). Hence (6) becomes

$$\phi^{n+1}(\lambda) = \sum_{a=0}^{n} \Pi^a(\lambda)(\lambda I + q)^{-1} \uparrow \phi(\lambda) \qquad \lambda \to \infty \tag{11}$$

Theorem 1. $f(t) \in \mathscr{P}_s(Q)$ and moreover $f(t)$ satisfies both Kolmogorov's backward and forward equations. Moreover, $f(t)$ is the minimal Q process, that is, for arbitrary $P(t) \in \mathscr{P}_s(Q)$, we have

$$p_{ij}(t) \geqslant f_{ij}(t) \qquad t \geqslant 0, i, j \in E \tag{12}$$

Theorem 1 can be expressed by the resolvent operators as follows:

Theorem 2. $\phi(\lambda) \in \mathscr{P}_s(Q)$ and moreover $\phi(\lambda)$ satisfies both the backward and forward equations. Furthermore, $\phi(\lambda)$ is the minimal Q process; that is, for any $\psi(\lambda) \in \mathscr{P}_s(Q)$, we have

$$\psi_{ij}(\lambda) \geqslant \phi_{ij}(\lambda) \qquad \lambda > 0, i, j \in E \tag{13}$$

Proof. It suffices to prove Theorem 2. While clearly (12) implies (13), the converse implication is not obvious. But suppose that (13) holds, i.e. that $\psi(\lambda) \geqslant \phi(\lambda)$ for $\lambda > 0$, and look back to the proof of Theorem 8.1. There (8.14) tells us that

$$(-\mathrm{d}/\mathrm{d}\lambda)^n(\psi(\lambda) - \phi(\lambda)) = n![\psi(\lambda)^{n+1} - \phi(\lambda)^{n+1}]$$

whence easily

$$0 \leqslant (-\mathrm{d}/\mathrm{d}\lambda)^n(\psi_{ij}(\lambda) - \phi_{ij}(\lambda)) \leqslant n!/\lambda^{n+1}$$

so that $\psi_{ij}(\lambda) - \phi_{ij}(\lambda)$ is the Laplace transform of a function $g(\cdot)$ satisfying $0 \leqslant g(t) \leqslant 1$ for almost all $t > 0$. Hence $p_{ij}(t) \geqslant f_{ij}(t)$ for almost all $t > 0$, so for all $t \geqslant 0$ by continuity.

For a more elegant argument, see Feller (1971), Chapter XIII.10, Theorem 3, p. 426; this leads to the result that $[(n/t)\psi(n/t)]^n \to p(t)$ and likewise $[(n/t)\phi(n/t)]^n \to f(t)$, so that obviously $\psi(\lambda) \geqslant \varphi(\lambda)$ implies $p(t) \geqslant f(t)$, we need to prove that $\phi(\lambda)$ satisfies the norm condition, the resolvent equation, the B condition and the F condition, and that (13) holds.

The non-negativity of $\phi(\lambda)$ is clear. By induction we have $\lambda \sum_j \phi_{ij}^n(\lambda) \leqslant 1$ for all n; thus the norm condition is satisfied.

To prove that the resolvent equation of $\phi(\lambda)$ holds, we only need to prove that for all n

$$\begin{aligned}\Pi^n(\lambda)(\lambda I + q)^{-1} &- \Pi^n(\mu)(\mu I + q)^{-1} \\ &= (\mu - \lambda) \sum_{a=0}^{n} \Pi^a(\lambda)(\lambda I + q)^{-1}\pi^{n-a}(\mu)(\mu I + q)^{-1}\end{aligned} \tag{14}$$

holds. By summing over n in the above expression, the resolvent equation follows.

To prove (14), denoting the right-hand side of (14) as A_n we have

$$\Pi(\lambda)A_n = A_{n+1} - (\mu - \lambda)(\lambda I + q)^{-1}\Pi^{n+1}(\mu)(\mu I + q)^{-1} \tag{15}$$

Thus (14) holds for $n = 0$. Suppose (14) holds for some n. Operating $\Pi(\lambda)$ on the left-hand side of (14) we have

$$\begin{aligned}\Pi(\lambda)A_n &= \Pi^{n+1}(\lambda)(\lambda I + q)^{-1} - \Pi(\lambda)\Pi^n(\mu)(\mu I + q)^{-1} \\ &= \Pi^{n+1}(\lambda)(\lambda I + q)^{-1} - \Pi^{n+1}(\mu)(\mu I + q)^{-1} \\ &\quad - (\mu - \lambda)(\lambda I + q)^{-1}\Pi^{n+1}(\mu)(\mu I + q)^{-1}\end{aligned}$$

By substituting the above expression into (15), it follows that (14) holds where n is replaced by $n + 1$. Hence (14) holds for all n.

By (4) and (5) we obtain

$$(\lambda + q_i)\phi_{ij}^{n+1}(\lambda) = \delta_{ij} + \sum_{k \neq i} q_{ik}\phi_{kj}^n(\lambda) \tag{16}$$

$$\phi_{ij}^{n+1}(\lambda)(\lambda + q_j) = \delta_{ij} + \sum_{k \neq j} \phi_{ik}^n(\lambda)q_{kj} \tag{17}$$

Letting $n \to \infty$, the B condition and the F condition of $\phi(\lambda)$ are satisfied.

Finally, let $\psi(\lambda) \in \mathscr{P}_s(Q)$. It is clear that $\psi_{ij}(\lambda) \geqslant \phi_{ij}^0(\lambda)$. From (8.10) and (4), by induction we can easily see $\psi_{ij}(\lambda) \geqslant \phi_{ij}^n(\lambda)$ for all n. Hence we obtain (13), and the proof is complete. QED

2.10 PROPERTIES OF THE MINIMAL SOLUTION

Lemma 1. Let a column vector $f \geqslant 0$ and a row vector $g \geqslant 0$. Then when $n \uparrow \infty$,

$$\xi^n \uparrow \phi(\lambda)f \tag{1}$$

$$\eta^n \uparrow g\phi(\lambda) \tag{2}$$

where ξ^n is determined by

$$\xi_i^0 = 0$$

$$\xi_i^{n+1} = \frac{f_i}{\lambda + q_i} + \sum_{k \neq i} \frac{q_{ik}}{\lambda + q_i}\xi_k^n \tag{3}$$

and η^n is defined as

$$\eta_j^0 = 0$$

$$\eta_j^{n+1} = \frac{g_j}{\lambda + q_j} + \sum_{k \neq j} \eta_k^n \frac{q_{kj}}{\lambda + q_j} \tag{4}$$

Proof. It suffices to set $\xi^n = \phi^n(\lambda) f$ and $\eta^n = g\phi^n(\lambda)$. Then (1) and (2) hold, and by (9.16) and (9.17) it follows that (3) and (4) hold. QED

Lemma 2. Assume that a column vector $f \geqslant 0$, while u satisfies

$$(\lambda + q_i) u_i = \sum_{i \neq j} q_{ij} u_j + f_i \qquad i \in E$$

$$0 \leqslant u_i \leqslant \infty \tag{5}$$

Then $u \geqslant \phi(\lambda) f$.

Suppose that a row vector $g \geqslant 0$, whereas v satisfies

$$v_j(\lambda + q_j) = \sum_{i \neq j} v_i q_{ij} + g_j \qquad j \in E$$

$$0 \leqslant v_i \leqslant \infty \tag{6}$$

Then $v \geqslant g\phi(\lambda)$.

Proof. It suffices to prove the first case. Because $u \geqslant \xi_i^0 = 0$, owing to (5) and (3) and by induction, we obtain $u \geqslant \xi^n$; hence $u \geqslant \phi(\lambda) f$. The proof is complete. QED

Set

$$A(\mu, \lambda) = I + (\mu - \lambda)\phi(\lambda) \qquad \lambda, \mu > 0 \tag{7}$$

Lemma 3. $A(\mu, \lambda)$ is bounded[1], and for arbitrary $\lambda, \mu, \nu > 0$,

$$A(\mu, \lambda) A(\lambda, \nu) = A(\mu, \nu) \tag{8}$$

$$\phi(\mu) A(\mu, \lambda) = A(\mu, \lambda)\phi(\mu) = \phi(\lambda) \tag{9}$$

In particular,

$$A(\mu, \lambda) A(\lambda, \mu) = I \tag{10}$$

That is, for $A(\mu, \lambda)$, there exists a bounded left inverse matrix $A(\lambda, \mu)$ and a bounded right inverse matrix $A(\lambda, \mu)$.

[1] A matrix $A = (a_{ij})$ is said to be bounded if $\sup_i \sum_j |a_{ij}| < \infty$.

Proof. By using the norm condition and the resolvent equation of $\phi(\lambda)$ it can be easily obtained that $A(\mu, \lambda)$ is bounded and (8) and (9) hold. QED

Denote the equations

$$(U_\lambda) \qquad \lambda u_i - \sum_j q_{ij} u_j = 0 \qquad i \in E \tag{11}$$

The class composed of all solutions $u \in m$ to equations (U_λ) will be denoted by $\mathscr{M}_\lambda$. The class composed of all non-negative solutions $u \in m$ is denoted by $\mathscr{M}_\lambda^+$. $\mathscr{M}_\lambda^+(K)$ denotes a subset of $\mathscr{M}_\lambda^+$ whose elements are bounded by K. Denote the equations

$$(V_\lambda) \qquad \lambda v_j - \sum_i v_i q_{ij} = 0 \qquad j \in E \tag{12}$$

The collection of all solutions $v \in l$ to equations (V_λ) will be denoted by $\mathscr{L}_\lambda$. The collection of all non-negative solutions $v \in \mathscr{L}_\lambda$ is denoted by $\mathscr{L}_\lambda^+$.

Lemma 4. If $f \in \mathscr{M}_\mu$ or $\mathscr{M}_\mu^+$, then $A(\mu, \lambda) f \in \mathscr{M}_\lambda$ or $\mathscr{M}_\lambda^+$. If $g \in \mathscr{L}_\lambda$ or $\mathscr{L}_\lambda^+$, then $gA(\mu, \lambda) \in \mathscr{L}_\lambda$ or $\mathscr{L}_\lambda^+$.

Proof. If $f \in \mathscr{M}_\mu$, then by the B condition of $\phi(\lambda)$,

$$\begin{aligned} QA(\mu, \lambda) f &= Qf + (\mu - \lambda) Q\phi(\lambda) f \\ &= \mu f + (\mu - \lambda)[\lambda\phi(\lambda) f - f] \\ &= \lambda A(\mu, \lambda) f \in m \end{aligned}$$

That is $A(\mu, \lambda) f \in \mathscr{M}_\lambda$.[1]

Let $f \in \mathscr{M}_\mu^+$. When $\lambda \leqslant \mu$, obviously

$$A(\mu, \lambda) f = f + (\mu - \lambda)\phi(\lambda) f \geqslant 0 \tag{13}$$

When $\lambda > \mu$, we have

$$(\lambda I - Q) f = (\lambda - \mu) f \geqslant 0$$

By Lemma 2, $f \geqslant \phi(\lambda)(\lambda - \mu) f$, that is, $A(\mu, \lambda) f \geqslant 0$.

Similarly, we can prove that $gA(\mu, \lambda) \in \mathscr{L}_\lambda$ or $\mathscr{L}_\lambda^+$. QED

Theorem 5. $\lambda\phi(\lambda)\mathbf{1} = \mathbf{1} - \phi(\lambda)\boldsymbol{d} - \bar{X}(\lambda)$, that is,

$$\lambda \sum_j \phi_{ij}(\lambda) = 1 - \sum_{a \in H} \phi_{ia}(\lambda) d_a - \bar{X}_i(\lambda) \tag{14}$$

[1] The above calculation needs careful justification, because $Q = (q_{ij})$ may fail to be bounded so that we cannot appeal to the associative law of multiplication to obtain

$$(\lambda I - Q)[\phi(\lambda) f] = f \tag{X}$$

from $(\lambda I - Q)\phi(\lambda) = I$ (the B condition for $\phi(\lambda)$). A strict proof of (X) will probably involve interchanging the order of repeated summations: see the proof of Lemma 11.6 below.

The reader will notice that this point occurs repeatedly later on.

where H is the set of non-conservative states, $\boldsymbol{d}=(d_i)$ is the non-conservative column vector of Q and $\bar{X}(\lambda)$ is the maximal solution of

$$(\lambda I-Q)u=0 \qquad 0\leqslant u\leqslant 1 \tag{15}$$

Moreover, $u^n=\Pi^n(\lambda)\mathbf{1}\downarrow\bar{X}(\lambda)$. We define u^n as follows:

$$u_i^0\equiv 0 \qquad u_i^{n+1}=\sum_{k\neq i}\frac{q_{ik}}{\lambda+q_i}u_k^n \tag{16}$$

Proof. By the B condition of $\phi(\lambda)$

$$(\lambda I-Q)\phi(\lambda)\boldsymbol{d}=\boldsymbol{d} \tag{17}$$

$$(\lambda I-Q)[\mathbf{1}-\lambda\phi(\lambda)\mathbf{1}]=(\lambda I-Q)\mathbf{1}-\lambda I=-Q\mathbf{1}=\boldsymbol{d} \tag{18}$$

By Lemma 2

$$\mathbf{1}-\lambda\phi(\lambda)\mathbf{1}\geqslant\phi(\lambda)\boldsymbol{d}$$

Thus by (17) and (18) we find that

$$\bar{X}(\lambda)=\mathbf{1}-\lambda\phi(\lambda)\mathbf{1}-\phi(\lambda)\boldsymbol{d}$$

is the solution of equation (15), and also we obtain (14).

By Lemma 1 it follows that $1-\bar{X}(\lambda)=\phi(\lambda)(\lambda\mathbf{1}+\boldsymbol{d})$ is the limit of the increasing sequence ξ^n, where

$$\xi_i^0=0 \qquad \xi_i^{n+1}=\frac{\lambda+d_i}{\lambda+q_i}+\sum_{k\neq i}\frac{q_{ik}}{\lambda+q_i}\xi_k^n \tag{19}$$

Thus $\bar{X}(\lambda)$ is the limit of the decreasing sequence $u^n=1-\xi^n$. By the above expression (16) follows.

Finally, suppose that u is the solution of equation (15). So by $u\leqslant\mathbf{1}$ we have $u=\Pi(\lambda)\,u\leqslant\Pi(\lambda)\mathbf{1}$. Therefore $u\leqslant\Pi^n(\lambda)\mathbf{1}$, and hence $u\leqslant\lim_{h\to\infty}\Pi^n(\lambda)\mathbf{1}=\bar{X}(\lambda)$, that is, $\bar{X}(\lambda)$ is the maximal solution of equation (15).

Lemma 6. If $u\in\mathscr{M}_\lambda$ and $|u|\leqslant 1$, then

$$-\bar{X}(\lambda)\leqslant u\leqslant\bar{X}(\lambda) \tag{20}$$

Proof. The right inequality has been proved in the proof of Theorem 5. The left inequality can be proved similarly. QED

Lemma 7. If $\bar{X}(\lambda)\neq 0$, then

$$\sup_i\bar{X}_i(\lambda)=1 \qquad \inf_i\sum_{a\in H}\phi_{ia}(\lambda)d_a=0 \tag{21}$$

$$\inf_i[\phi(\lambda)u]_i=0 \qquad u\in m \tag{22}$$

Proof. Let $\sup_i\bar{X}_i(\lambda)=a$, then $0<a\leqslant 1$. Since $a^{-1}\bar{X}(\lambda)\in\mathscr{M}_\lambda^+(1)$, by the maximum of $\bar{X}(\lambda)$ we have $a^{-1}\bar{X}(\lambda)\leqslant\bar{X}(\lambda)$, so $1\leqslant a$; hence $a=1$, namely $\sup_i\bar{X}_i(\lambda)=1$. From this and (14) we obtain $\inf_i\sum_{a\in H}\phi_{ia}(\lambda)d_a=0$ and $\inf_i[\lambda\phi(\lambda)\mathbf{1}]_i=0$.Thus we get $\inf_i[\phi(\lambda)u]_i=0$ for $u\in m$. The proof is complete. QED

2.11 EXIT FAMILY AND ENTRANCE FAMILY

Definition 1. $(\xi(\lambda),\lambda>0)$ is called an exit family if $0\leqslant\xi(\lambda)\in m$ and

$$\xi(\mu)=A(\lambda,\mu)\xi(\lambda) \qquad \lambda,\mu>0 \tag{1}$$

An exit family $(\xi(\lambda),\lambda>0)$ is said to be harmonic if $\xi(\lambda)\in\mathscr{M}_\lambda^+$.

$(\eta(\lambda),\ \lambda>0)$ is called an entrance family if $0\leqslant\eta(\lambda)\in l$ and

$$\eta(\mu)=\eta(\lambda)A(\lambda,\mu) \qquad \lambda,\mu>0 \tag{2}$$

An entrance family $(\eta(\lambda),\lambda>0)$ is said to be harmonic if $\eta(\lambda)\in\mathscr{L}_\lambda^+$.

Obviously, for an exit family $\xi(\lambda)$ or an entrance family $\eta(\lambda)$ we have

$$\xi(\lambda)\downarrow \qquad \eta(\lambda)\downarrow \qquad (\lambda\uparrow\infty) \tag{3}$$

Moreover if $\xi(\lambda)=0$ or $\eta(\lambda)=0$ for some λ, then they hold for all $\lambda>0$.

Definition 2. We call

$$\xi=\lim_{\lambda\downarrow 0}\xi(\lambda) \tag{4}$$

the standard image of the exit family $\xi(\lambda)$ $(\lambda>0)$. We call

$$\eta=\lim_{\lambda\downarrow 0}\eta(\lambda) \tag{5}$$

the standard image of the entrance family $\eta(\lambda)$ $(\lambda>0)$.

Lemma 1. Let ξ be the standard image of exit family $\xi(\lambda)$, then[1]

$$\xi=\xi(\mu)+\mu\Gamma\xi(\mu) \qquad \mu>0 \tag{6}$$

$$\xi=\xi(\lambda)+\lambda\phi(\lambda)\xi \qquad \lambda>0 \tag{7}$$

where

$$\Gamma=\lim_{\lambda\downarrow 0}\phi(\lambda)=\sum_{n=0}^{\infty}\Pi^n q^{-1} \tag{8}$$

Let η be the standard image of entrance family $\eta(\lambda)$, then

$$\eta=\eta(\mu)+\mu\eta(\mu)\Gamma \qquad \mu>0 \tag{9}$$

$$\eta=\eta(\lambda)+\lambda\eta\phi(\lambda) \qquad \lambda>0 \tag{10}$$

[1] Agree on $0.\infty=\infty$. $0=0$, $1/0=\infty$.

Proof. Notice that

$$\xi(\lambda) = \xi(\mu) + (\mu - \lambda)\phi(\lambda)\xi(\mu) \tag{11}$$

$$\xi(\mu) = \xi(\lambda) + (\lambda - \mu)\phi(\lambda)\xi(\mu) \tag{12}$$

If $\sum_j \Gamma_{ij}\xi_j(u) < \infty$, (6) follows by letting $\lambda \downarrow 0$ in (11). If $\sum_j \Gamma_{ij}\xi_j(u) = \infty$, let $\lambda \downarrow 0$ in (11); by Fatou's lemma we have

$$\xi_i \geqslant \xi_i(\mu) + \mu \sum_j \Gamma_{ij}\xi_j(\mu) = \infty$$

Therefore (6) still holds. In the same way, let $\mu \downarrow 0$ in (12), and we obtain (7). Similarly, we can prove (9) and (10), and the proof is complete. QED

Corollary

Given a fixed $i, \xi_i < \infty$ if and only if for some (hence for all) $\lambda > 0$ we have

$$[\lambda\phi(\lambda)\xi]_i = [\lambda\Gamma\xi(\lambda)]_i < \infty$$

In that case, we have

$$[\lambda\phi(\lambda)\xi]_i = \xi_i - \xi_i(\lambda) \qquad \lambda > 0$$

Similarly, given a fixed $j, \eta_j < \infty$ if and only if for some (hence for all) $\lambda > 0$ we have

$$[\lambda\eta\phi(\lambda)]_j = [\lambda\eta(\lambda)\Gamma]_j < \infty$$

In that case, we have

$$[\lambda\eta\phi(\lambda)]_j = \eta_j - \eta_j(\lambda) \qquad \lambda > 0$$

Lemma 2. Denote $X^0 = \lim_{\lambda\downarrow 0} \lambda\phi(\lambda)\mathbf{1}$. Then

$$\sum_{a\in H} X^a + \bar{X} + X^0 = 1 \tag{13}$$

where

$$X_i^a = \Gamma_{ia} d_a \qquad a \in H \tag{14}$$

is the standard image of the exit family

$$X_i^a(\lambda) = \phi_{ia}(\lambda) d_a \qquad a \in H, \quad \lambda > 0 \tag{15}$$

Thus

$$\lambda\phi(\lambda)X^a = X^a - X^a(\lambda) \qquad a \in H \tag{16}$$

$\bar{X}$ is the standard image of the exit family $\bar{X}(\lambda)$. Hence

$$\lambda\phi(\lambda)\bar{X} = \bar{X} - \bar{X}(\lambda) \qquad \lambda > 0 \tag{17}$$

$\bar{X}$ is a solution of the equation

$$\Pi u = u \qquad 0 \leqslant u \leqslant 1 \tag{18}$$

while

$$\lambda\phi(\lambda)X^0 = X^0 \tag{19}$$

and X^0 is the maximal one of the solutions that satisfy the condition $\lambda\phi(\lambda)u = u$ and equation (18).

Proof. From the resolvent equation of $\phi(\lambda)$ it follows that $X^a(\lambda)$ $(a \in H)$ is an exit family whose standard image is

$$X_i^a = \lim_{\lambda\downarrow 0} X_i^a(\lambda) = \lim_{\lambda\downarrow 0} \phi_{ia}(\lambda)d_a = \Gamma_{ia}d_a \tag{20}$$

On account of (10.14) $\sum_{a\in H} X^a(\lambda) \leqslant 1$ and therefore $\sum_{a\in H} X^a \leqslant 1$. Hence (16) follows from (7).

By (10.14) and (16) we obtain

$$\lambda\phi(\lambda)\left(\mathbf{1} - \sum_{a\in H} X^a\right) = \mathbf{1} - \sum_{a\in H} X^a - \bar{X}(\lambda) \tag{21}$$

From this and the resolvent equation of $\phi(\lambda)$ it follows that $\bar{X}(\lambda)$ $(\lambda > 0)$ is an exit family. And since $\bar{X}(\lambda) \leqslant 1$, we know its standard image $\leqslant 1$. Hence by (7) we obtain (17). Also because

$$(\lambda I - Q)\bar{X}(\lambda) = 0 \qquad 0 \leqslant \bar{X}(\lambda) \leqslant \mathbf{1}$$

and consequently, letting $\lambda \downarrow 0$, we find that $\bar{X}$ is a solution of equation (18).

By (10.14), $\sum_{a\in H} X^a(\lambda) + \bar{X}(\lambda) \leqslant 1$. Letting $\lambda \downarrow 0$, we obtain $\lambda\phi(\lambda)\mathbf{1} \downarrow X^0$ and

$$X^0 = \mathbf{1} - \sum_{a\in H} X^a - \bar{X} \geqslant 0$$

From this, (13) follows. By (17) and (21) we obtain (19). By the B condition of $\phi(\lambda)$ we have

$$(\lambda I - Q)\lambda\phi(\lambda)\mathbf{1} = \lambda \tag{22}$$

and $\lambda\phi(\lambda)\mathbf{1} \downarrow X^0$ $(\lambda \downarrow 0)$. In the foregoing expression, let $\lambda \downarrow 0$, and we obtain $QX^0 = 0$, that is, X^0 satisfies (18). Suppose that u satisfies (18) and $\lambda\phi(\lambda)u = u$. Then $u = \lambda\phi(\lambda)\, u \leqslant \lambda\phi(\lambda)\mathbf{1}$, hence $u \leqslant X^0$.

Definition 3. $\bar{X}$ is called the maximal exit solution of matrix Q, and X^0 is called the maximal passive solution of matrix Q.

Lemma 3. (i) $\eta(\lambda)$ $(\lambda > 0)$ is an entrance family if and only if there exists a Riesz decomposition as follows:

$$\eta(\lambda) = \alpha\phi(\lambda) + \bar{\eta}(\lambda) \tag{23}$$

where the row vector $\alpha \geqslant 0$, and there exists some (hence all) $\lambda > 0$ so that

$\alpha\phi(\lambda)\in l$, that is

$$\left[\alpha, 1-\sum_{a\in H} X^a(\lambda)-\bar{X}(\lambda)\right]<\infty \tag{24}$$

$\bar{\eta}(\lambda)\in\mathscr{L}_\lambda^+$ is a harmonic entrance family. The vector α and hence $\bar{\eta}(\lambda)$ are uniquely determined by $\eta(\lambda)$:

$$\eta(\lambda)(\lambda I-Q)=\alpha \tag{25}$$

$$\eta(\lambda)\downarrow 0 \qquad \lambda\eta(\lambda)\to\alpha \qquad (\lambda\uparrow\infty) \tag{26}$$

(ii) $\xi(\lambda)$ $(\lambda>0)$ is an exit family if and only if there exists a Riesz decomposition as follows:

$$\xi(\lambda)=\phi(\lambda)\beta+\bar{\xi}(\lambda) \tag{27}$$

where the column vector $\beta\geqslant 0$, and there exists some (hence all) $\lambda>0$ so that $\phi(\lambda)\beta\in m$. $\bar{\xi}(\lambda)\in\mathscr{M}_\lambda^+$ is a harmonic exit family. Vector β and hence $\bar{\xi}(\lambda)$ are uniquely determined by $\xi(\lambda)$:

$$(\lambda I-Q)\xi(\lambda)=\beta \tag{28}$$

$$\xi(\lambda)\downarrow \mathbf{0} \qquad \lambda\xi(\lambda)\to\beta \qquad (\lambda\uparrow\infty) \tag{29}$$

Proof. We first prove (i) and proceed to prove the necessity. Since $(\eta(\lambda), \lambda>0)$ is an entrance family it follows that for arbitrary $v>0$, $\lambda>0$, we have

$$\eta(v)+(v-\lambda)\eta(v)\phi(\lambda)=\eta(\lambda)\geqslant\mathbf{0}$$

$$\eta(v)\geqslant(\lambda-v)\eta(v)\phi(\lambda)$$

$$\eta_j(v)\geqslant(\lambda-v)\eta_j(v)\lambda^{-1}+(\lambda-v)\sum_i\eta_i(v)[\phi_{ij}(\lambda)-\lambda^{-1}\delta_{ij}]$$

$$v\eta_j(v)\geqslant(1-v\lambda^{-1})\sum_i\eta_i(v)\lambda[\lambda\phi_{ij}(\lambda)-\delta_{ij}]$$

In the summation on the right-hand side of the above expression, the terms for $i\neq j$ are non-negative. Hence according to the Q condition of $\phi(\lambda)$ and Fatou's lemma, let $\lambda\to\infty$, and we obtain

$$v\eta_j(v)\geqslant\sum_i\eta_i(v)q_{ij}$$

Hence there exists a finite non-negative row vector $\alpha(v)$ so that

$$\eta(v)(vI-Q)=\alpha(v) \tag{30}$$

By Lemma 10.2, $\eta(v)\geqslant\alpha(v)\phi(v)$, therefore $\alpha(v)\phi(v)\in l$ is valid for all $v>0$. From the F condition of $\phi(\lambda)$ it follows that for any non-negative row vector α, provided $\alpha\phi(v)\in l$, there exists

$$\alpha\phi(v)(vI-Q)=\alpha \tag{31}$$

Hence

$$\eta(v)=\alpha(v)\phi(v)+\bar{\eta}(v) \tag{32}$$

where $\bar{\eta}(v)\in\mathscr{L}_\lambda^+$. Multiplying the above expression from the right by $A(v,\lambda)$ and noticing (10.9), and (2) as well, we obtain

$$\eta(\lambda)=\alpha(v)\phi(\lambda)+\bar{\eta}(v)A(v,\lambda) \tag{33}$$

According to Lemma 10.4, $\bar{\eta}(v)A(v,\lambda)\in\mathscr{L}_\lambda^+$. Substituting (33) into (30) and noticing (31) we obtain $\alpha(v)=\alpha(\lambda)$, that is, $\alpha(\lambda)=\alpha$ is independent of λ. Thus $\bar{\eta}(v)A(v,\lambda)=\bar{\eta}(\lambda)$. In this way (32) becomes (23), where there exists non-negative row vector α so that $\alpha\phi(\lambda)\in l$ for all $\lambda>0$, and $(\bar{\eta}(\lambda),\lambda>0)$ is a harmonic entrance family. From (3) and (25) we derive the first expression of (26). From this and the dominated convergence theorem, letting $\lambda\to\infty$ in (25), we obtain the second expression of (26). The necessity is proved. The sufficiency is obvious.

Now we are going to prove (ii). Suppose that $(\xi(\lambda),\lambda>0)$ is an exit family. Noticing (2.6), for arbitrary $v>0$, the column vector

$$\beta(v)\equiv(vI-Q)\xi(v) \tag{34}$$

is finite. On account of (2.6) the B condition of $\phi(\lambda)$ is

$$\begin{aligned}(vI-Q)\xi(v)&=(vI-Q)\{[I+(\lambda-v)\phi(v)]\xi(\lambda)\}\\&=\{(vI-Q)[I+(\lambda-v)\phi(v)]\}\xi(\lambda)\\&=[vI-Q+(\lambda-v)I]\xi(\lambda)\\&=(\lambda I-Q)\xi(\lambda)\end{aligned}$$

Therefore, $\beta(\lambda)=\beta$ is independent of $\lambda>0$, and hence (28) holds. From (3) and (28) we derive the first expression of (29). Because of this and the dominated convergence theorem, let $\lambda\to\infty$ in (28) and we obtain the second expression of (29). The limit $\beta\geqslant 0$ in the second expression of (29) because $\xi(\lambda)\geqslant 0$. According to Lemma 10.2, from (28) we get $\xi(v)\geqslant\phi(v)\beta$, hence $\phi(v)\beta\in m$ is valid for all $v>0$. By the B condition of $\phi(\lambda)$ it follows that for any non-negative column vector β, provided $\phi(v)\beta\in m$, we have

$$(vI-Q)\phi(v)\beta=\beta \tag{35}$$

Hence from the above expression, (28) and lemma 10.2, we have

$$\xi(v)=\phi(v)\beta+\bar{\xi}(v) \tag{36}$$

where $\xi(v)\in\mathscr{M}_v^+$. Multiplying the above expression from the left by $A(v,\lambda)$ and noticing (10.9) and (1) we obtain

$$\xi(\lambda)=\phi(\lambda)\beta+A(v,\lambda)\bar{\xi}(v) \tag{37}$$

Therefore $\bar{\xi}(\lambda)=A(v,\lambda)\bar{\xi}(v)$. Hence $(\bar{\xi}(\lambda),\lambda>0)$ is a harmonic exit family. Thus the necessity is proved. The sufficiency is obvious. QED

Corollary

When $\lambda \uparrow \infty$,

$$X^a(\lambda) \downarrow 0 \qquad \lambda X_i^a(\lambda) \to \delta_{ia} d_a \qquad a \in H \tag{38}$$

$$\bar{X}(\lambda) \downarrow 0 \qquad \lambda \bar{X}(\lambda) \to 0 \tag{39}$$

Lemma 4. If $(\eta(\lambda), \lambda > 0)$ is an entrance family, then $\sigma^0 = \lambda[\eta(\lambda), X^0] < \infty$ is independent of $\lambda > 0$. If $(\xi(\lambda), \lambda > 0)$ is an exit family and its standard image $\xi \in m$, then

$$(\lambda - \mu)[\eta(\lambda), \xi(\mu)] = \lambda[\eta(\lambda), \xi] - \mu[\eta(\mu), \xi] \tag{40}$$

$$\lambda[\eta(\lambda), \xi] \uparrow V \leqslant \infty \tag{41}$$

In particular, when $a \in H$

$$V_\lambda^a = \lambda[\eta(\lambda), X^a] \uparrow V^a < \infty \tag{42}$$

where

$$V^a = V_\mu^a + \eta_a(\mu) d_a \qquad \text{is independent of } \mu \tag{43}$$

Suppose that η denotes the standard image of the entrance family $(\eta(\lambda), \lambda > 0)$. Then

$$[\eta(\lambda), \xi] = [\eta, \xi(\lambda)] \tag{44}$$

If $[\eta, X^0] < \infty$, then

$$[\eta, X^0] = 0 \tag{45}$$

Proof. By (2), (7) and $\xi \in m$,

$$\begin{aligned} \lambda[\eta(\lambda), \xi] &= \lambda[\eta(\mu) A(\mu, \lambda), \xi] \\ &= \lambda[\eta(\mu), A(\mu, \lambda)\xi] \\ &= \lambda[\eta(\mu), \xi] + (\mu - \lambda)[\eta(\mu), \lambda\phi(\lambda)\xi] \\ &= \lambda[\eta(\mu), \xi] + (\mu - \lambda)[\eta(\mu), \xi - \xi(\lambda)] \\ &= \mu[\eta(\mu), \xi] + (\lambda - \mu)[\eta(\mu), \xi(\lambda)] \end{aligned} \tag{46}$$

From this, (40) follows, and therefore (41) is obtained. Similarly, by using (19), it can be btained that σ^0 is independent of λ.

When $a \in H$, $\xi(\lambda) = X^a(\lambda)$, (40) becomes

$$V_\lambda^a = V_\mu^a + (\lambda - \mu)[\eta(\mu), X^a(\lambda)] \tag{47}$$

Since $\lambda X^a(\lambda) \leqslant d_a$, by (37) and the dominated convergence theorem,

$$[\eta(\mu), \lambda X^a(\lambda)] \to \eta_a(\mu) d_a \qquad \lambda \uparrow \infty, \quad a \in H \tag{48a}$$

Hence let $\lambda \to \infty$ in (47), and (42) and (43) follow.

When $\mu \to 0$,

$$(\lambda - \mu)[\eta(\lambda), \xi(\mu)] = \lambda[\eta(\lambda), \xi(\mu)] - \mu[\eta(\lambda), \xi(\mu)] \to \lambda[\eta(\lambda), \xi]$$

From this, let $\mu \to 0$ in (40) and we have

$$\mu[\eta(\mu), \xi] \to 0 \qquad \text{if} \quad \mu \to 0 \tag{48b}$$

Thus let $\lambda \to 0$ in (40), and we obtain (44).

By (39), if $[\eta, X^0] < \infty$, then $\sigma^0 \leqslant \lambda[\eta, X^0] \to 0$, as $\lambda \to 0$. Hence $\sigma^0 = 0$, $[\eta(\lambda), X^0] = 0$. Letting $\lambda \to 0$, we obtain (45). QED

Lemma 5. If Q is non-conservative, then

$$\lambda \sum_{a \in H} X^a(\lambda) \to \boldsymbol{d} \qquad \lambda \to \infty \tag{49}$$

Proof. We have

$$Z(\lambda) = \mathbf{1} - \lambda\phi(\lambda)\mathbf{1} = \sum_{a \in H} X^a(\lambda) + \bar{X}(\lambda)$$

is an exit family, and moreover $(\lambda I - Q)Z(\lambda) = d$. Hence by Lemma 3, $\lambda Z(\lambda) \to d$. Noticing (38), (49) follows. QED

Lemma 6. If, for the row vector $\alpha, \alpha\phi(\lambda) = 0$ (that is, $\sum_i \alpha_i \phi_{ij}(\lambda) = 0$ for every j, and moreover the series is absolutely convergent), then $\alpha = 0$.

If, for the column vector β, $\phi(\lambda)\beta = 0$ (that is, $\sum_j \phi_{ij}\beta_j = 0$ for every i, and furthermore the series is absolutely convergent), then $\beta = 0$.

Proof. Since $\phi(\lambda)$ satisfies the system of backward equations, it follows that

$$\lambda\phi_{ij}(\lambda) = \delta_{ij} + \sum_k q_{ik}\phi_{kj}(\lambda) \tag{50}$$

Fix i. Multiplying both sides by β_j and summing the resulting expression for j, we obtain

$$0 = \beta_i + \sum_j \sum_k q_{ik}\phi_{kj}(\lambda)\beta_j = \beta_i + \sum_k q_{ik}\left(\sum_j \phi_{kj}(\lambda)\beta_j\right) = \beta_i$$

The order of summation can be interchanged because

$$\begin{aligned} \sum_j\left(\sum_k |q_{ik}|\phi_{kj}(\lambda)|\beta_j|\right) &= \sum_j |\beta_j|\left(\sum_k |q_{ik}|\phi_{kj}(\lambda)\right) = \sum_j |\beta_j|\left(\sum_k q_{ik}\phi_{kj}(\lambda) + 2q_i\phi_{ij}(\lambda)\right) \\ &= \sum_j |\beta_j|[\lambda\phi_{ij}(\lambda) - \delta_{ij} + 2q_i\phi_{ij}(\lambda)] \leqslant (\lambda + 2q_i)\sum_i \phi_{ij}(\lambda)|\beta_j| < \infty \end{aligned}$$

Applying the F condition of $\phi(\lambda)$,

$$\lambda\phi_{ij}(\lambda)=\delta_{ij}+\sum_k \phi_{ik}(\lambda)q_{kj} \tag{51}$$

we can prove similarly that $\alpha=0$. QED

2.12 GENERAL FORM OF Q PROCESSES

If $\psi(\lambda)\in\mathscr{P}_s(Q)$ then $\psi(\lambda)-\phi(\lambda)\geqslant 0$. As $\psi(\lambda)$ satisfies the system of backward inequalities (8.10) and (8.12), whereas $\phi(\lambda)$ satisfies the B conditon, it follows that $(\lambda I-Q)[\psi(\lambda)-\phi(\lambda)]\geqslant 0$. Moreover if $i\in E-H$, then equality holds. More precisely, for fixed $j, u_i=v_{ij}(\lambda)-\phi_{ij}(\lambda)$ satisfies

$$\lambda u_i-\sum_k q_{ik}u_k=d_iF^i_j(\lambda) \tag{1}$$

where $F^i_j(\lambda)\geqslant 0$ and d is the non-conservative quantity. By Lemma 10.2

$$B_{ij}(\lambda)=\psi_{ij}(\lambda)-\phi_{ij}(\lambda)-\sum_{a\in H}\phi_{ia}(\lambda)d_aF^a_j(\lambda)\geqslant 0$$

If j is fixed, $B_{\cdot j}(\lambda)\in\mathscr{M}^+_\lambda(1/\lambda)$. Thus the first part of the following theorem is derived.

Theorem 1. Any Q process $\psi(\lambda)\in\mathscr{P}_s(Q)$ must take the following form:

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+\sum_{a\in H}X^a_i(\lambda)F^a_j(\lambda)+B_{ij}(\lambda) \tag{2}$$

where $X^a_i(\lambda)=\phi_{ia}(\lambda)d_a$ $(a\in H)$. If i is fixed, then $0\leqslant B_{i\cdot}(\lambda)\in l$; if j is fixed, $B_{\cdot j}(\lambda)\in\mu^+_\lambda(1/\lambda)$. Moreover,

$$F^a(\lambda)\geqslant 0 \qquad \lambda[F^a(\lambda),1]\leqslant 1 \qquad a\in H \tag{3}$$

If $\psi(\lambda)$ satisfies the system of backward equations, and Q is non-conservative, then $F^a(\lambda)=0$ $(a\in H)$.

If $\psi(\lambda)$ satisfies the system of forward equations, then when i is fixed, $B_{i\cdot}(\lambda)\in\mathscr{L}^+_\lambda$. And if Q is non-conservative, then $F^a(\lambda)\in\mathscr{L}^+_\lambda$ $(a\in H)$.

Proof. Since $\psi_{i\cdot}(\lambda)\in l$ when i is fixed, so $F^a(\lambda)\in l$ $(a\in H)$, $B_{i\cdot}(\lambda)\in l$.

Suppose $\psi(\lambda)$ satisfies the system of backward equations, and Q is non-conservative. By Theorem 8.4,

$$\sum_{a\in H}X^a_i(\lambda)F^a_j(\lambda)=\sum_{a\in H}\phi_{ia}(\lambda)d_aF^a_j(\lambda)=0$$

By Lemma 11.6, this is equivalent to $F^a(\lambda)=0$ $(a\in H)$.

If $\psi(\lambda)$ satisfies the system of forward equations, since both $\psi(\lambda)$ and $\phi(\lambda)$ satisfy the F condition, it follows that if we set $G(\lambda)=\{d_iF^i_j(\lambda)\}$ $(i,j\in E)$, then

$$[\phi(\lambda)G(\lambda)+B(\lambda)](\lambda I-Q)=0$$

Multiplying from the left by $(\lambda I-Q)$ we obtain

$$G(\lambda)(\lambda I-Q)=0$$

Hence $B(\lambda)(\lambda I-Q)=0$. That is, for fixed $i, B_{i\cdot}(\lambda)\in\mathscr{L}^+_\lambda$, $F^a(\lambda)\in\mathscr{L}^+_\lambda$ $(a\in H)$.

Now let us prove (3). In section 7.18 we shall also prove (3). Here we shall give another proof.

Operating $\lambda I-Q$ on both sides of (2) and summing (2) over j, we obtain

$$(\lambda I-Q)(\lambda\psi(\lambda)\mathbf{1})_i=\lambda+\sum_{a\in H}\delta_{ia}d_a\lambda[F^a(\lambda),\mathbf{1}] \tag{4}$$

If $i=a$,

$$(\lambda I-Q)(\lambda\psi(\lambda)\mathbf{1})_a=\lambda+d_a\lambda[F^a(\lambda),\mathbf{1}] \tag{5}$$

Taking Laplace transforms on both sides of (7.3) we have

$$(\lambda I-Q)\lambda\psi(\lambda)\mathbf{1}\leqslant\lambda\mathbf{1}+\boldsymbol{d} \tag{6}$$

By (5) and (6), (3) follows. QED

Theorem 2. Each Q process $\psi(\lambda)\in\mathscr{P}_s(Q)$ must have the following form:

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+\sum_k H_{ik}(\lambda)\phi_{kj}(\lambda)+C_{ij}(\lambda) \tag{7}$$

where $H(\lambda)=\{H_{ik}(\lambda)\}\geqslant 0$ and $C(\lambda)=\{C_{ij}(\lambda)\}\geqslant 0$. When i is fixed, $C_{i\cdot}(\lambda)\in\mathscr{L}^+_\lambda$; when j is fixed, $C_{\cdot j}(\lambda)\in m$, $H_{\cdot j}(\lambda)\in m$.

$\psi(\lambda)$ satisfies the system of backward equations if and only if $H_{\cdot j}(\lambda)\in\mathscr{M}^+_\lambda$, $C_{\cdot j}(\lambda)\in\mathscr{M}^+_\lambda(1/\lambda)$.

$\psi(\lambda)$ satisfies the system of forward equations if and only if $H(\lambda)=0$.

Proof. By the system of forward inequalities (8.11) and the F condition of $\phi(\lambda)$, we have

$$[\psi(\lambda)-\phi(\lambda)](\lambda I-Q)=H(\lambda)\geqslant 0$$

Therefore, by Lemma 10.2

$$C(\lambda)\equiv\psi(\lambda)-\phi(\lambda)-H(\lambda)\phi(\lambda)\geqslant 0$$

and $C_{i\cdot}(\lambda)\in\mathscr{L}^+_\lambda$; hence we have (7). By (7) we obtain $C_{\cdot j}(\lambda)\in m$. Since

$$H_{\cdot j}(\lambda)\lambda\phi_{jj}(\lambda)\leqslant\lambda\psi_{\cdot j}(\lambda)\leqslant 1$$

it follows that $H_{\cdot j}(\lambda)\in m$.

The B condition of $\psi(\lambda)$ is equivalent to

$$(\lambda I-Q)[H(\lambda)\phi(\lambda)+C(\lambda)]=0$$

Multiplying the above expression form the right by $(\lambda I - Q)$, we have $(\lambda I - Q)H(\lambda) = 0$. Thus by the above expression we have $(\lambda I - Q)C(\lambda) = 0$. Hence the B conditionis equivalent to $C_{\cdot j}(\lambda) \in \mathcal{M}_\lambda^+(1/\lambda)$, $H_{\cdot j}(\lambda) \in \mathcal{M}_\lambda^+$.

The F condition of $\psi(\lambda)$ is equivalent to

$$[H(\lambda)\phi(\lambda) + C(\lambda)](\lambda I - Q) = 0$$

That is, $H(\lambda) = 0$. The proof is complete. QED

CHAPTER 3

Construction of Q Processes in Simple Cases

3.1 INTRODUCTION

Q is assumed to be conservative in many works on construction of Q processes such as Reuter (1959), Feller (1940, 1945, 1957), Doob (1945), Chung (1963, 1966), Williams (1964, 1966) and Xiang-qun Yang (1966a) because in that case any Q process satisfies the system of backward equations. Only a small number of articles like Reuter (1962), Feller (1957b, 1971) and Xiang-qun Yang (1965a) are concerned with non-conservative Q. If we just construct the Q processes satisfying the system of backward equations, we may do so in principle by using Theorem 2.7.7 for $P(t)$. But this construction is not an immediate one. Moreover this way of construction does not suit the construction of Q processes satisfying the system of forward equations.

In this chapter we consider construction of Q processes in simple cases, and it is not necessary to assume Q to be conservative. If Q is conservative and in single exit, all Q processes have been constructed in Reuter (1959). As for non-conservative Q, a class of honest Q processes that do not satisfy the system of backward equations are constructed in Reuter (1962). In section 3.2 all Q processes satisfying the system of backward equations are directly constructed in the case of single exit. A class of processes containing the results in Reuter (1962) are constructed in section 3.3 for non-conservative Q. In particular, when Q is single non-conservative and in null exit, we construct all Q processes. And in section 3.4 all Q processes satisfying the system of forward equations are constructed when the minimal solution is a stopping process and furthermore in single exit. The results of this chapter are seen in Xiang-qun Yang (1981a).

3.2 SINGLE EXIT CASE: CONSTRUCTION OF Q PROCESSES SATISFYING THE BACKWARD EQUATIONS

By Lemma 2.10.4, the dimension m^+ of the solution space $\mathcal{U}_\lambda^+$ of the system of equations

$$(\lambda I - Q)u = 0 \qquad 0 \leqslant u \in m \tag{1}$$

is independent of λ. When $m^+ = 0$, that is, if $\mathscr{M}_\lambda^+$ constains only the zero element, Q is called null exit. When $m^+ = 1$, Q is called single exit. And if m^+ is finite, Q is called finite exit.

Suppose that Q is single exit. By Lemma 2.10.6, $\bar{X}(\lambda) \neq 0$; on account of Theorem 2.10.5, the minimal solution $\phi(\lambda)$ is a stopping process. According to Theorem 2.12.1, every Q process $\psi(\lambda)$ satisfying the system of backward equations possesses the following form:

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \bar{X}_i(\lambda)F_j(\lambda) \tag{2}$$

Under the conditions of $m^+ > 0$ we shall determine the conditions under which $F(\lambda)$ makes $\psi(\lambda) \in \mathscr{P}_s(Q)$, that is, $\psi(\lambda)$ satisfies the norm condition and resolvent equation because the B condition is always satisfied for $\psi(\lambda)$ defined by (2).

Theorem 1. Let $m^+ > 0$. For $\psi(\lambda)$ defined by (2) to be a Q process it is necessary and sufficient that either $\psi(\lambda) = \phi(\lambda)$ or $\psi(\lambda)$ can be obtained as follows:

Take a row vector $\alpha \geqslant 0$ such that $\alpha\phi(\lambda) \in l$, and take a harmonic entrance family $\bar{\eta}(\lambda) \in \mathscr{L}_\lambda^+$. Moreover

$$\eta(\lambda) = \alpha\phi(\lambda) + \bar{\eta}(\lambda) \neq 0 \tag{3}$$

Take a constant c such that

$$[\alpha, X^0] + \bar{\sigma}^0 + \sum_{a \in H}([\alpha, X^a] + \bar{V}^a) \leqslant c \tag{4}$$

is satisfied, where $X^0, \bar{X}, X^a (a \in H)$ are defined by Lemma 2.11.2, while

$$\bar{\sigma}^0 = \lambda[\bar{\eta}(\lambda), X^0] < \infty \qquad \text{is independent of } \lambda \tag{5}$$

$$\bar{V}_\lambda^a = \lambda[\bar{\eta}(\lambda), X^a] \uparrow \bar{V}^a < \infty \qquad \lambda \uparrow \infty, a \in H \tag{6}$$

$$\bar{V}^a = \bar{V}_\lambda^a + \bar{\eta}_a(\lambda)d_a \qquad \text{is independent of } \lambda \tag{7}$$

Finally set

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \bar{X}_i(\lambda)\frac{\sum_k \alpha_k \phi_{kj}(\lambda) + \bar{\eta}_j(\lambda)}{c + [\alpha, \bar{X} - \bar{X}(\lambda)] + \lambda[\bar{\eta}(\lambda), \bar{X}]} \tag{8}$$

The process $\psi(\lambda)$ is honest if and only if

$$Q \text{ is conservative and, moreover, } [a, X^0] + \sigma^0 = c \tag{9}$$

The process $\psi(\lambda)$ satisfies the system of backward equations. The necessary and sufficient condition under which this process satisfies the system of forward equations is that $\alpha = 0$.

When $m^+ = 1$ the above-stated processes include all the Q processes that satisfy the system of backward equations. When Q is conservative and, furthermore, $m^+ = 0$, the processes above include all Q processes.

Proof. By minimality of $\phi(\lambda), \psi(\lambda) \geqslant 0$ is equivalent to $F(\lambda) \geqslant 0$. Upon noting (2.10.14), the norm condition is equivalent to

$$F(\lambda) \geqslant 0 \qquad \bar{X}(\lambda)\lambda[F(\lambda), \mathbf{1}] \leqslant \bar{X}(\lambda) + \sum_{a \in H} X^a(\lambda) \tag{10}$$

Because of (2.10.14),

$$\sum_{a \in H} X^a(\lambda) = \phi(\lambda)\mathbf{d} = \sum_{n=0}^{\infty} \Pi^n(\lambda)(\lambda + q)^{-1}\mathbf{d} \leqslant 1 \tag{11}$$

Therefore,

$$\Pi^n(\lambda)\left(\sum_{a \in H} X^a(\lambda)\right) = \sum_{a=n}^{\infty} \Pi^a(\lambda)(\lambda I + q)^{-1}\mathbf{d} \to 0 \qquad n \to \infty \tag{12}$$

Multiplying (10) from the left by $\Pi^n(\lambda)$ and taking its limit, we obtain

$$\bar{X}(\lambda)\lambda[F(\lambda), \mathbf{1}] \leqslant \bar{X}(\lambda)$$

Hence $\lambda[F(\lambda), 1] \leqslant 1$. Consequently, the norm condition is equivalent to

$$F(\lambda) \geqslant 0 \qquad \lambda[F(\lambda), 1] \leqslant 1 \tag{13}$$

$\psi(\lambda)$ is honest if and only if

$$Q \text{ is conservative and, moreover, } \lambda[F(\lambda), \mathbf{1}] = 1 \tag{14}$$

Since $\phi(\lambda)$ satisfies the resolvent equation, substituting $\psi(\lambda)$ in the resolvent equation, and observing that $\bar{X}(\lambda)$ is an exit family, we obtain that the resolvent equation of $\psi(\lambda)$ is equivalent to

$$F(\lambda)A(\lambda, \mu) = \{1 + (\mu - \lambda)[F(\lambda), \bar{X}(\mu)]\}F(\mu) \tag{15}$$

Or by (2.10.10), upon multiplying the above formula by $A(\mu, \lambda)$ from the right, we find that (15) is equivalent to

$$F(\lambda) = \{1 + (\mu - \lambda)[F(\lambda), \bar{X}(\mu)]\}F(\mu)A(\mu, \lambda) \tag{16}$$

If, for some $\mu > 0, F(\mu) = 0$, then from the formula above we know that for all $\lambda > 0, F(\lambda) = 0$. Hence $\psi(\lambda) = \phi(\lambda)$.

Otherwise, for all $\mu > 0, F(\mu) \neq 0$. Since $\lambda[F(\lambda), \bar{X}(\mu)] \leqslant \lambda[F(\lambda), \mathbf{1}] \leqslant 1$, it follows that $1 + (\mu - \lambda)[F(\lambda), \bar{X}(\mu)] > 0$. From (16) it can be seen that $F(\mu)A(\mu, \lambda) \geqslant 0$. Therefore, if one $\mu > 0$ is fixed, then $\eta(\lambda) = F(\mu)A(\mu, \lambda)$ is an entrance family. Thus, (16) is equivalent to

$$F(\lambda) = m_\lambda \eta(\lambda) \qquad m_\lambda > 0, \eta(\lambda) \neq 0 \tag{17}$$

where the quantity m_λ satisfies

$$m_\lambda = m_\mu + (\mu - \lambda)m_\lambda[\eta(\lambda), \bar{X}(\mu)]m_\mu \tag{18}$$

where $\eta(\lambda)$ is a non-zero entrance family. According to Lemma 2.11.3, $\eta(\lambda)$ has Riesz representation (3).

Dividing (18) on both sides by $m_\lambda m_\mu$, we obtain

$$m_\mu^{-1} = m_\lambda^{-1} + (\mu-\lambda)[\eta(\lambda), \bar{X}(\mu)] \tag{19}$$

But by (2.11.40),

$$(\mu-\lambda)[\eta(\lambda), \bar{X}(\mu)] = \mu[\eta(\mu), \bar{X}] - \lambda[\eta(\lambda), \bar{X}] \tag{20}$$

Consequently (19) becomes

$$m_\lambda^{-1} - \lambda[\eta(\lambda), \bar{X}] = c \qquad \text{(constant)} \tag{21}$$

Hence by (2.11.17),

$$\begin{aligned} m_\lambda &= \frac{1}{c+\lambda[\eta(\lambda), \bar{X}]} = \frac{1}{c+\lambda[\alpha\phi(\lambda), \bar{X}] + \lambda[\bar{\eta}(\lambda), \bar{X}]} \\ &= \frac{1}{c+[\alpha, \bar{X}-\bar{X}(\lambda)] + \lambda[\bar{\eta}(\lambda), \bar{X}]} \end{aligned} \tag{22}$$

Furthermore, every deduction from (18) to (22) can be inverted.

Substituting (17) (22) in (13), we have

$$\lambda[\eta(\lambda), \mathbf{1}-\bar{X}] \leqslant c \tag{23}$$

But by (2.11.13), (2.11.16)–(2.11.19) and Lemma 2.11.4,

$$\begin{aligned} \lambda[\eta(\lambda), 1-\bar{X}] &= \lambda[\alpha\phi(\lambda), \mathbf{1}-\bar{X}] + \lambda[\bar{\eta}(\lambda), \mathbf{1}-\bar{X}] \\ &= \lambda\left[\alpha\phi(\lambda), X^0 + \sum_{a\in H} X^a\right] + \lambda\left[\bar{\eta}(\lambda), X^0 + \sum_{a\in H} X^a\right] \\ &= \left[\alpha, \lambda\phi(\lambda)\left(X^0 + \sum_{a\in H} X^a\right)\right] + \bar{\sigma}^0 + \sum_{a\in H} \bar{V}^a_\lambda \\ &= [\alpha, X^0] + \bar{\sigma}^0 + \sum_{a\in H}([\alpha, X^a - X^a(\lambda)] + \bar{V}^a_\lambda) \\ &\uparrow [\alpha, X^0] + \bar{\sigma}^0 + \sum_{a\in H}([\alpha, X^a] + \bar{V}^a) \qquad \lambda\uparrow\infty \end{aligned} \tag{24}$$

The last step is arrived at on account of (2.11.37). Therefore, the norm condition (23) becomes (4) whereas (14) becomes (9). Operating on (8) from the right by $(\lambda I - Q)$, we find that the F condition holds if and only if $\alpha = 0$. The other conclusions of the theorem are quite clear, and the proof is complete. QED

In the proof of Theorem 1, if, in (2), we replace $\bar{x}(\lambda)$ by a solution $\bar{\xi}(\lambda)$ to the equation (2.10.15), which is a non-null harmonic exit family, and slightly modify it, the theorem remains valid. For this we have the following theorem.

Theorem 2. Let $m^+ > 0$. Let $\bar{\xi}(\lambda) \in \mathscr{M}_\lambda^+(1)$ be a non-null harmonic exit family and let $\bar{\xi}$ be the standard image of $\bar{\xi}(\lambda)$, $\operatorname{Sup}_i \bar{\xi}_i(\lambda) = 1$. For $\psi(\lambda) \in \mathscr{P}_s(Q)$ defined by

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \bar{\xi}_i(\lambda)F_j(\lambda) \tag{25}$$

it is necessary and sufficient that either $\psi(\lambda) = \phi(\lambda)$ or $\psi(\lambda)$ can be obtained as follows: Take a row vector $\alpha \geqslant 0$ so that $\alpha\phi(\lambda) \in l$; take a harmonic entrance family $\bar{\eta}(\lambda) \in \mathscr{L}_\lambda^+$ so that (3) holds and satisfies

$$\begin{aligned} &[\alpha, \bar{X} - \bar{\xi}] < \infty \\ &W_\lambda \equiv \lambda[\bar{\eta}(\lambda), \bar{X} - \bar{\xi}] \uparrow W < \infty \qquad \lambda\uparrow\infty \end{aligned} \tag{26}$$

Take a constant c such that

$$[\alpha, X^0) + \bar{\sigma}^0 + [\alpha, \bar{X} - \bar{\xi}] + W + \sum_{a\in H}([\alpha, X^a] + \bar{V}^a) \leqslant c \tag{27}$$

is satisfied, where the notation is the same as in Theorem 1. Finally, set

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \bar{\xi}_i(\lambda)\frac{\sum_k \alpha_k\phi_{kj}(\lambda) + \bar{\eta}_j(\lambda)}{c + [\alpha, \bar{\xi} - \bar{\xi}(\lambda)] + \lambda[\bar{\eta}(\lambda), \bar{\xi}]} \tag{28}$$

The process $\psi(\lambda)$ satisfies the system of backward equations. For the process to be honest, it is necessary and sufficient that Q is conservative, and

$$\bar{\xi}(\lambda) = \bar{X}(\lambda), [\alpha, X^0] + \sigma^0 = c \tag{29}$$

3.3 NULL EXIT CASE: CONSTRUCTION OF Q PROCESSES WITH ONE NON-CONSERVATIVE STATE

Suppose that Q is non-conservative, and moreover,

$$Z(\lambda) = 1 - \lambda\phi(\lambda)\mathbf{1} = \sum_{a\in H} X^a(\lambda) + \bar{X}(\lambda) \tag{1}$$

Obviously $Z(\lambda)$ is a non-zero exit family, whose standard image is

$$Z = \sum_{a\in H} X^a + \bar{X} = 1 - X^0 \tag{2}$$

Now let us find the necessary and sufficient conditions on $F(\lambda)$ under which $\psi(\lambda) \in \mathscr{P}_s(Q)$ can be defined by

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + Z_i(\lambda)F_j(\lambda) \tag{3}$$

Since $Z(\lambda) \notin \mathscr{M}_\lambda^+$ it follows that the B condition does not hold. Thus we need to consider the norm condition, resolvent equation and Q condition.

Theorem 1. For $\psi(\lambda)$ defined by (3) to be a non-minimal Q process it is necessary and sufficient that $\psi(\lambda)$ can be obtained as follows:

Take a row vector $\alpha \geqslant 0$ such that $\alpha\phi(\lambda)\in l$, and take a harmonic entrance family $\bar{\eta}(\lambda)\in\mathscr{L}^{+}_{\lambda}$; furthermore, (2.3) holds, and also it is necessary to satisfy

$$[\alpha, Z] + U = \infty \qquad \text{if } \alpha \neq 0 \tag{4}$$

or, equivalently,

$$[\alpha, \mathbf{1}] + Y = \infty \qquad \text{if } \alpha \neq 0 \tag{5}$$

where

$$U_\lambda = \lambda[\bar{\eta}(\lambda), Z]\uparrow U \qquad \lambda\uparrow\infty \tag{6}$$

$$Y_\lambda = \lambda[\bar{\eta}(\lambda), \mathbf{1}]\uparrow Y \qquad \lambda\uparrow\infty \tag{7}$$

And take a constant c so that

$$[\alpha, X^0] + \bar{\sigma}^0 \leqslant c \tag{8}$$

is satisfied, where σ^0 is the same as in (2.5). Finally, set

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + Z_i(\lambda)\frac{\sum_k \alpha_k\phi_{kj}(\lambda) + \bar{\eta}_i(\lambda)}{c + [\alpha, Z - Z(\lambda)] + \lambda[\bar{\eta}(\lambda), Z]}$$

The process $\psi(\lambda)$ does not satisfy the system of backward equations. It satisfies the system of forward equations if, and only if, $\alpha = 0$. For the process $\psi(\lambda)$ to be honest, it is necessary and sufficient that

$$[\alpha, X^0] + \bar{\sigma}^0 = c \tag{10}$$

Proof. Following Theorem 2.1, in order that the norm condition and resolvent equation may hold, it is necessary and sufficient that (2.17) holds, where $\eta(\lambda)$ processes representation (2.3), and m_λ is determined by (2.22) (by replacing $X, X(\lambda)$ with $Z, Z(\lambda)$). Substituting (2.17) and replacing (2.22) in the norm condition (2.13), we have

$$\lambda[\eta(\lambda), \mathbf{1} - Z] \leqslant c \tag{11}$$

But

$$\lambda[\eta(\lambda), \mathbf{1} - Z] = \lambda[\eta(\lambda), X^0] = [\alpha, X^0] + \bar{\sigma}^0 \tag{12}$$

Consequently (11) becomes (8).

For $\psi(\lambda)$ in (9) to be a Q process, the Q condition is yet to be checked, that is,

$$\lim_{\lambda\to\infty} \lambda Z_i(\lambda)\frac{\lambda\eta_j(\lambda)}{c + \lambda[\eta(\lambda), Z]} = 0 \tag{13}$$

Noting (2.11.38), from (2) and Lemma 2.11.5 we have

$$\lambda Z_i(\lambda) \to d_i \qquad (\lambda\to\infty) \tag{14}$$

By Lemma 2.11.3, $\lambda\eta(\lambda)\to\alpha$. Again by $Z_i(\lambda)\downarrow 0$ it follows that

$$\begin{aligned}\lambda[\eta(\lambda), Z] &= [\alpha, Z - Z(\lambda)] + \lambda[\bar{\eta}(\lambda), Z] \\ &\uparrow [\alpha, Z] + U \qquad \lambda\uparrow\infty\end{aligned} \tag{15}$$

Hence (13) becomes

$$d_i \frac{\alpha_j}{c + [\alpha, Z] + U} = 0 \tag{16}$$

and this is equivalent to (4). Since $Z = 1 - X^0$, by (2.11.39),

$$\sigma^0 = \lambda[\eta(\lambda), X^0] = [\alpha, X^0] + \bar{\sigma}^0 < \infty \tag{17}$$

It follows that (4) is equivalent to (5), and the proof is concluded. QED

According to Lemma 2.10.4, the dimension n^+ of the solution space $\mathscr{L}^{+}_{\lambda}$ of the equation

$$v(\lambda I - Q) = 0 \qquad 0 \leqslant v \in l \tag{18}$$

is independent of λ. When $n^+ = 0$, the matrix Q is called null entrance. If $n^+ = 1$, the matrix Q is called single entrance. If n^+ is finite, the matrix Q is called finite entrance.

Suppose Q is non-conservative and non-null entrance, then we may take $\alpha = 0$ and the non-zero harmonic entrance family $\bar{\eta}(\lambda)\in\mathscr{L}^{+}_{\lambda}$ in Theorem 1. Hence the non-minimal Q process $\psi(\lambda)$ in Theorem 1 exists.

If Q is non-conservative and null entrance, then for the non-minimal Q process $\psi(\lambda)$ in Theorem 1 to exist, we must have $\alpha \geqslant 0$ so that $\alpha\phi(\lambda)\in l$ and, moreover, $[\alpha, 1] = \infty$. This condition can be given by the following lemma (which is also found in the work by Zhen-ting Hou (1974, Lemma 12.2.4)).

Lemma 2. There exists $\alpha \geqslant 0$ so that $\alpha\phi(\lambda)\in l$ while $[\alpha, 1] = \infty$ if, and only if, for some (hence all) $\lambda > 0$,

$$\inf_i \lambda\sum_j \phi_{ij}(\lambda) = 0 \tag{19}$$

Proof. Suppose that there exists $\alpha \geqslant 0$ such that for some $\lambda > 0$, we have $\alpha\phi(\lambda)\in l$ and, moreover, $[\alpha, 1] = \infty$. From the resolvent equation and the norm condition of $\phi(\lambda)$ we know that $\alpha\phi(\lambda)\in l$ holds for all $\lambda > 0$. Since

$$\begin{aligned}\lambda[\alpha\phi(\lambda), \mathbf{1}] &= [\alpha, \lambda\phi(\lambda)\mathbf{1}] \\ &\geqslant [\alpha, \mathbf{1}]\inf_i \lambda\sum_i \phi_{ij}(\lambda)\end{aligned}$$

it follows that (19) is valid for all $\lambda > 0$.

Assume that (19) holds. Since $\lambda\phi_{ii}(\lambda) > 0$, it follows that $\lambda\sum_j\phi_{ij}(\lambda) > 0$. Hence it is deduced from (19) that there exist mutually different $i_k \in E, k = 1, 2, 3, \ldots$, such that

$$\lambda \sum_j \phi_{i_k j}(\lambda) < 1/2^k$$

Take

$$\alpha_j = \begin{cases} 1 & \text{if } j \in \{i_1, i_2, \ldots\} \\ 0 & \text{if } j \notin \{i_1, i_2, \ldots\} \end{cases}$$

and we have

$$\lambda[\alpha\phi(\lambda), \mathbf{1}] = [\alpha, \lambda\phi(\lambda)\mathbf{1}] < \sum_{k=1}^{\infty} 1/2^k = 1$$

$$[\alpha, \mathbf{1}] = \sum_{k=1}^{\infty} \alpha_{i_k} = \sum_{k=1}^{\infty} 1 = \infty$$

The lemma is proved. QED

The condition (19) is equivalent to

$$\sup_i Z_i(\lambda) = 1 \tag{20}$$

If $\bar{X}(\lambda) \neq 0$, then by Lemma 2.10.7,

$$\sup_i \bar{X}_i(\lambda) = 1 \tag{21}$$

Therefore (20) is satisfied.

Suppose $\bar{X}(\lambda) = 0$; then (20) becomes

$$\sup_i \sum_{a \in H} X_i^a(\lambda) = 1 \tag{22}$$

If Q is single non-conservative, that is, H contains only one state a, then (22) becomes

$$\sup_i X_i^a(\lambda) = 1 \tag{23}$$

or, equivalently

$$\sup_i \phi_{ia}(\lambda) = 1/d_a \tag{24}$$

Theorem 3. Assume Q to be null exit and, furthermore, non-conservative only in one state a. Then

$$Z_i(\lambda) = \phi_{ia}(\lambda)d_a \qquad Z_i = \Gamma_{ia}d_a = 1 - X_i^0 \tag{25}$$

where Γ is defined by (2.10.8). If Q is null entrance, and also

$$\sup_i \phi_{ia}(\lambda) < 1/d_a$$

then the Q process is unique.

If Q is non-zero entrance, or if Q is null entrance and, moreover, (24) holds, then the Q process is not unique. In this case every non-minimal Q process can be obtained by means of Theorem 1.

3.4 SINGLE ENTRANCE CASE: CONSTRUCTION OF Q PROCESSES SATISFYING THE FORWARD EQUATIONS

In this section Q is not required to be conservative, but we assume that the minimal solution is a stopping process, and that $n^+ > 0$. Then we can select a non-zero harmonic entrance family $\bar{\eta}(\lambda) \in \mathscr{L}_\lambda^+$. If the Q process $\psi(\lambda)$ has the form

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + F_i(\lambda)\bar{\eta}_j(\lambda) \tag{1}$$

then $\psi(\lambda)$ satisfies the system of forward equations. Conversely, if $n^+ = 1$, then every Q process $\psi(\lambda)$ satisfying the system of forward equations must take the form (1). Under the condition $n^+ > 0$ we shall determine $F(\lambda)$ so that $\psi(\lambda)$ in (1) is a Q process. As the F condition is always satisfied, we only need to investigate the norm condition and the resolvent equation.

Theorem 1. Suppose that the minimal solution $\phi(\lambda)$ is a stopping process and $n^+ > 0$. For $\psi(\lambda)$ defined by (1) to be a Q process it is necessary and sufficient that either $\psi(\lambda) = \phi(\lambda)$ or $\psi(\lambda)$ can be derived as follows: Take a constant $\delta \geqslant 0$ and a harmonic exit family $\bar{\xi}(\lambda) \in \mathscr{N}_\lambda^{\prime}(1)$, whose standard image is $\bar{\xi}$. If $\delta > 0$, it is also demanded that $\sup_i \bar{\xi}_i(\lambda) = 1$ is satisfied. If Q is non-conservative, then for each $a \in H$, take a quantity $\beta^a \geqslant 0$ such that $\sum_{a\in H}\beta^a X^a(\lambda) \in m$, and, moreover,

$$\xi(\lambda) = \sum_{a \in H} \beta^a X^a(\lambda) + \delta\xi(\lambda) \neq 0 \tag{2}$$

and

$$\begin{aligned} &\lambda[\bar{\eta}(\lambda), \xi] < \infty \\ &\bar{W}_\lambda = \lambda[\bar{\eta}(\lambda), \bar{X} - k\delta\bar{\xi}] \uparrow \bar{W} < \infty \qquad \lambda \uparrow \infty \end{aligned} \tag{3}$$

where[1]

$$\xi = \sum_{a \in H} \beta^a X^a + \delta\bar{\xi} \neq 0$$

$$k = \inf\{1/\delta, 1/\beta^a, (a \in H)\} \tag{4}$$

[1] Agree on $1/0 = \infty$.

Take a constant c such that

$$\bar{\sigma}^0 + \bar{W} + \sum_{a\in H} (1 - k\beta^a)\bar{V}^a \leqslant kc \tag{5}$$

is satisfied, where $\bar{\sigma}^0$, $\bar{V}^a$ are defined by (2.5) and (2.6). Finally, let us set

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \frac{(\sum_{a\in H}\beta^a X_i^a(\lambda) + \delta\bar{\xi}_i(\lambda))\bar{\eta}_j(\lambda)}{c + \sum_{a\in H}\beta^a \lambda[\bar{\eta}(\lambda), X^a] + \delta\lambda[\bar{\eta}(\lambda), \bar{\xi}]} \tag{6}$$

The process $\psi(\lambda)$ is honest if, and only if,

$$\bar{\xi} = \bar{X} \qquad \beta^a = \delta \qquad (a\in H) \qquad \delta^{-1}c = \bar{\sigma}^0 \tag{7}$$

The process $\psi(\lambda)$ satisfies the system of forward equations. For the process $\psi(\lambda)$ to satisfy the system of backward equations, it is necessary and sufficient that $\beta^a = 0$ $(a\in H)$. When $n^+ = 1$, the processes obtained above include all the Q processes satisfying the system of forward equations.

Proof. On account of (2.10.14), the norm condition is equivalent to

$$0 \leqslant F(\lambda) \qquad F(\lambda)\lambda[\bar{\eta}(\lambda), \mathbf{1}] \leqslant \sum_{a\in H} X^a(\lambda) + \bar{X}(\lambda) \tag{8}$$

In analogy with (2.15), the resolvent equation is equivalent to

$$A(\mu, \lambda)F(\mu) = F(\lambda) + (\lambda - \mu)F(\lambda)[\bar{\eta}(\lambda), F(\mu)] \tag{9}$$

or, equivalently,

$$F(\mu) = \{1 + (\lambda - \mu)[\bar{\eta}(\lambda), F(\mu)]\}A(\lambda, \mu)F(\lambda) \tag{10}$$

so that if $F(\lambda) = 0$ for some $\lambda > 0$; then $F(\lambda) = 0$ for all $\lambda > 0$, therefore $\psi(\lambda) = \phi(\lambda)$. Otherwise (10) is equivalent to

$$F(\lambda) = m_\lambda \xi(\lambda), \qquad m_\lambda > 0, \xi(\lambda) \neq 0 \tag{11}$$

where the quantity $m_\lambda > 0$ satisfies

$$m_\mu = m_\lambda\{1 + (\lambda - \mu)[\bar{\eta}(\lambda), \xi(\mu)]m_\mu\} \tag{12}$$

whereas $\xi(\lambda)$ is a non-zero exit family. According to Lemma 2.11.3, $\xi(\lambda)$ has Riesz representation

$$\xi(\lambda) = \phi(\lambda)\beta + \tilde{\xi}(\lambda) \neq 0 \tag{13}$$

where the column vector $\beta \geqslant 0$ is such that $\phi(\lambda)\beta\in m$, and the exit family $\tilde{\xi}(\lambda)\in\mathscr{M}_\lambda^+$. Using Lemmas 2.10.6 and 2.10.7, it is easy to prove that $\delta = \sup_i \tilde{\xi}_i(\lambda)$ is independent of $\lambda > 0$. If $\delta = 0$, set $\bar{\xi}(\lambda) = 0$; if $\delta > 0$, set $\bar{\xi}(\lambda) = \delta^{-1}\tilde{\xi}(\lambda)$. Then $(\bar{\xi}(\lambda), \lambda > 0)$ is a harmonic exit family. Let its standard image be $\bar{\xi}$; hence

$$\sup_i \bar{\xi}(\lambda) = 1 \qquad \text{if } \delta > 0 \tag{14}$$

comparing this with (2.12.2), we obtain $\beta_j = 0$ $(j\in E - H)$. Upon setting $\beta^a = \beta_a/d_a$ $(a\in H)$, (13) becomes

$$\xi(\lambda) = \sum_{a\in H} \beta^a X^a(\lambda) + \delta\bar{\xi}(\lambda) \neq 0 \tag{15}$$

whose standard image is

$$\xi = \sum_{a\in H} \beta^a X^a + \delta\bar{\xi} \neq 0 \tag{16}$$

We are going to prove $\lambda[\bar{\eta}(\lambda), \xi] < \infty$. If not, upon dividing (12) on both sides by $m_\lambda m_\mu$, we have

$$m_\lambda^{-1} = m_\mu^{-1} + (\lambda - \mu)[\bar{\eta}(\lambda), \xi(\mu)]$$

Setting $\mu\downarrow 0$, we obtain

$$m_\lambda^{-1} \geqslant \lim_{\mu\downarrow 0} (\lambda - \mu)[\bar{\eta}(\lambda), \xi(\mu)] = \lambda[\bar{\eta}(\lambda), \xi] = \infty$$

Thus $m_\lambda = 0$, which contradicts (11).

By using (2.11.40),

$$(\mu - \lambda)[\bar{\eta}(\lambda), \xi(\mu)] = \mu[\bar{\eta}(\mu), \xi] - \lambda[\bar{\eta}(\lambda), \xi] \tag{17}$$

Therefore, upon dividing (12) by $m_\lambda m_\mu$, we find that there exists a constant c such that

$$m_\lambda = \{c + \lambda[\bar{\eta}(\lambda), \xi]\}^{-1} \tag{18}$$

Upon substituting (11) and (15) in (1) and comparing the result with (2.12.2), we obtain $F_{(\lambda)}^a = \beta^a m_\lambda \bar{\eta}(\lambda)$.

Observing (2.12.3), we have

$$\beta^a m_\lambda \lambda[\bar{\eta}(\lambda), \mathbf{1}] \leqslant 1 \qquad a\in H. \tag{19}$$

By substituting (15) and (11) in the norm condition (8), we find

$$\left(\sum_{a\in H} \beta^a X^a(\lambda) + \delta\bar{\xi}(\lambda)\right) m_\lambda \lambda[\bar{\eta}(\lambda), \mathbf{1}] \leqslant \sum_{a\in H} X^a(\lambda) + \bar{X}(\lambda) \tag{20}$$

Operating on the formula above by $\Pi^n(\lambda)$, setting $n\to\infty$, and considering (2.12), we obtain

$$\delta\bar{\xi}(\lambda)m_\lambda\lambda[\bar{\eta}(\lambda), \mathbf{1}] \leqslant \bar{X}(\lambda). \tag{21}$$

Hence it can be seen that the norm condition (20) is equivalent to (19) and (21).

By (14) and

$$\bar{\xi}(\lambda) \leqslant \bar{X}(\lambda) \tag{22}$$

we obtain that (21) is equivalent to

$$\delta m_\lambda \lambda[\bar{\eta}(\lambda), \mathbf{1}] \leq 1 \tag{23}$$

Thus the norm conditions (21) and (19) become

$$m_\lambda \lambda[\bar{\eta}(\lambda), \mathbf{1}] \leq k \tag{24}$$

By (16) and (4), $0 < k < \infty$. Upon substituting (15) and (18) in (24) we find

$$\lambda[\bar{\eta}(\lambda), X^0] + \sum_{a \in H} \lambda[\bar{\eta}(\lambda), X^a] + \lambda[\bar{\eta}(\lambda), \bar{X}]$$

$$\leq kc + k \sum_{a \in H} \beta^a \lambda[\bar{\eta}(\lambda), X^a] + k\delta\lambda[\bar{\eta}(\lambda), \bar{\xi}]$$

that is,

$$\bar{\sigma}^0 + \bar{W}_\lambda + \sum_{a \in H} (1 - k\beta^a)\bar{V}_\lambda^a \leq kc \tag{25}$$

Setting $\lambda \uparrow \infty$, we find that the norm condition is equivalent to (5).

If $\psi(\lambda)$ is honest, we must establish an equality in the second formula in (8), that is to say, the equalities in (21) and (19) hold. From this it follows that $\beta^a = \delta$ $(a \in H)$, $\bar{\xi}(\lambda) = \bar{X}(\lambda)$, i.e. $\bar{\xi} = \bar{X}$. Hence $k = \delta^{-1}$, and the equality in (5) becomes $\bar{\sigma}^0 = \delta^{-1}c$. Therefore, for $\psi(\lambda)$ to be honest, the necessary and sufficient condition is (7). The other conclusions to the theorem are easily seen, and the proof is complete. QED

CHAPTER 4

Uniqueness

4.1 INTRODUCTION

Given that Q satisfies (2.2.6), there always exist Q processes. This chapter discusses the problem of uniqueness. Section 4.2 provides the necessary and sufficient conditions for uniqueness of Q processes satisfying the system of backward equations. Section 4.3 gives those for uniqueness of Q processes satisfying the system of forward equations. The content of these two sections is derived from Reuter (1957). The proof of the Hou–Reuter theorem in section 4.4, that is, the criterion for uniqueness of Q processes, is obtained from Reuter (1976), but now it has been simplified and improved. Thus, the problem of uniqueness in construction theory has been completely solved.

4.2 UNIQUENESS THEOREM: THE SYSTEM OF BACKWARD EQUATIONS

When we consider the problem of uniqueness of the Q processes satisfying the system of backward equations, the equation

$$(U_\lambda) \qquad \lambda u - Qu = 0 \tag{1}$$

will play a prominent role. Recall that the set of solutions $u \in m$ of the equation (U_λ) is denoted by $\mathscr{M}_\lambda$, the set of non-negative solutions $u \in m$ is written as $\mathscr{M}_\lambda^+$, and the set of solutions in $\mathscr{M}_\lambda^+$ which are bounded by K is denoted by $\mathscr{M}_\lambda^+(K)$. The dimension of $\mathscr{M}_\lambda^+$ is denoted by m^+.

Theorem 1. The following conditions are equivalent:

(i) The Q process satisfying the system of backward equations is unique.

(ii) For some $\eta > 0$ (hence all $\lambda > 0$), $\mathscr{M}_\lambda$ consists of the zero solution only.

(iii) For some $\lambda > 0$ (hence all $\lambda > 0$), $\mathscr{M}_\lambda^+$ consists of the zero solution only.

If one of the above-stated conditions does not hold, then there are infinitely many Q processes satisfying the system of backward equations. If Q is conservative, there exist infinitely many honest Q processes satisfying the system

of backward equations; if Q is non-conservative, then all Q processes satisfying the system of backward solutions are stopping.

Proof. (i)$\Rightarrow$(ii). Suppose (ii) does not hold. By Lemma 2.10.6, $\bar{X}(\lambda) \neq 0$. The process $\psi(\lambda)$ obtained by taking $\bar{\eta}(\lambda) = 0, \alpha \geqslant 0, [\alpha, 1] = 1$, and $c = [\alpha, X^0]$ in (3.2.8) satisfies the system of backward equations. If Q is conservative, such $\psi(\lambda)$ is still honest. But there are infinitely many choices of α, which therefore is in contradiction with (i). Hence (ii) is valid.

(ii)$\Rightarrow$(iii) need not be proved. We are going to prove (iii)$\Rightarrow$(i). Assume that (iii) holds. Since $\bar{X}(\lambda) \in \mathcal{M}_\lambda^+(1)$, it follows that $\bar{X}(\lambda) = 0$. And since $\bar{X}(\lambda)$ is an exit family, it follows that, for all $\lambda > 0, \bar{X}(\lambda) = 0$.

If $\psi(\lambda)$ satisfies the system of backward equations according to Theorem 2.12.1, it follows that $F^a(\lambda) = 0 (a \in H)$ in (2.12.2). Nevertheless when j is fixed, $u_i = \lambda B_{ij}(\lambda) \in \mathcal{M}_\lambda^+(1)$. By maximality of $\bar{X}(\lambda), \lambda B_{ij}(\lambda) \leqslant \bar{X}_i(\lambda)$, hence $B_{ij}(\lambda) = 0$. Thus, (2.12.2) becomes $\psi(\lambda) = \phi(\lambda)$, that is, (i) is obtained.

If there exists an honest Q process $\psi(\lambda)$ satisfying the system of backward equations, then by summing for j in the B condition of $\psi(\lambda)$, we find that Q is conservative, and the proof is complete. QED

4.3 UNIQUENESS THEOREM: THE SYSTEM OF FORWARD EQUATIONS

When the problem of uniqueness of Q processes satisfying the system of forward equations is taken into consideration, the system of equations

$$(V_\lambda) \qquad \lambda v - vQ = 0 \tag{1}$$

will play an important role. Let us recall that the set of solutions $v \in l$ of the equation (V_λ) is written as $\mathcal{L}_\lambda$, all non-negative solutions $v \in l$ are denoted by $\mathcal{L}_\lambda^+$, and the dimension of $\mathcal{L}_\lambda^+$ is denoted by n^+.

Theorem 1. (i) If the minimal solution is honest, or if it is a stopping process but $n^+ = 0$, then the Q process satisfying the system of forward equations is unique.

(ii) Suppose that the minimal solution is a stopping process and $n^+ = 1$. Then we have infinitely many Q processes that satisfy the system of forward equations, and only one of them is honest.

(iii) If the minimal solution is a stopping process and $n^+ > 1$, there exist infinitely many Q processes satisfying the system of forward equations, of which infinitely many are honest.

Proof. (i) When the minimal solution is honest, uniqueness is obvious. Let the minimal solution be stopping and moreover let $n^+ = 0$. If $\psi(\lambda)$ is a Q process satisfying the system of forward equations, by Theorem 2.12.1, it follows that $F^a(\lambda) \in \mathcal{L}_\lambda^+ (a \in H)$ in (2.12.2), $u_j = B_{ij}(\lambda) \in \mathcal{L}_\lambda^+$, hence $F^a(\lambda) = 0 (a \in H)$, $B_{ij}(\lambda) = 0$. Equation (2.12.2) turns into $\psi(\lambda) = \phi(\lambda)$, that is, the Q process satisfying the system of forward equations is unique.

(ii) Theorem 3.4.1 has described the construction.

(iii) The preceding part has already been answered by Theorem 3.4.1. Since $n^+ > 1$, it follows that we may select the exit family $\bar{\eta}a(\lambda) \in \mathcal{L}_\lambda^+$ $(a = 1, 2)$, so that $\bar{\eta}^1(\lambda), \bar{\eta}^2(\lambda)$ become linearly independent. Arbitrarily take two constants $p^a \geqslant 0$ $(a = 1, 2)$ such that

$$\bar{\eta}(\lambda) = p^1\bar{\eta}^1(\lambda) + p^2\bar{\eta}^2(\lambda) \neq 0$$

For $\bar{\eta}(\lambda)$, there exists an honest Q process satisfying the system of forward equations according to Theorem 3.4.1. But there are infinitely many ways of selecting $p^a (a = 1, 2)$ so that $\bar{\eta}(\lambda)$ is made different (constant factors not being considered). Therefore there exist infinitely many honest Q processes that satisfy the system of forward equations, and the proof is terminated. QED

4.4 CRITERION FOR UNIQUENESS: THE HOU–REUTER THEOREM

The criterion for uniqueness of Q processes is given by Zhen-ting Hou (1974) and summarized in his book (Zhen-ting Hou, 1982). In addition these discuss the existence of combinations of various cases and the problem of uniqueness, namely the so-called qualitative theory. Reuter (1976) has simplified the proof in Zhen-ting Hou (1974). Here the simplified proof of Reuter is adopted and further improved.

Theorem 1. Let a given matrix Q satisfy (2.2.6). Then for the Q process to be unique it is necessary and sufficient that the minimal Q process $\phi(\lambda)$ is honest, or that it is stopping and moreover satisfies the two conditions below:

$$\text{(i)} \qquad \inf_i \lambda \sum_j \phi_{ij}(\lambda)_\lambda > 0 \qquad \lambda > 0 \tag{1}$$

(ii) $n^+ = 0$, that is, the equation (V_λ) has no non-zero and non-negative solution $v \in l$.

We point out that condition (i) implies $m^+ = 0$. In fact, by (2.10.14), (1) becomes

$$\sup_i \left(\sum_{a \in H} X_i^a(\lambda) + \bar{X}_i(\lambda) \right) < 1 \qquad \lambda > 0 \tag{2}$$

since, if $\bar{X}(\lambda) \neq 0$, (2.10.21) holds. From (2) it necessarily follows that $\bar{X}(\lambda) = 0$, and hence $m^+ = 0$.

Thus the condition (i) is equivalent to the following two conditions:

$$(\mathrm{i}_1) \qquad m^+ = 0$$

$$(\mathrm{ii}_2) \qquad \sup_i \sum_{a\in H} \phi_{ia}(\lambda) d_a < 1 \qquad \lambda > 0$$

Proof of the theorem. Necessity: Assume that the minimal Q process is stopping and moreover the Q process is unique; hence a Q process of F type is unique. By Theorem 3.1, $n^+ = 0$. Now let us assume that

$$\inf_i \lambda \sum_j \phi_{ij}(\lambda) = 0 \tag{3}$$

According to Lemma 3.2 there exists a row vector $\alpha \geqslant 0$ such that $[\alpha, 1] = \infty$ and furthermore $\alpha\phi(\lambda)\in l$, and for this α, by Theorem 3.3.1,

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + Z_i(\lambda)\frac{\sum_k \alpha_k \phi_{kj}(\lambda)}{c + [\alpha, Z - Z(\lambda)]} \tag{4}$$

is a Q process where $Z(\lambda) = 1 - \lambda\phi(\lambda)1 \neq 0$, $c \geqslant [\alpha, X^0]$. Since the selection of constant c may not be unique, it follows that the Q process is not unique. This is in contradiction with the hypothesis of the necessary conditions. Hence (i) is valid.

Sufficiency: Let (i) and (ii) hold and let $\psi(\lambda)$ be a Q process. From (i_1) it follows that, in Theorem 2.12.1, $B(\lambda) = 0$; hence (2.12.2) becomes

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \sum_{a\in H} X_i^a(\lambda) F_j^a(\lambda) \tag{5}$$

If H is empty, from the formula above we know $\psi(\lambda) = \phi(\lambda)$, consequently the Q process is unique. In what follows we suppose $H \neq \varnothing$. Substituting the formula (5) in the resolvent equation of $\psi(\lambda)$, noting that $\phi(\lambda)$ satisfies the resolvent equation, and that because of Lemma 2.11.6, $X^a(\lambda)$, $a\in H$ are linearly independent, we obtain

$$F^a(\lambda)A(\lambda,\mu) = F^a(\mu) + (\mu - \lambda)\sum_{b\in H}[F^a(\lambda), X^b(\mu)]F^b(\mu) \qquad a\in H \tag{6}$$

Since $F^a(\lambda) \geqslant 0$, $\lambda[F^a(\lambda), 1] \leqslant 1$, it can be seen from the above formula that for arbitrary $\lambda, \mu > 0$, $F^a(\lambda)A(\lambda,\mu)\in l$. Hence fix a and $\lambda > 0$ as well, and by (2.10.8), $\eta(\mu) = F^a(\lambda)A(\lambda,\mu)$ $(\mu > 0)$ is an entrance family. According to Lemma 2.11.3,

$$\eta(\mu) = \alpha\phi(\mu) + \bar{\eta}(\mu)$$

where $\alpha \geqslant 0$ is independent of μ so that $\alpha\phi(\mu)\in l$ and $(\bar{\eta}(\mu), \mu > 0)$ is a harmonic entrance family. By (ii) $n^+ = 0$, therefore $\bar{\eta}(\mu) = 0$. Again since α depends on a and λ, it follows that $\alpha = \alpha^a(\lambda)$, i.e.

$$F^a(\lambda)A(\lambda,\mu) = \alpha^a(\lambda)\phi(\mu) \tag{7}$$

In particular when $\mu = \lambda$,

$$F^a(\lambda) = \alpha^a(\lambda)\phi(\lambda) \tag{8}$$

On account of condition (i)

$$1 \geqslant \lambda[F^a(\lambda), 1] = \lambda[\alpha^a(\lambda)\phi(\lambda), 1]$$
$$= [\alpha^a(\lambda), \lambda\phi(\lambda)1] \geqslant \eta_\lambda[\alpha^a(\lambda), 1]$$

so that

$$[\alpha^a(\lambda), 1] \leqslant 1/\eta_\lambda. \tag{9}$$

Upon substituting (8) in (6) and observing (2.10.8) and Lemma 2.11.6, we have

$$\alpha^a(\lambda) = \alpha^a(\mu) + (\mu - \lambda)\sum_{b\in H}[\alpha^a(\lambda), \phi(\lambda)X^b(\mu)]\alpha^b(\mu) \tag{10}$$

or owing to the fact that $(X^a(\lambda), \lambda > 0)$ is an exit family, we have

$$\alpha^a(\lambda) = \alpha^a(\mu) + \sum_{b\in H}[\alpha^a(\lambda), X^b(\lambda) - X^b(\mu)]\alpha^b(\mu) \tag{11}$$

Because of (10), $\alpha^a(\lambda)$ decreases as λ increases. Now let us proceed to prove that

$$\alpha^a(\lambda)\downarrow 0 \qquad \lambda\uparrow\infty \tag{12}$$

Actually since both $\psi(\lambda)$ and $\phi(\lambda)$ satisfy Q conditions it follows by (5) and (8) that

$$\lim_{\lambda\to\infty}\sum_{a\in H}\lambda X_i^a(\lambda)[\lambda\alpha^a(\lambda)\phi(\lambda)]_j = 0$$

Hence

$$\lim_{\lambda\to\infty}\lambda X_i^a(\lambda)\alpha_j^a(\lambda)\lambda\phi_{jj}(\lambda) = 0 \qquad a\in H$$

By the continuity condition of $\phi(\lambda)$, that is,

$$\delta_{ia}d_a \lim_{\lambda\to\infty}\alpha_j^a(\lambda)\delta_{jj} = 0$$

taking $i = a$ we obtain (12).

Because, if $\lambda > \mu$, by (9),

$$\sum_{b\in H}[\alpha^a(\lambda), X^b(\lambda)]\alpha^b(\mu) \leqslant \sum_{b\in H}[\alpha^a(\lambda), X^b(\mu)]\alpha^b(\mu)$$
$$\leqslant \sum_{b\in H}[\alpha^a(\mu), X^b(\mu)]\alpha^b(\mu) \leqslant \sum_{b\in H}[\alpha^a(\mu), X^b(\mu)]1/\eta_\mu$$
$$\leqslant \left[\alpha^a(\mu), \sum_{b\in H}X^b(\mu)\right]1/\eta_\mu \leqslant [\alpha^a(\mu), 1]1/\eta_\mu \leqslant 1/\eta_\mu^2 < \infty.$$

Therefore (11) may be written as

$$\alpha^a(\lambda) + \sum_{b\in H} [\alpha^a(\lambda), X^b(\mu)]\alpha^b(\mu) = \alpha^a(\mu) + \sum_{b\in H} [\alpha^a(\lambda), X^b(\lambda)]\alpha^b(\mu)$$

and, moreover, when $\lambda \to \infty$, by using the theorem of dominated convergence we have

$$0 + \sum_{b\in H} [0, X^b(\mu)]\alpha^b(\mu) = \alpha^a(\mu) + \sum_{b\in H} [0, 0]\alpha^b(\mu)$$

Therefore $\alpha^a(\mu) = 0$ $(a\in H, \mu > 0)$. Thus $F^a(\lambda) = 0$ $(a\in H, \lambda > 0)$. Hence $\psi(\lambda) = \phi(\lambda)$, and so the Q process is unique. The theorem is proved. QED

PART II CONSTRUCTION THEORY OF BIRTH–DEATH PROCESSES

In order to study further the construction theory of processes, it is very useful to study the construction of two kinds of special processes, i.e. bilateral birth–death processes and unilateral birth–death processes. The bilateral birth–death process that we consider is conservative; for the unilateral birth–death process there possibly exists a non-conservative state. The dimension of the solution space $\mathscr{M}_\lambda^+$ for the bilateral birth–death process is $m^+ \leqslant 2$; for the unilateral birth–death process, it is $m^+ \leqslant 1$. The birth–death processes are important because they have very important theoretical significance and application value. By studying them, many varied and deep results can be derived. More importantly, the study of birth–death processes is often the source that gives the ideas and methods for solving problems of general processes.

CHAPTER 5

Bilateral Birth–Death Processes

5.1 INTRODUCTION

If the state space E is taken to be the set of all integers, and $Q = (q_{ij})$ has the following form:

$$\begin{array}{lll} q_{ij} = 0 & \text{if } |i-j| > 1 & \\ q_{i,i-1} = a_i > 0 & q_{i,i+1} = b_i > 0 & q_i = -q_{ii} = a_i + b_i \end{array} \tag{1}$$

we call the Q processes bilateral birth–death processes.

In this chapter, the Q processes will always refer to bilateral birth–death processes. The bilateral birth–death processes are conservative and, therefore, they surely satisfy the system of backward equations.

In this chapter the construction problem of the bilateral birth–death processes is satisfactorily solved. In other words, all the bilateral birth–death processes are constructed. The bilateral birth–death processes are looked on as diffusion, and the method used is the analytical method. The content of this chapter is derived from Xiang-qun Yang (1964b)

5.2 NATURAL SCALE AND STANDARD MEASURE

For Q of the form (1.1), we call

$$\begin{aligned} z_i &= -b_0\left(1 + \frac{b_{-1}}{a_{-1}} + \frac{b_{-1}b_{-2}}{a_{-1}a_{-2}} + \cdots + \frac{b_{-1}b_{-2}\cdots b_{i+1}}{a_{-1}a_{-2}\cdots a_{i+1}}\right) \qquad \text{if } i < -1 \\ z_{-1} &= -b_0 \qquad z_0 = 0 \qquad z_1 = a_0 \\ z_i &= a_0\left(1 + \frac{a_1}{b_1} + \frac{a_1a_2}{b_1b_2} + \cdots + \frac{a_1a_2\cdots a_{i-1}}{b_1b_2\cdots b_{i-1}}\right) \qquad \text{if } i > 1 \end{aligned} \tag{1}$$

the natural scale, we call

$$r_1 = \lim_{i \to -\infty} z_i \qquad r_2 = \lim_{i \to +\infty} z_i \tag{2}$$

the boundary points and we call

$$\mu_i = \frac{a_{-1}a_{-2}\cdots a_{i+1}}{b_0 b_{-1} b_{-2}\cdots b_{i+1} b_i} \qquad \text{if } i < -1$$

$$\mu_{-1} = \frac{1}{b_0 b_{-1}} \qquad \mu_0 = \frac{1}{a_0 b_0} \qquad \mu_1 = \frac{1}{a_0 a_1}$$

$$\mu_i = \frac{b_1 b_2 \cdots b_{i-1}}{a_0 a_1 a_2 \cdots a_{i-1} a_i} \qquad \text{if } i > 1 \tag{3}$$

the canonical measure.

5.3 CLASSIFICATION OF BOUNDARY POINTS

The boundary points may be classified by means of the natural scale and the canonical measure. The boundary point r_2 is said to be

(a) regular if r_2 is finite and $\sum_{i\geq 0}\mu_i$ is finite;
(b) exit if r_2 is irregular but finite, and $\sum_{i\geq 0}(r_2 - z_i)\mu_i$ is finite;
(c) entrance if r_2 is irregular, but $\sum_{i\geq 0} z_i\mu_i$ is finite; and
(d) natural for all other cases.

Similarly, r_1 may be classified.

If we set

$$R_1 = \sum_{i\leq 0}(z_i - r_1)\mu_i = \sum_{i\leq 0}(z_i - z_{i-1})\sum_{i\leq j\leq 0}\mu_j$$

$$S_1 = -\sum_{i\leq 0} z_i \mu_i$$

$$R_2 = \sum_{i\geq 0}(r_2 - z_i)\mu_i = \sum_{i\geq 0}(z_{i+1} - z_i)\sum_{0\leq j\leq i}\mu_j$$

$$S_2 = \sum_{i\geq 0} z_i\mu_i \tag{1}$$

then we find that r_2 is finite when R_2 is finite; and that $\sum_{i\geq 0}\mu_i$ is finite when S_2 is finite. Therefore, the above classification of the boundary points is comprehensive and, furthermore, if r_a is entrance then r_a is finite.

Theorem 1. The boundary point r_a is

(a) regular if and only if $R_a < \infty$, $S_a < \infty$;
(b) exit if and only if $R_a < \infty$, $S_a = \infty$;
(c) entrance if and only if $R_a = \infty$, $S_a < \infty$;
(d) natural if and only if $R_a = \infty$, $S_a = \infty$.

Proof. Prove the theorem for $a = 2$. Obviously,

$$R_2 \leq r_2 \sum_{i\geq 0}\mu_i \qquad S_2 \leq r_2 \sum_{i\geq 0}\mu_i$$

Hence if r_2 is regular, then $R_2 < \infty$, $S_2 < \infty$. Conversely, if $R_2 < \infty$, $S_2 < \infty$, it follows that r_2 must be finite by the definition of R_2. Besides

$$\sum_{i\geq 0}\mu_i = \frac{1}{r_2}(S_2 + R_2) < \infty$$

If r_2 is exit, by the definition we obtain $R_2 < \infty$, $\sum_{i\geq 0}\mu_i = \infty$. From $S_2 \geq z_1\sum_{i\geq 1}\mu_i$ follows $S_2 = \infty$. Conversely, if $R_2 < \infty$, $S_2 = \infty$. By what is proved in the first section it follows obviously that r_2 is irregular. However by $R_2 < \infty$ it follows obviously that r_2 is finite. Therefore, r_2 is exit.

If r_2 is entrance, by the definition we obtain $S_2 < \infty$; owing to the fact that r_2 is irregular and according to what is proved in the first section, we surely get $R_2 = \infty$. Conversely, if $R_2 = \infty$, $S_2 < \infty$, then by what is verified in the first section it follows that r_2 is irregular. Therefore r_2 is entrance.

By the proof of the above three sections it follows immediately that for r_2 to be natural it is necessary and sufficient that $R_2 = \infty$, $S_2 = \infty$, and the proof is complete. QED

5.4 SECOND-ORDER DIFFERENCE OPERATOR

Let μ be the column vector on E, and define u^+ and $D_\mu u^+$ as follows:

$$u_i^+ = \frac{u_{i+1} - u_i}{z_{i+1} - z_i} \qquad i\in E$$
$$(D_\mu u^+)_i = \frac{u_i^+ - u_{i-1}^+}{\mu_i} \qquad i\in E \tag{1}$$

Theorem 1. For an arbitrary column vector u,

$$Qu = D_\mu u^+ \tag{2}$$

that is,

$$a_i u_{i-1} - (a_i + b_i)u_i + b_i u_{i+1} = (D_\mu u^+)_i \tag{3}$$

Proof. Since

$$a_i = \frac{1}{(z_i - z_{i-1})u_i}$$

$$b_i = \frac{1}{(z_{i+1} - z_i)u_i} \tag{4}$$

it follows that

$$
\begin{aligned}
(D_\mu u^+)_i &= \left(\frac{u_{i+1}-u_i}{z_{i+1}-z_i} - \frac{u_i-u_{i-1}}{z_i-z_{i-1}}\right)\Big/ u_i \\
&= b_i(u_{i+1}-u_i) - a_i(u_i-u_{i-1}) \\
&= a_i u_{i-1} - (a_i+b_i)u_i + b_i u_{i+1}
\end{aligned}
$$

and the proof is concluded. QED

Let u be a column vector; $u\mu$ represents the row vector with components $u_j\mu_j$. Conversely, if v is a row vector, $v\mu^{-1}$ represents the column vector with components $v_i\mu_i^{-1}$.

Theorem 2. Assume that v is a row vector and $u = v\mu^{-1}$. Then

$$vQ = (Qu)\mu \tag{5}$$

Proof. By observing

$$\mu_{i-1}b_{i-1}\mu_i^{-1} = a_i \qquad a_{i+1}\mu_{i+1}\mu_i^{-1} = b_i \tag{6}$$

we see that

$$
\begin{aligned}
(Qu)_i &= a_i v_{i-1}\mu_{i-1}^{-1} - (a_i+b_i)v_i\mu_i^{-1} + b_i v_{i+1}\mu_{i+1}^{-1} \\
&= v_{i-1}b_{i-1}\mu_i^{-1} - (a_i+b_i)v_i\mu_i^{-1} + v_{i+1}a_{i+1}\mu_i^{-1} \\
&= [v_{i-1}b_{i-1} - v_i(a_i+b_i) + v_{i+1}a_{i+1}]\mu_i^{-1} \\
&= (vQ)_i\mu_i^{-1}
\end{aligned}
$$

QED

Corollary

Suppose that u and f are column vectors, and that $v = u\mu$ and $g = f\mu$. Then u satisfies

$$Qu = f \tag{7}$$

or

$$\lambda u - Qu = f \qquad \lambda > 0 \tag{8}$$

if and only if v satisfies

$$vQ = g \tag{9}$$

or

$$\lambda v - vQ = g \tag{10}$$

Lemma 3. The solution of the system of equations

$$
\begin{aligned}
&u_i = f_i \\
&a_k u_{k-1} - (a_k+b_k)u_k + b_k u_{k+1} = -f_k \qquad i<k<n \\
&u_n = f_n
\end{aligned}
\tag{11}
$$

is

$$u_k = f_i\frac{z_n-z_k}{z_n-z_i} + f_n\frac{z_k-z_i}{z_n-z_i} + \frac{z_n-z_k}{z_n-z_i}\sum_{j=i+1}^{k-1}(z_i-z_j)f_j u_j + \frac{z_k-z_i}{z_n-z_i}\sum_{j=k}^{n-1}(z_n-z_j)f_j u_j \tag{12}$$

Proof. By Theorem 1, the system (11) of equations becomes

$$
\begin{aligned}
&u_i = f_i \\
&u_k^+ - u_{k-1}^+ = -f_k\mu_k \qquad i<k<n \\
&u_n = f_n
\end{aligned}
\tag{13}
$$

By the above expression we obtain

$$u_k^+ = u_i^+ + \sum_{l=i+1}^{k}(u_l^+ - u_{l-1}^+) = u_i^+ - \sum_{l=i+1}^{k} f_l\mu_l$$

and from this we get

$$
\begin{aligned}
u_k &= u_i + \sum_{l=i}^{k-1}(u_{l+1}-u_l) = u_i + \sum_{l=i}^{k-1}u_l^+(z_{l+1}-z_l) \\
&= u_i + \sum_{l=i}^{k-1}u_i^+(z_{l+1}-z_l) - \sum_{l=i}^{k-1}\left(\sum_{j=i+1}^{l} f_j u_j\right)(z_{l+1}-z_l) \\
&= f_i + u_i^+(z_k-z_i) - \sum_{j=i+1}^{k-1}\sum_{l=i}^{k-1} f_j u_j(z_{l+1}-z_l)
\end{aligned}
$$

Therefore

$$u_k = f_i + u_i^+(z_k-z_i) - \sum_{j=i+1}^{k-1}(z_k-z_j)f_j\mu_j \tag{14}$$

In particular, if $k = n$, we have

$$f_n = f_i + u_i^+(z_n-z_i) - \sum_{j=i+1}^{n-1}(z_n-z_j)f_j\mu_j$$

so that

$$u_i^+ = \frac{f_n-f_i}{z_n-z_i} + \frac{1}{z_n-z_i}\sum_{j=i+1}^{n-1}(z_n-z_j)f_j\mu_j \tag{15}$$

Upon substituting (15) into (14) and rearranging it, we obtain (12), and the proof is terminated. QED

5.5 SOLUTION OF THE EQUATION $\lambda u - D_\mu u^+ = 0$

Theorem 1. Let u and v be two solutions of the equation

$$\lambda u - D_\mu u^+ = 0 \qquad \lambda > 0 \tag{1}$$

Then

$$W(u, v) \equiv u^+ v - u v^+ = \text{constant} \tag{2}$$

Proof. First note that for arbitrary vectors s, t

$$\begin{aligned} s_i t_i - s_{i-1} t_{i-1} &= s_i(t_i - t_{i-1}) + t_{i-1}(s_i - s_{i-1}) \\ &= s_{i-1}(t_i - t_{i-1}) + t_i(s_i - s_{i-1}) \end{aligned}$$

so that

$$\begin{aligned} [D_\mu(st)]_i &= s_i(D_\mu t)_i + t_{i-1}(D_\mu s)_i \\ &= s_{i-1}(D_\mu t)_i + t_i(D_\mu s)_i \end{aligned} \tag{3}$$

Therefore

$$\begin{aligned} D_\mu W(u, v) &= D_\mu(u^+ v) - D_\mu(u v^+) \\ &= v_i(D_\mu u^+)_i + u^+_{i-1}(D_\mu v)_i \\ &\quad - u_i(D_\mu v^+)_i - v^+_{i-1}(D_\mu u)_i \\ &= \lambda v_i u_i + u^+_{i-1}(D_\mu v)_i - \lambda u_i v_i - v^+_{i-1}(D_\mu u)_i \\ &= \frac{u_i - u_{i-1}}{z_i - z_{i-1}} \frac{v_j - v_{i-1}}{u_i} - \frac{v_i - v_{i-1}}{z_i - z_{i-1}} \frac{u_i - u_{i-1}}{u_i} \\ &= 0 \end{aligned}$$

QED

Equation (1) can be rewritten as follows:

$$u^+_i - u^+_{i-1} = \lambda u_i \mu_i \qquad \lambda > 0 \tag{4}$$

Consequently if u is a solution of equation (1), then for $i > 0$

$$\begin{aligned} u^+_i &= u^+_0 + \sum_{k=1}^{i} (u^+_k - u^+_{k-1}) \\ &= u^+_0 + \lambda \sum_{k=1}^{i} u_k \mu_k \qquad i > 0 \end{aligned} \tag{5}$$

$$\begin{aligned} u_i &= u_0 + \sum_{k=0}^{i-1} (u_{k+1} - u_k) \\ &= u_0 + \sum_{k=0}^{i-1} u^+_k (z_{k+1} - z_k) \qquad i > 0 \end{aligned} \tag{6}$$

Substituting (5) into (6) and rearranging, we obtain

$$u_i = u_0 + u^+_0(z_i - z_0) + \lambda \sum_{k=1}^{i-1} u_k(z_i - z_k)\mu_k \qquad i > 0 \tag{7}$$

Similarly, for $i < 0$

$$u^+_i = u^+_0 - \lambda \sum_{i+1 \leq k \leq 0} u_k \mu_k \qquad i < 0 \tag{8}$$

$$u_i = u_0 - u^+_0(z_0 - z_i) + \lambda \sum_{i+1 \leq h < 0} u_k(z_k - z_i)\mu_k \qquad i < 0 \tag{9}$$

Conversely, arbitrarily fixing the values of u_0, u^+_0, we can determine u_1, u^+_1, u_2, $u^+_2, \ldots$, by (5) and (6); similarly, we can determine u_{-1}, u^+_{-1}, u_{-2}, $u^+_{-2}, \ldots$. Obviously u determined in this manner is a solution of equation (1).

Given $u_0 = 1$ and $u^+_0 = 0$ the solution of equation (1) is denoted by v. Given $u_0 = 0$, $u^+_0 = 1$ we denote the solution of equation (1) by s. From (7) and (9) we can see that v_i and s_i are positive and strictly increasing as $0 < i \uparrow +\infty$; and that v_i and $-s_i$ are positive and strictly increasing with increase of the absolute value of i, as $0 > i \downarrow -\infty$. Moreover, by Theorem 1,

$$W(s, v) = s^+_0 v_0 - s_0 v^+_0 = 1 \tag{10}$$

Lemma 2. When $i > 0$,

$$v_i/s_i > v^+_i/s^+_i$$

Proof.

$$\frac{v_i}{s_i} - \frac{v^+_i}{s^+_i} = \frac{W(s, v)}{s_i s^+_i} = \frac{1}{s_i s^+_i} > 0$$

QED

Lemma 3. v_i/s_i is strictly decreasing as $0 < i \uparrow +\infty$.

Proof.

$$\begin{aligned} \left(\frac{v}{s}\right)^+_i &= \left(\frac{v_{i+1}}{s_{i+1}} - \frac{v_i}{s_i}\right) \frac{1}{z_{i+1} - z_i} \\ &= -\frac{W(s, v)}{s_i s_{i+1}} < 0 \end{aligned}$$

QED

Lemma 4. v^+_i/s^+_i is strictly increasing as $0 < i \uparrow +\infty$.

Proof.

$$\begin{aligned} \left[D_\mu\left(\frac{v^+}{s^+}\right)\right]_i &= \left(\frac{v^+_i}{s^+_i} - \frac{v^+_{i-1}}{s^+_{i-1}}\right)\mu_i^{-1} \\ &= \frac{s^+_{i-1} v^+_i - s^+_i v^+_{i-1}}{s^+_i s^+_{i-1} \mu_i} \end{aligned}$$

$$= \frac{s_{i-1}^+(D_\mu v^+)_i - v_{i-1}^+(D_\mu s^+)_i}{s_i^+ s_{i-1}^+}$$

$$= \frac{\lambda(s_{i-1}^+ v_i - v_{i-1}^+ s_i)}{s_i^+ s_{i-1}^+}$$

$$= \frac{\lambda[(s_i^+ - \lambda s_i)v_i - (v_i^+ - \lambda v_j)s_i]}{s_i^+ s_{i-1}^+}$$

$$= \frac{\lambda W(s,v)}{s_i^+ s_{i-1}^+} = \frac{\lambda}{s_i^+ s_{i-1}^+} > 0 \qquad \text{QED}$$

Lemma 5. Assume that u is a solution of equation (1), and that u_i is positive and strictly increasing as $0 < i \uparrow +\infty$. Then u_i^+ is also strictly increasing as $0 < i \uparrow +\infty$. $u(r_2) = \lim_{i\to\infty} u_i < \infty$ if and only if r_2 is regular or exit; $u^+(r_2) = \lim_{i\to\infty} u_i^+ < \infty$ if and only if r_2 is regular or entrance.

Proof. From (5) it can be seen that u_i^+ $(i > 0)$ is positive and strictly increasing.
Suppose $u(r_2) < \infty$. By (7) we have

$$u_i > \lambda u_0 \sum_{k=1}^{i-1} (z_i - z_k)\mu_k \tag{11}$$

We see $R_2 < \infty$ as $i \to +\infty$, that is, r_2 is regular or exit. Conversely, assume $R_2 < \infty$. By (5)

$$u_{i+1} - u_i < u_0^+(z_{i+1} - z_i) + \lambda u_i(z_{i+1} - z_i)\sum_{k=1}^{i} \mu_k$$

$$\frac{u_{i+1}}{u_i} - 1 < \frac{u_0^+}{u_0}(z_{i+1} - z_i) + \lambda(z_{i+1} - z_i)\sum_{k=1}^{i} \mu_k$$

The right-hand side of the above expression is the ith term of a convergent series, so that $\sum_{i>0} \log(u_{i+1}/u_i)$ converges. Thus $\lim_{i\to+\infty} u_i < \infty$. Suppose $u^+(r_2) < \infty$. By (7) we have $u_i > u_0^+ z_i$. Therefore, by (5)

$$u_i^+ > u_0^+ + u_0^+ \lambda \sum_{k=1}^{i} z_k \mu_k$$

Therefore $s_2 < \infty$, that is, r_2 is regular or entrance. Conversely if $s_2 < \infty$, by (6)

$$u_i < u_0 + u_{i-1}^+(z_i - z_0) = u_0 + u_{i-1}^+ z_i$$

so that

$$u_i^+ - u_{i-1}^+ = \lambda u_i \mu_i < \lambda u_0 \mu_i + \lambda u_{i-1}^+ z_i \mu_i$$

$$\frac{u_i^+}{u_{i-1}^+} - 1 < \frac{\lambda u_0}{u_0^+} u_i + \lambda z_i \mu_i$$

Since $s_2 < \infty$ implies $\sum_{i\geq 0} \mu_i < \infty$, it follows from the above expression that $\sum_{i\geq 0} \log(u_i^+/u_{i-1}^+)$ converges. Therefore, $\lim_{i\to+\infty} u_i^+ < \infty$, and the proof is terminated. QED

Lemma 6. When $0 < i \uparrow +\infty$

$$\frac{v_i}{s_i} - \frac{v_i^+}{s_i^+} = \frac{1}{s_i s_i^+} \to \begin{cases} 0 & \text{if } r_2 \text{ is not regular;} \\ c > 0 & \text{if } r_2 \text{ is regular.} \end{cases}$$

Proof. It can be deduced from Lemmas 2 and 5.

By Lemmas 2 to 4, we can set

$$\bar{\theta} = \lim_{i\to+\infty} \frac{v_i}{s_i} \qquad \underline{\theta} = \lim_{i\to+\infty} \frac{v_i^+}{s_i^+} \tag{12}$$

Moreover, $\underline{\theta} \leq \bar{\theta}$. $\underline{\theta} = \bar{\theta}$ if and only if r_2 is irregular. QED

Theorem 7. For u to be a positive strictly decreasing solution of equation (1), satisfying the condition $u_0 = 1$, it is necessary and sufficient that u has the following form:

$$u = v - \theta s \tag{13}$$

where $\underline{\theta} \leq \theta \leq \bar{\theta}$. If r_2 is regular, then there exist infinitely many solutions stated above and they are between $\underline{u} = v - \bar{\theta}s$ and $\bar{u} = v - \underline{\theta}s$. If r_2 is irregular, the above mentioned solution u is unique.

Proof. Since v and s are two linearly independent solutions, it follows that every solution u is a linear combination of v and s, so that the solution u satisfying $u_0 = 1$ must have the form (13).
Suppose that u is a strictly decreasing positive solution, then $u = v - \theta s > 0$, $u^+ = v^+ - \theta s^+ > 0$ and, therefore, $\underline{\theta} \leq \theta \leq \bar{\theta}$. Conversely, if $\underline{\theta} \leq \theta \leq \bar{\theta}$, then $u_i > 0$ and $u_i^+ < 0$ for $i > 0$. That is, u is positive and strictly decreasing in $i > 0$. In $i < 0$, since v and $(-s)$ are positive and strictly increasing with the increase of absolute value of i as $0 \geq i \downarrow -\infty$, it follows that u is positive and strictly decreasing on E, and the proof is concluded. QED

Lemma 8. For $u, \underline{u}$ and $\bar{u}$ in Theorem 7,

$$u(r_2) = \begin{cases} 0 & \text{if } r_2 \text{ is exit or natural; when } r_2 \text{ is regular } \underline{u}(r_2) = 0 \\ 1/s^+(r_2) & \text{if } r_2 \text{ is entrance; when } r_2 \text{ is regular, } \bar{u}(r_2) = 1/s^+(r_2) \end{cases}$$

$$u^+(r_2) = \begin{cases} 0 & \text{if } r_2 \text{ is entrance or natural; when } r_2 \text{ is regular, } \bar{u}^+(r_2) = 0 \\ -1/s(r_2) & \text{if } r_2 \text{ is exit; when } r_2 \text{ is regular, } \underline{u}^+(r_2) = -1/s(r_2) \end{cases}$$

Proof. When r_2 is exit or regular, by Lemma 5 we have $v(r_2) < \infty$, $s(r_2) < \infty$. Therefore, $\underline{u}(r_2) = v(r_2) - \bar{\theta}s(r_2) = 0$.

When r_2 is regular,

$$\bar{u}(r_2) = v(r_2) - \underline{\theta}s(r_2) = \frac{v(r_2)s^+(r_2) - v^+(r_2)s(r_2)}{s^+(r_2)}$$

$$= \frac{1}{s^+(r_2)}$$

When r_2 is entrance or natural, by

$$u_i = \bar{u}_i = v_i - \underline{\theta}s_i \leqslant v_i - \frac{v_i^+}{s_i^+}s_i = \frac{1}{s_i^+}$$

If r_2 is natural, then since $s^+(r_2) = \infty$, it follows that $u(r_2) = \bar{u}(r_2) = 0$. If r_2 is entrance, then

$$u(r_2) = \bar{u}(r_2) \leqslant 1/s^+(r_2)$$

For an arbitrary $\varepsilon > 0$, when i is sufficiently large,

$$u(r_2) + \varepsilon > v_i - \underline{\theta}s_i$$

Upon fixing i, when $j(> i)$ is large enough,

$$u(r_2) + \varepsilon > v_i - \frac{v_j^+}{s_j^+}S_i$$

But when j is fixed,

$$\left(v - \frac{v_j^+}{s_j^+}s\right)_j^+ = v_i^+ - \frac{v_j^+}{s_j^+}s_i^+ = \left(\frac{v_i^+}{s_i^+} - \frac{v_j^+}{s_j^+}\right)s_i^+ < 0$$

so that

$$u(r_2) + \varepsilon > v_j - \frac{v_j^+}{s_j^+}s_j = \frac{1}{s_j^+} \to \frac{1}{s^+(r_2)}$$

Since ε is arbitrary, $u(r_2) \geqslant 1/s^+(r_2)$. Therefore, when r_2 is entrance,

$$u(r_2) = 1/s^+(r_2).$$

The proof for $u^+(r_2)$ is similar and can be left out. The proof is complete. QED

Theorem 9. In equation (1) there exist a strictly decreasing positive solution $u_1(\lambda)$ and a strictly increasing positive solution $u_2(\lambda)$ which have the following properties:

(i) $u_1^+(\lambda) < 0$ is strictly increasing; $u_2^+(\lambda) > 0$ is strictly increasing

$$u_2^+(\lambda)u_1(\lambda) - u_2(\lambda)u_1^+(\lambda) = 1 \qquad \lambda > 0 \tag{14}$$

(ii) $u_a(r_a, \lambda) \equiv \lim_{z_i \to r_a} u_{ai}(\lambda)$ is finite if and only if r_a is regular or exit $u_a^+(r_a, \lambda) = \lim_{z_i \to r_a} u_{ai}(\lambda)$ is finite, or equivalently $\sum_i u_{ai}(\lambda)\mu_i < \infty$ if and only if r_a is regular or entrance.

(iii) If r_a is not entrance, then $u_b(r_a, \lambda) = 0$ $(b \neq a)$. If r_a is entrance or natural, then $u_b^+(r_a, \lambda) = 0$ $(b \neq a)$.

Proof. By Theorem 7, it follows that the strictly decreasing positive solution $u_1(\lambda)$ of equation (1) exists; similarly, the strictly increasing positive solution $u_2(\lambda)$ also exists. By (5) and (8) it can be seen that both $u_1^+(\lambda) < 0$ and $u_2^+(\lambda) > 0$ are strictly increasing. Obviously, $W(u_2(\lambda), u_1(\lambda)) > 0$. And (14) is satisfied after properly normalizing.

We shall prove (ii), (iii) for $= 2$. By Lemma 5 we deduce (ii). However,

$$\lambda \sum_i u_{2i}(\lambda)\mu_i = u_2^+(r_2, \lambda) - u_2^+(r_1, \lambda) \tag{15}$$

Obviously, $u_2^+(r_1, \lambda)$ is finite, therefore, that $u_2^+(r_2, \lambda)$ is finite is equivalent to $\sum_i u_{2i}(\lambda)u_i < \infty$. By Lemma 8, $u_1(\lambda)$ and $u_2(\lambda)$ can be selected such that (iii) is satisfied and the proof is complete. QED

Theorem 10. Both $u_1(\lambda)\mu$ and $u_2(\lambda)\mu$ are two linearly independent solutions of the equation

$$(V_\lambda) \qquad \lambda v - vQ = 0 \qquad \lambda > 0 \tag{16}$$

Each solution of the equation (v_λ) is their linear combination.

Proof. This proof follows from the corollary to Theorem 4.2. QED

5.6 MINIMAL SOLUTION

From now on, we shall denote the solutions of Theorem 5.9. by $u_1(\lambda)$ and $u_2(\lambda)$. Set

$$\phi_{ij}(\lambda) = \begin{cases} u_{2i}(\lambda)u_{1j}(\lambda)\mu_j & \text{if } i \leqslant j \\ u_{1i}(\lambda)u_{2j}(\lambda)\mu_j & \text{if } i > j \end{cases} \tag{1}$$

Then

$$\mu_i\phi_{ij}(\lambda) = \mu_j\phi_{ji}(\lambda) \tag{2}$$

Let f be a column vector, and g be a row vector. Then

$$[\phi(\lambda)f]_i = \sum_j \phi_{ij}(\lambda)f_j = u_{1i}(\lambda)\sum_{j \leqslant i} u_{2j}(\lambda)f_j\mu_j + u_{2i}(\lambda)\sum_{j > i} u_{1j}(\lambda)f_j\mu_j \tag{3}$$

$$[g\phi(\lambda)]_j = \sum_i g_i\phi_{ij}(\lambda) = u_{1j}(\lambda)u_j\sum_{i \leqslant j} g_iu_{2i}(\lambda) + u_{2j}(\lambda)\mu_j\sum_{i > j} g_iu_{1i}(\lambda) \tag{4}$$

If $g = f\mu$, then

$$g\phi(\lambda) = [\phi(\lambda)f]\mu \tag{5}$$

Theorem 1. The equation

$$\lambda \sum_i \phi_{ij}(\lambda) = 1 - \frac{u_{1i}(\lambda)}{u_1(r_1, \lambda)} - \frac{u_{2j}(\lambda)}{u_2(r_2, \lambda)} \tag{6}$$

holds. (If $u_a(r_a, \lambda) = \infty$, then the corresponding term in (6) is understood as zero.)

Proof. By (3) and (5.4)

$$\lambda \sum_j \phi_{ij}(\lambda) = u_{1i}(\lambda) \sum_{j \leqslant i} \lambda u_{2j}(\lambda)\mu_j + u_{2i}(\lambda) \sum_{j > i} \lambda u_{1j}(\lambda)\mu_j$$

$$= u_{1i}(\lambda) \sum_{j \leqslant i} [u_{2j}^+(\lambda) - u_{2j-1}^+(\lambda)] + u_{2i}(\lambda) \sum_{j > i} [u_{1j}^+(\lambda) - u_{1,j-1}^+(\lambda)]$$

$$= u_{1i}(\lambda)[u_{2i}^+(\lambda) - u_2^+(r_1, \lambda)] + u_{2i}(\lambda)[u_1^+(r_2, \lambda) - u_{1i}^+(\lambda)]$$

$$= u_{1i}(\lambda)u_{2i}^+(\lambda) - u_{2i}(\lambda)u_{1i}^+(\lambda) - u_{1i}(\lambda)u_2^+(r_1, \lambda) + u_{2i}(\lambda)u_1^+(r_2, \lambda) \tag{7}$$

If we can prove

$$u_1^+(r_2, \lambda) = -\frac{1}{u_2(r_2, \lambda)}, \qquad u_2^+(r_1, \lambda) = \frac{1}{u_1(r_1, \lambda)} \tag{8}$$

then by (5.14), follows (6) from (7).

We prove only the first expression of (8). In fact, if r_2 is entrance or natural, by Theorem 5.9(ii), (iii), we obtain $u_2(r_2, \lambda) = \infty$, $u_1^+(r_2, \lambda) = 0$. Therefore, the first expression of (8) obviously holds. If r_2 is regular or exit, by (5.14), we only need prove

$$\lim_{x_i \to r_2} u_{1i}(\lambda)u_{2i}^+(\lambda) = 0 \tag{9}$$

When r_2 is regular, obviously the above expression holds since $u_2^+(r_2, \lambda) < \infty$, $u_1(r_2, \lambda) = 0$. When r_2 is exit, since $u_1(r_2, \lambda) = 0$ and $u_2^+(\lambda)$ is increasing, $-u_1^+(\lambda)$ is decreasing, it follows that

$$0 \leqslant u_{1i}(\lambda)u_{2i}^+(\lambda) = u_{2i}^+(\lambda)[u_{1i}(\lambda) - u_1(r_2, \lambda)]$$

$$= u_{2i}^+(\lambda) \sum_{j \geqslant i} [-u_{1j}^+(\lambda)(z_{j+1} - z_j)]$$

$$\leqslant -u_{1i}^+(\lambda) \sum_{j \geqslant i} u_{2j}^+(\lambda)(z_{j+1} - z_j)$$

$$= -u_{1i}^+(\lambda)[u_2(r_2, \lambda) - u_{2i}(\lambda)]$$

$$\to -u_1^+(r_2, \lambda)[u_2(r_2, \lambda) - u_2(r_2, \lambda)]$$

$$= 0 \qquad (z_i \to r_2)$$

and the proof is terminated. QED

Theorem 2. If $f \in m$, $g \in l$ then $\phi(\lambda)f \in m$, $g\phi(\lambda) \in l$ and

$$\lambda\phi(\lambda)f - Q[\phi(\lambda)f] = f \qquad \lambda > 0 \tag{10}$$

$$\lambda g\phi(\lambda) - [g\phi(\lambda)]Q = g \qquad \lambda > 0 \tag{11}$$

Proof. By Theorem 1 we can get that $\phi(\lambda)f \in m$, $g\phi(\lambda) \in l$. On account of (5) and the corollary to Theorem 4.2, we only need to prove (10). In fact, by (3) we have

$$[\phi(\lambda)f]_i^+ = u_{1i}^+(\lambda) \sum_{j \leqslant i} u_{2j}(\lambda)f_j\mu_j + u_{2i}^+(\lambda) \sum_{j > i} u_{1j}(\lambda)f_j\mu_j \tag{12}$$

$$D_\mu[\phi(\lambda)f]_i^+ = D_\mu u_{1i}^+(\lambda) \sum_{j \leqslant i} u_{2j}(\lambda)f_j u_j + D_\mu u_{2i}^+(\lambda) \sum_{j > i} u_{1j}(\lambda)f_j\mu_j$$

$$+ [u_{1,i-1}^+(\lambda)u_{2i}(\lambda) - u_{2,i-1}^+(\lambda)u_{1i}(\lambda)]f_i$$

$$= \lambda u_{1i}(\lambda) \sum_{j \leqslant i} u_{2j}(\lambda)f_j\mu_j + \lambda\mu_{2i}(\lambda) \sum_{j > i} u_{1j}(\lambda)f_j\mu_j$$

$$- [u_{2i}^+(\lambda)u_{1i}(\lambda) - u_{2i}(\lambda)u_{1i}^+(\lambda)]f_i = \lambda[\phi(\lambda)f]_i - f_i \qquad \text{QED}$$

Lemma 3. Let $f \in m$ and r_a be regular or exit. Then

$$[\phi(\lambda)f](r_a) = \lim_{z_i \to r_a} [\phi(\lambda)f]_i = 0 \tag{13}$$

Proof. Since $u_a(r_a, \lambda) < \infty$ when r_a is regular or exit and

$$\frac{u_{ai}(\lambda)}{u_a(r_a, \lambda)} \to 1 \qquad u_{bi}(\lambda) \to 0 \qquad (b \neq a) \tag{14}$$

as $z_i \to r_a$, by Theorem 1, $[\phi(\lambda)1](r_a) = 0$. Equation (13) follows from this. QED

Theorem 4. $\phi(\lambda)$ is the minimal Q process. For $\phi(\lambda)$ to be honest it is necessary and sufficient that both r_1 and r_2 are entrance or natural.

Proof. $\phi(\lambda) \geqslant 0$ is trivial. The norm condition follows from (6), and for the equality of (6) to hold it is necessary and sufficient that both r_1 and r_2 are

entrance or natural. From (10) and (11) it can be seen that the B condition and F condition for $\phi(\lambda)$ hold.

Let $f \in m$ and set $F(\lambda) = \phi(\lambda)f$. By Theorem 2 it follows that $F(\lambda) - F(\mu) + (\lambda - \mu)\phi(\lambda)F(\mu) \in m$ and is the solution of equation (5.1). Therefore, it is a linear combination of $u_1(\lambda)$ and $u_2(\lambda)$, that is

$$F(\lambda) - F(\mu) + (\lambda - \mu)\phi(\lambda)F(\mu) = c_1 u_1(\lambda) + c_2 u_2(\lambda) \tag{15}$$

where c_1 and c_2 are constants.

If r_2 is regular or exit, by Lemma 3, letting $z_i \to r_2$ in (15), we obtain

$$0 = c_1 u_1(r_2, \lambda) + c_2 u_2(r_2, \lambda) = c_2 u_2(r_2, \lambda)$$

Therefore, $c_2 = 0$. If r_2 is entrance or natural, since $u_2(r_2, \lambda) = \infty$, $u_1(r_2, \lambda) < \infty$ and the left-hand side of (15) is bounded, then $c_2 = 0$ follows, too. Similarly, we can prove $c_1 = 0$. Thus (15) becomes

$$F(\lambda) - F(\mu) + (\lambda - \mu)\phi(\lambda)F(\mu) = 0$$

Letting $f_i = \delta_{ij}$, we find the above expression becomes the resolvent equation for $\phi(\lambda)$.

Thus $\phi(\lambda)$ is a Q process satisfying the system of backward and forward equations. Next we proceed to prove the minimality of $\phi(\lambda)$.

Let $\psi(\lambda)$ be a Q process. Since Q is conservative, and $\psi(\lambda)$ satisfies the B condition, it follows that, for fixed j, $\psi_{ij}(\lambda) - \phi_{ij}(\lambda)$ is a solution of equation (5.1), so that

$$\psi_{ij}(\lambda) - \phi_{ij}(\lambda) = c_1 u_{1i}(\lambda) + c_2 u_{2i}(\lambda) \tag{16}$$

where c_1 and c_2 are constants independent of i.

If r_2 is regular or exit, by $\psi(\lambda) \geqslant 0$ and Lemma 3, after letting $z_i \to r_2$ in (16) it follows that $c_1 u_1(r_2, \lambda) + c_2 u_2(r_2, \lambda) = c_2 u_2(r_2, \lambda) \geqslant 0$. Therefore, $c_2 \geqslant 0$. If r_2 is natural or entrance, then the left-hand side of (16) is bounded; but $u_1(r_2, \lambda) < \infty$, $u_2(r_2, \lambda) = \infty$, and it follows that $c_2 = 0$. Therefore, we always have $c_2 \geqslant 0$. similarly, we can prove $c_1 \geqslant 0$. Hence $\psi(\lambda) \geqslant \phi(\lambda)$, and the proof is complete.

QED

5.7 SEVERAL LEMMAS

From now on, we shall simply write

$$X_i^1(\lambda) = \frac{u_{1i}(\lambda)}{u_1(r_1, \lambda)} \qquad X_i^2(\lambda) = \frac{u_{2i}(\lambda)}{u_2(r_2, \lambda)} \tag{1}$$

$$X_i^1 = \frac{r_2 - z_i}{r_2 - r_1} \qquad X_i^2 = \frac{z_i - r_1}{r_2 - r_1} \tag{2}$$

When r_a is regular or exit, $X^a(\lambda) \neq 0$. When r_a is entrance or natural, $X^a(\lambda) = 0$. If r_1 is finite, and r_2 is infinite, we shall define $X^1 = 1$. If r_1 and r_2 are infinite, we shall define $X^1 = 0$.

We can make similar comments for X^2. X^a is a solution of equation (2.11.18). Equation (6.6) becomes

$$\lambda\phi(\lambda)1 = 1 - X^1(\lambda) - X^2(\lambda) \tag{3}$$

Lemma 1. $X^a(\lambda) \in \mathscr{M}_\lambda^+(1)(a = 1, 2)$ are exit families, and

$$\lambda\phi(\lambda)X^a = X^a - X^a(\lambda) \qquad a = 1, 2 \tag{4}$$

Proof. Obviously, $X^a(\lambda) \in \mathscr{M}_\lambda^+(1)$. From (4) and the resolvent equation for $\phi(\lambda)$ we know that $X^a(\lambda)$ is an exit family. Next we prove (4).

Obviously, if both r_1 and r_2 are infinite, then of course (4) holds. If r_a is finite, and $r_b (b \neq a)$ is infinite, then $X^b = X^b(\lambda) = 0$, $X^a = 1$, and (3) becomes

$$\lambda\phi(\lambda)1 = 1 - X^a(\lambda) \tag{5}$$

Therefore, (4) holds.

If r_1 and r_2 are finite, it suffices to prove that (4) holds for $a = 2$; (4) for $a = 1$ can be proved similarly. By (6.3)

$$\begin{aligned} \lambda \sum_j \phi_{ij}(\lambda)(r_2 - z_j) &= u_{1i}(\lambda) \sum_{j \leqslant i} \lambda u_{2j}(\lambda)\mu_j \sum_{k \geqslant j} (z_{k+1} - z_k) \\ &\quad + u_{2i}(\lambda) \sum_{j > i} \lambda \mu_{1j}(\lambda)\mu_j \sum_{k \geqslant j} (z_{k+1} - z_k) \end{aligned} \tag{6}$$

The first term on the right-hand side is given by

$$\begin{aligned} &u_{1i}(\lambda) \sum_{j \leqslant i} \lambda u_{2j}(\lambda)\mu_j \sum_{k \geqslant j} (z_{k+1} - z_k) \\ &= u_{1i}(\lambda)\left(\sum_{k < i} (z_{k+1} - z_k) \sum_{j \leqslant k} \lambda u_{2j}(\lambda)\mu_j + \sum_{k \geqslant i} (z_{k+1} - z_k) \sum_{j \leqslant i} \lambda u_{2j}(\lambda)\mu_j \right) \\ &= u_{1i}(\lambda)\left(\sum_{k < i} (z_{k+1} - z_k)[u_{2k}^+(\lambda) - u_2^+(r_1, \lambda)] \right. \\ &\quad \left. + \sum_{k \geqslant i} (z_{k+1} - z_k)[u_{2i}^+(\lambda) - u_2^+(r_1, \lambda)] \right) \\ &= u_{1i}(\lambda)\left(u_{2i}(\lambda) - u_2(r_1, \lambda) + u_{2i}^+(\lambda)(r_2 - z_i) - \sum_k (z_{k+1} - z_k)u_2^+(r_1, \lambda) \right) \end{aligned}$$

The second term is given by

$$\begin{aligned}
&u_{2i}(\lambda)\sum_{j>i}\lambda u_{ij}(\lambda)\mu_j\sum_{k\geqslant j}(z_{k+1}-z_k)\\
&=u_{2i}(\lambda)\sum_{k>i}(z_{k+1}-z_k)\sum_{i<j\leqslant k}\lambda u_{1j}(\lambda)\mu_j\\
&=u_{2i}(\lambda)\sum_{k>i}(z_{k-1}-z_k)[u_{1k}^+(\lambda)-u_{1i}^+(\lambda)]\\
&=u_{2i}(\lambda)\sum_{k\geqslant i}(z_{k+1}-z_k)[u_{1k}^+(\lambda)-u_{1i}^+(\lambda)]\\
&=u_{2i}(\lambda)[u_1(r_1,\lambda)-u_{1i}(\lambda)-u_{1i}^+(\lambda)(r_2-z_i)]\\
&=u_{2i}(\lambda)[-u_{1i}(\lambda)-u_{1i}^+(\lambda)(r_2-z_i)]
\end{aligned}$$

Hence substituting them into (6) and noting (5.14), we obtain

$$\lambda\sum_j\phi_{ij}(\lambda)(r_2-z_j)=(r_2-z_i)-u_{1i}(\lambda)u_2(r_1,\lambda)-u_{2i}(\lambda)(r_2-r_1)u_2^+(r_1,\lambda) \tag{7}$$

Since r_1 being finite implies that r_2 must not be entrance, it follows that $u_2(r_1,\lambda)=0$. Again noting (6.8), and multiplying both sides of (7) by (r_2-r_1), we find that (4) holds for $a=2$, and the proof is finished. QED

Lemma 2. Let r_1 be entrance and r_2 be regular or exit. Set

$$\eta_{1j}=(r_2-z_j)\mu_j \qquad \eta_{1j}(\lambda)=-\frac{u_{1j}(\lambda)\mu_j}{u_1^+(r_1,\lambda)} \tag{8}$$

Then $\eta_1(\lambda)\in\mathscr{L}_\lambda^+$ is an entrance family and moreover

$$\lambda\eta_1\phi(\lambda)=\eta_1-\eta_1(\lambda) \tag{9}$$

Proof. By the corollary to Theorem 4.2 we obtain $\eta_1(\lambda)\in\mathscr{L}_\lambda^+$. If we can prove

$$u_2(r_1,\lambda)=-\frac{1}{u_1^+(r_1,\lambda)} \tag{10}$$

then by (7) and by noting (6.5), we obtain (9).

To prove (10), by (5.14) it follows that we need only to prove

$$\lim_{z_i\to r_1}[u_{1i}(\lambda)u_{2i}^+(\lambda)]=0 \tag{11}$$

Since $u_2^+(r_1,\lambda)=0$, it follows that

$$\begin{aligned}
0&\leqslant u_{1i}(\lambda)u_{2i}^+(\lambda)=u_{1i}(\lambda)[u_{2i}^+(\lambda)-u_2^+(r_1,\lambda)]\\
&=u_{1i}(\lambda)\sum_{j\leqslant i}\lambda u_{2j}(\lambda)\mu_j\\
&\leqslant u_{2i}(\lambda)\sum_{j\leqslant i}\lambda u_{1j}(\lambda)\mu_i\\
&=u_{2i}(\lambda)[u_{1i}^+(\lambda)-u_1^+(r_1,\lambda)]\to 0 \qquad (z_i\to r_1)
\end{aligned}$$

That is (11). From (9) and the resolvent equation for $\phi(\lambda)$ it follows that $\eta_1(\lambda)$ is an exit family, and the proof is complete. QED

Lemma 3. Let r_2 be regular or exit. For the entrance family $\bar\eta(\lambda)\in\mathscr{L}_\lambda^+$ $(\lambda>0)$ it is necessary and sufficient that $\bar\eta(\lambda)$ has the following Riesz representation:

$$\bar\eta(\lambda)=p_1\Phi_1(\lambda)+p_2X^2(\lambda)\mu \tag{12}$$

where the constant $p_a\geqslant 0\,(a=1,2)$, $p_a=0$ if r_a is exit or natural, and

$$\Phi_1(\lambda)=\begin{cases}\eta_1(\lambda) & \text{if } r_1 \text{ is entrance}\\ X^1(\lambda)\mu & \text{if } r_2 \text{ is regular}\end{cases} \tag{13}$$

Proof. By Theorem 5.10 it follows that $\bar\eta(\lambda)\in\mathscr{L}_\lambda^+$ has the form $\bar\eta(\lambda)=c_{1\lambda}u_1(\lambda)\mu+c_{2\lambda}u_2(\lambda)\mu$. Since $\bar\eta(\lambda)\in l$ by Theorem 5.9(ii) it follows that $c_{a\lambda}=0$ if r_a is exist or natural. Therefore, $\bar\eta(\lambda)=p_{1\lambda}\Phi_1(\lambda)+p_{2\lambda}X^2(\lambda)\mu$ and, moreover, $p_{a\lambda}=0$ if r_a is exist or natural. Because $X^a(\lambda)$ is an exit family if r_a is regular, so $X^a(\lambda)\mu$ is an entrance families. Now that $\bar\eta(\lambda)$, $\Phi_1(\lambda)$, $X^2(\lambda)\mu$ are all entrance families, it follows that $p_{a\lambda}=p_a$ is independent of λ. Hence $\bar\eta(\lambda)$ has the representation (12). Since $\bar\eta(\lambda)\geqslant 0$, by Theorem 5.9(iii) we get $p_a\geqslant 0\,(a=1,2)$. Conversely, it is clear that $\bar\eta(\lambda)$ given by (12) is an entrance family, and $\bar\eta(\lambda)\in\mathscr{L}_\lambda^+$. QED

Lemma 4. Let r_1 and r_2 be regular or exit. If r_a is regular then

$$U_\lambda^{ab}\equiv\lambda[X^a(\lambda)\mu,X^b]\uparrow U^{ab}=\begin{cases}+\infty & \text{if } b=a\\ 1/(r_2-r_1) & \text{if } b\neq a\end{cases} \tag{14}$$

as $\lambda\uparrow\infty$.

Proof. As in (2.11.46), we can obtain

$$U_\lambda^{ab}-U_\nu^{ab}=(\lambda-\nu)[X^a(\nu)\mu,X^b(\lambda)] \tag{15}$$

Thus the monotonicity of U_λ^{ab} is proved. Next we might as well suppose that r_2 is regular. By (4) and (6.12),

$$\begin{aligned}
[\lambda\phi(\lambda)X^b]_i^+&=\lambda u_{1i}^+(\lambda)\sum_{j\leqslant i}u_{2j}(\lambda)X_j^b\mu_j+\lambda\mu_{2i}^+(\lambda)\sum_{j>i}u_{1j}(\lambda)X_j^b\mu_j\\
&=\frac{(-1)^b}{r_2-r_1}-[X_i^b(\lambda)]^+
\end{aligned} \tag{16}$$

Letting $z_i \to r_2$ we obtain

$$-U_\lambda^{2b} = \frac{(-1)^b}{r_2 - r_1} - X^{b+}(r_2, \lambda) \tag{17}$$

where

$$X^{b+}(r_2, \lambda) = \lim_{z_i \to r_2} [X_i^b(\lambda)]^+$$

To prove (14), we only need to prove

$$\lim_{\lambda \to \infty} X^{2+}(r_2, \lambda) = +\infty \tag{18}$$

$$\lim_{\lambda \to \infty} X^{1+}(r_2, \lambda) = 0 \tag{19}$$

Since both $X^2(\lambda)$ and $[X^2(\lambda)]^+$ are increasing functions, it follows that

$$\frac{X^2(r_2, \lambda) - X_i^2(\lambda)}{r_2 - z_i} < X^{2+}(r_2, \lambda)$$

By (2.11.29) we have $X^a(\lambda) \downarrow 0 (\lambda \uparrow \infty, a = 1, 2)$. From the expression above, we obtain

$$\frac{1}{r_2 - z_i} \leqslant \varliminf_{\lambda \to \infty} X^{2+}(r_2, \lambda)$$

Letting $z_i \to r_2$, (18) follows. Furthermore,

$$0 \leqslant -X^{1+}(r_2, \lambda) < -[X_0^1(\lambda)]^+ = \frac{X_0^1(\lambda) - X_1^1(\lambda)}{z_1 - z_0}$$

Then (19) follows by $X^1(\lambda) \downarrow 0$ again, and the proof is complete. QED

Lemma 5. If r_a is exit or regular, then the standard image of $X^a(\lambda)$ is X^a.

Proof. We shall prove the lemma for $a = 2$. Since $X^2(\lambda) \leqslant X^2$, the standard image of $X^2(\lambda)$, $\bar{X}^2$ is $\leqslant X^2$, so that

$$\lambda\phi(\lambda)\bar{X}^a = \bar{X}^2 - X^2(\lambda) \tag{20}$$

Moreover $u \equiv X^2 - \bar{X}^2$ is a solution of equation (2.11.18). By (4) and (20), we have

$$\lambda\phi(\lambda)u = u \tag{21}$$

Let r_1 be infinite. Since $r_2 - z_i$ and z_i are two linearly independent solutions of $Qu = 0$, it follows that

$$u = c_1(r_2 - z_i) + c_2 z_i = c_1 r_2 + (c_2 - c_1)z_i$$

Since the left-hand side is bounded, surely $c_1 = c_2$. Thus $u = c_1 r_2$ and,

$$\lambda\phi(\lambda)u = c_1 r_2 \lambda\phi(\lambda)1 = c_1 r_2[1 - X^2(\lambda)] = u - c_1 r_2 X^2(\lambda)$$

Comparing (21) it follows that $c_1 = 0$. Consequently $u = 0$, $X^2 = \bar{X}^2$.

Let r_1 be finite. Then $u = c_1 X^1 + c_2 X^2$, since X^1 and X^2 are two linearly independent solutions of (2.11.18). By (4) and (20) we derive

$$\lambda\phi(\lambda)u = u - c_1 X^1(\lambda) - c^2 X^2(\lambda)$$

Comparing (21) we have

$$c_1 X^1(\lambda) + c_2 X^2(\lambda) = 0$$

If r_1 is exit or regular, then $c_1 = c_2 = 0$ and hence $u = 0$; if r_1 is natural (r_1 cannot be entrance because r_1 is finite) then $c_2 = 0$, so that $0 \leqslant c_1 X_i^1 = u_i \leqslant X_i^2$. Letting $i \to -\infty$ we have $c_1 = 0$. Therefore $u = 0$ and the proof is complete.

QED

5.8 ONE OF r_1 AND r_2 IS ENTRANCE OR NATURAL, THE OTHER IS EXIT OR REGULAR

From this section on, we shall construct all Q processes. By Theorem 6.4 it follows that if both r_1 and r_2 are entrance or natural, then the minimal solution $\phi(\lambda)$ is honest, so that the Q process is unique. In this section, we suppose that one boundary point is regular or exit (for example r_2), and that the other boundary point is entrance or natural (say, r_1); whence $u_1(r_1, \lambda) = \infty$, $u_2(r_2, \lambda) < \infty$. Hence equation (5.1) possesses only one linearly independent non-null non-negative bounded solution $\bar{X}(\lambda) = X^2(\lambda)$, that is, the dimension of $\mathscr{M}_\lambda^+$, $m^+ = 1$. In this case, the construction problem is already solved by Theorem 3.2.1. Certainly, we take a relatively special form in the present case.

If we denote $c - [\alpha, X^0] - \bar{\sigma}^0$ of Theorem 3.2.1 by $\bar{c}$, then

$$\begin{aligned} c &+ [a, \bar{X} - \bar{X}(\lambda)] + \lambda[\bar{\eta}(\lambda), \bar{X}] \\ &= \bar{c} + [a, X^0 + \bar{X} - \bar{X}(\lambda)] + \lambda[\bar{\eta}(\lambda), X^0 + \bar{X}] \\ &= \bar{c} + [\alpha, 1 - \bar{X}(\lambda)] + \lambda[\bar{\eta}(\lambda), 1] \end{aligned} \tag{1}$$

In addition, if we also rewrite $\bar{c}$ as c, then Theorem 3.2.1 now takes the form below.

Theorem 1. Let r_1 be entrance or natural, and r_2 be exit or regular. For $\psi(\lambda)$ to be a Q process it is necessary and sufficient that either $\psi(\lambda) = \phi(\lambda)$ or $\psi(\lambda)$ can be obtained as follows. Take a row vector $\alpha \geqslant 0$ such that $\alpha\phi(\lambda) \in l$ and select constants $p_a \geqslant 0$ $(a = 1, 2)$ and, moreover, $p_a = 0$ if r_a is exit or natural. According to (7.12) take $\bar{\eta}(\lambda)$ satisfying

$$\eta(\lambda) = \alpha\phi(\lambda) + \bar{\eta}(\lambda) \neq 0 \tag{2}$$

Take a constant $c \geq 0$. Finally, set

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + X_i^2(\lambda)\frac{\sum_k \alpha_k \phi_{kj}(\lambda) + \bar{\eta}_j(\lambda)}{c + [\alpha, 1 - X^2(\lambda)] + \lambda[\bar{\eta}(\lambda), 1]} \tag{3}$$

For the process $\psi(\lambda)$ to be honest it is necessary and sufficient that $c = 0$. For the process $\psi(\lambda)$ to satisfy the system of forward equations it is necessary and sufficient that $\alpha = 0$.

5.9 BOTH r_1 AND r_2 ARE REGULAR OR EXIT: LINEARLY DEPENDENT CASE

In this section we suppose that both r_1 and r_2 are regular or exit; whence non-null $X^a(\lambda) \in \mathcal{M}_\lambda^+(1)$ $(a = 1, 2)$ are exit families. Let $\psi(\lambda)$ be a Q process. Since both $\psi(\lambda)$ and $\phi(\lambda)$ satisfy the B condition, thus for fixed j, $\psi_{ij}(\lambda) - \phi_{ij}(\lambda)$ is a solution of equation (5.1), so that

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + X_i^1(\lambda)F_j^1(\lambda) + X_i^2(\lambda)F_j^2(\lambda) \tag{1}$$

Letting $z_i \to r_a$, we obtain $F^a(\lambda) \geq 0$.

We shall determine $F^a(\lambda)$ $(a = 1, 2)$ so that $\psi(\lambda)$ of (1) is a Q process. We only need consider the norm conditions and the resolvent equation because $\psi(\lambda)$ in (1) satisfies the B condition.

Because of (7.3) and by observing that $X^a(r_a, \lambda) = 1$, $X^a(r_b, \lambda) = 0\ (b \neq a)$, we can easily see that the norm condition of $\psi(\lambda)$ is equivalent to

$$F^a(\lambda) \geq 0 \qquad \lambda[F^a(\lambda), 1] \leq 1 \qquad a = 1, 2 \tag{2}$$

Since $\phi(\lambda)$ satisfies the resolvent equation, upon substituting $\psi(\lambda)$ of (1) into the resolvent equation, noticing that $X^a(\lambda)$ $(a = 1, 2)$ are exit families, and noting their linear independence, we find that the resolvent equation for $\psi(\lambda)$ is equivalent to

$$F^a(\lambda)A(\lambda, v) = F^a(v) + (v - \lambda)\sum_{b=1}^{2}[F^a(\lambda), X^b(v)]F^b(v) \qquad a = 1, 2, \lambda, v > 0 \tag{3}$$

To begin with we suppose that $F^1(v)$, $F^2(v)$ are linearly dependent for some $v > 0$, that is,

$$F^a(v) = m_{av}\eta(v) \qquad m_{av} \geq 0, \quad \eta(v) \geq 0, \quad a = 1, 2 \tag{4}$$

By (2.10.8), multiplying both sides of (3) from the right by $A(v, \lambda)$, we have

$$F^a(\lambda) = \left(m_{av} + (v - \lambda)\sum_{b=1}^{2}[F^a(\lambda), X^b(v)]m_{bv}\right)\eta(v)A(v, \lambda)$$

Therefore, $F^{(a)}_{(\lambda)}$ $(a = 1, 2)$ are linearly dependent for arbitrary $\lambda > 0$. Thus (4) holds for all $v > 0$.

On account of (2.10.8), under (2) and (4), (3) is equivalent to

$$F^a(\lambda) = m_{a\lambda}\eta(\lambda) \qquad \text{quantities } m_{a\lambda} \geq 0, \text{ row vector } \eta(\lambda) \geq 0 \tag{5}$$

$$\eta(\lambda)\ (\lambda > 0) \qquad \text{are entrance families} \tag{6}$$

$$m_{a\lambda} = m_{av} + (v - \lambda)\sum_{b=1}^{2} m_{av}[\eta(\lambda), X^b(v)]m_{bv}, \qquad a = 1, 2 \tag{7}$$

By (7), if $m_{a\lambda} = 0$ for some a and some $\lambda > 0$, then $m_{a\lambda} = 0$ for all $\lambda > 0$. Hence either $m_{1\lambda} = m_{2\lambda} = 0$ and then $\psi(\lambda) = \psi(\lambda)$, which is trivial; or $\eta(\lambda) \neq 0$, $m_{1\lambda} + m_{2\lambda} > 0$. We discuss the latter case. By (6) and Lemma 7.3.

$$\eta(\lambda) = \alpha\phi(\lambda) + p_1 X^1(\lambda)\mu + p_2 X^2(\lambda)\mu \neq 0 \tag{8}$$

where constant $p_a \geq 0$ and $p_a = 0$ if r_a is exist; the row vector $\alpha \geq 0$ is such that $\alpha\phi(\lambda) \in l$. Next, by (7) we derive $m_{1\lambda}m_{2\lambda} = m_{1v}m_{2v}$. Therefore, there exist constants $d_a \geq 0$, $d_1 + d_2 > 0$ such that

$$d_1 m_{2\lambda} = d_2 m_{1\lambda} \tag{9}$$

We may as well suppose $d_2 > 0$ without loss of generality; hence $m_{2\lambda} > 0$, whereas (7) becomes

$$d_2 m_{2\lambda} = d_2 m_{2v} + (v - \lambda)m_{2\lambda}\sum_{b=1}^{2} d_b[\eta(\lambda), X^b(v)]m_{2v}$$

Dividing the above formula by $m_{2\lambda}m_{2v}$, we obtain

$$d_2 m_{2v}^{-1} = d_2 m_{2\lambda}^{-1} + (v - \lambda)\sum_{b=1}^{2} d_b[\eta(\lambda), X^b(v)] \tag{10}$$

Owing to (2.11.46) we have

$$(v - \lambda)[\eta(\lambda), X^b(v)] = v[\eta(v), X^b] - \lambda[\eta(\lambda), X^b] \tag{11}$$

Therefore

$$d_2 m_{2\lambda}^{-1} - \sum_{b=1}^{2} d_b\lambda[\eta(\lambda), X^b] = c \qquad (c \text{ is constant}) \tag{12}$$

From (9) it follows that

$$m_{a\lambda} = d_a\left(c + \sum_{b=1}^{2} d_b\lambda[\eta(\lambda), X^b]\right)^{-1} \qquad a = 1, 2 \tag{13}$$

By substituting (5) into (2) and noting that $X^1 + X^2 = 1$, the norm condition becomes

$$\begin{aligned} (d_1 - d_2)\lambda[\eta(\lambda), X^2] &\leq c \\ (d_2 - d_1)\lambda[\eta(\lambda), X^1] &\leq c \end{aligned} \tag{14}$$

The equalities hold if and only if $d_1 = d_2$, $c = 0$. By (8) and (7.4), we have

$$\lambda[\eta(\lambda), X^a] = \lambda[\alpha\phi(\lambda), X^a] + \sum_{b=1}^{2} p_b \lambda[X^b(\lambda)\mu, X^a]$$
$$= [\alpha, X^a - X^a(\lambda)] + p_1 U_\lambda^{1a} + p_2 U_\lambda^{2a}$$

Since $X^a(\lambda) \downarrow 0 (\lambda \uparrow \infty)$ and on account of (7.14)

$$\lambda[\eta(\lambda), X^a] \uparrow W_a \equiv [\alpha, X^a] + p_1 U^{1a} + p_2 U^{2a} \qquad \lambda \uparrow \infty \tag{15}$$

By (7.14) W_a is finite if and only if

$$[\alpha, X^a] < \infty \qquad p_a = 0 \tag{16}$$

and then

$$W_a = [\alpha, X^a] + \frac{p_b}{r_2 - r_1} \qquad (b \neq a) \tag{17}$$

Therefore, (14) is equivalent to

$$\begin{aligned} &c \geq 0 && \text{if } d_1 = d_2 \\ &c \geq (d_1 - d_2)W_2 && \text{if } d_1 > d_2 \\ &c \geq (d_2 - d_1)W_1 && \text{if } d_1 < d_2 \end{aligned} \tag{18}$$

Theorem 1. Suppose that both r_1 and r_2 are exist or regular and that $F^a(\lambda)$ $(a = 1, 2)$ are linearly dependent for some (hence all) $\lambda > 0$.

For $\psi(\lambda)$ of (1) to be a Q process it is necessary and sufficient that either $\psi(\lambda) = \phi(\lambda)$ or $\psi(\lambda)$ is obtained as follows: Select constants $d_a \geq 0$, $d_1 + d_2 > 0$, $p_a \geq 0$ ($p_a = 0$ if r_a is exit); take a row vector $\alpha \geq 0$ such that $\alpha\phi(\lambda) \in l$ and moreover (8) holds. It is also required that (16) holds if $d_a < d_b (b \neq a)$. Choose a constant c satisfying (18) (where W_a is the same as in (17)). Finally, set

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \frac{[d_1 X_i^1(\lambda) + d_2 X_i^2(\lambda)][\sum_k \alpha_k \phi_{kj}(\lambda) + p_1 X_j^1(\lambda)\mu_j + p_2 X_j^2(\lambda)\mu_j]}{c + \sum_{b=1}^{2} d_b\{[\alpha, X^b - X^b(\lambda)] + \lambda[p_1 X^1(\lambda)\mu + p_2 X^2(\lambda)\mu, X^b)\}} \tag{19}$$

For the process $\psi(\lambda)$ to be honest it is necessary and sufficient that $d_1 = d_2$, $c = 0$. For this process to satisfy the system of forward equations it is necessary and sufficient that $\alpha = 0$.

5.10 BOTH r_1 AND r_2 ARE REGULAR OR EXIT: LINEARLY INDEPENDENT CASE

In the previous section, we assumed that $F^a(\lambda)$ $(a = 1, 2)$ were linearly dependent. In this section, we suppose that $F^a(\lambda)$ $(a = 1, 2)$ are linearly independent.

It will be convenient to use matrix notation. We write

$$[y] = \begin{pmatrix} y^1 \\ \overline{y^2} \end{pmatrix} \qquad [y]' = (y^1, y^2)$$

where y^1 and y^2 are quantities or vectors. If y^a, v^a $(a = 1, 2)$ are vectors, the symbol $\{[y, v]\}$ will denote the second-order square matrix whose elements are $[y^a, v^a]$. The second-order identity matrix will be denoted by I.

Using these notations, (9.1), (9.2) and (9.3) may be written as

$$\psi(\lambda) = \phi(\lambda) + [X(\lambda)]'[F(\lambda)] \tag{1}$$
$$[F(\lambda)] \geq [0] \qquad \{\lambda[F(\lambda), X]\}[1] \leq [1] \tag{2}$$
$$[F(\lambda)A(\lambda, v)] = [F(v)] + (v - \lambda)\{[F(\lambda), X(v)]\}[F(v)] \tag{3}$$

Lemma 1. Let $\psi(\lambda)$ be a Q process having form (1). Then there exist two row vectors $\alpha^a \geq 0$ $(a = 1, 2)$ such that $\alpha^a\phi(\lambda) \in l$, and there exist two second-order square matrices $\mathscr{R}_\lambda = (r_\lambda^{ab}) \geq 0$ and $\mathscr{M}_\lambda = (M_\lambda^{ab}) \geq 0$ ($M_\lambda^{1a} = M_\lambda^{2a} = 0$ if r_a is exist). Moreover,

$$[F(\lambda)] = \mathscr{R}_\lambda[\alpha\phi(\lambda)] + \mathscr{M}_\lambda[X(\lambda)\mu] \tag{4}$$

Proof. Since $\psi(\lambda)$ is a Q process, (2) and (3) hold.

From (2) and (3) we can see that for arbitrary $\lambda, v > 0$, $[F(\lambda)A(\lambda, v)] \geq [0]$. Now fix, for the moment, a and $\lambda > 0$ and set $\eta(v) = F^a(\lambda)A(\lambda, v)$. Then $\eta(v)$ is an entrance family. Hence, by Lemmas 2.11.3 and 7.3, there exists a row vector $\beta_\lambda^a \geq 0$ independent of v (but dependent on a and λ), such that $\beta_\lambda^a\phi(\lambda) \in l$. Moreover,

$$vF^a(\lambda)A(\lambda, v) - F^a(\lambda)A(\lambda, v)Q = \beta_\lambda^a \tag{5}$$
$$F^a(\lambda)A(\lambda, v) = \beta_\lambda^a\phi(v) + M_\lambda^{a1}X^1(\lambda)\mu + M_\lambda^{a2}X^2(\lambda)\mu \tag{6}$$

where $\mathscr{M}_\lambda = (M_\lambda^{ab}) \geq 0$ is independent of v and $M_\lambda^{1a} = M_\lambda^{2a} = 0$ if r_a is exit. In particular, when $v = \lambda$ we have

$$\lambda F^a(\lambda) - F^a(\lambda)Q = \beta_\lambda^a \tag{7}$$

$$[F(\lambda)] = [\beta_\lambda^a\phi(\lambda)] + \mathscr{M}_\lambda[X(\lambda)\mu] \tag{8}$$

Therefore, we should like to prove the following. There exist two row vectors $\alpha^a \geq 0$ $(a = 1, 2)$, independent of λ, such that

$$\beta_\lambda^a = r_\lambda^{a1}\alpha^1 + r_\lambda^{a2}\alpha^2 \tag{9}$$

where $\mathscr{R}_\lambda = (r_\lambda^{ab}) \geq 0$. Then substituting (9) into (8) we obtain (4), and from $\beta_\lambda^a\phi(\lambda) \in l$ we can deduce $\alpha^a\phi(\lambda) \in l$ (if $r_\lambda^{1b} \equiv r_\lambda^{2b} \equiv 0$ for some b, then we can take α^b to be 0 independent of the value of β_λ^a. If we can prove this, the lemma is thus proved.

By (5) and (7), multiplying both sides of (3) from the left by $(\nu I - Q)$ we obtain

$$[\beta_\lambda] = [\beta_\nu] + (\nu - \lambda)\{[F(\lambda), X(\nu)]\}[\beta_\nu] \tag{10}$$

From this we can see $\beta_\lambda^a \downarrow (\lambda \uparrow)$, and if $\beta_\lambda^a = 0$ for some $\lambda > 0$ and some a, then $\beta_\nu^1 = \beta_\nu^2 = 0$ for $\nu > \lambda$; whence $\beta_\lambda^1 = \beta_\lambda^2 = 0$ for all $\lambda > 0$. In this case, (9) is trivial, we only need to choose $\alpha^1 = \alpha^2 = 0, \mathscr{R}_\lambda = 0$. We assume $\beta_\lambda^a \neq 0$ $(a = 1, 2)$ for all $\lambda > 0$. Set

$$\mathscr{T}_{\lambda\nu} \equiv I + (\nu - \lambda)\{[F(\lambda), X(\nu)]\}$$

its elements being $t_{\lambda\nu}^{ab}$, and then (10) can be rewritten as

$$\beta_\lambda^a = t_{\lambda\nu}^{a1}\beta_\nu^1 + t_{\lambda\nu}^{a2}\beta_\nu^2 \qquad a = 1, 2 \tag{11}$$

And when $\nu > \lambda$

$$0 \leqslant t_{\lambda\nu}^{ab}\beta_\nu^a \leqslant \beta_\lambda^b \qquad a, b = 1, 2 \tag{12}$$

If, for arbitrary $\lambda > 0$, when $\nu \to \infty$, we have

$$t_{\lambda\nu}^{a1}\beta_\nu^1 \to 0 \qquad a = 1, 2 \tag{13}$$

then, by (11), for arbitrary $\lambda > 0$, we have

$$t_{\lambda\nu}^{a2}\beta_\nu^2 \to \beta_\lambda^a \neq 0 \qquad a = 1, 2 \tag{14}$$

as $\nu \to \infty$. Fix an arbitrary $\lambda_0 > 0$, and let $\alpha^2 = \beta_{\lambda_0}^1$. By (11), we have

$$\beta_\lambda^a = t_{\lambda\nu}^{a1}\beta_\nu^1 + \frac{t_{\lambda\nu}^{a2}}{t_{\lambda_0\nu}^{12}} t_{\lambda_0\nu}^{12}\beta_\nu^2$$

The left-hand side of the above expression is independent of ν; by (13) it follows that the first term of the right-hand side converges to zero as $\nu \to \infty$. By (14) it follows that the second fraction on the right-hand side of the above formula must converge to some r_λ^{a2} that is finite and non-negative. Therefore, upon taking a limit as $\nu \to \infty$ we obtain

$$\beta_\lambda^a = r_\lambda^{a2}\alpha^2$$

Then we certainly have, the form (9).

If (13) does not hold, then there exist some $\lambda_1 > 0$ and some a_1 (we had better say $a_1 = 1$ without loss of generality) and a subsequence $\nu_n \to \infty$ such that

$$t_{\lambda_1\nu_n}^{11}\beta_{\lambda_1}^1 \to \alpha^1 \neq 0 \qquad 0 \leqslant \alpha^1 \leqslant \beta_{\lambda_1}^1. \tag{15}$$

If, for every $\lambda > 0$, we have

$$t_{\lambda\nu_n}^{a2}\beta_{\nu_n}^2 \to 0 \qquad a = 1, 2$$

as $\nu_n \to \infty$, then in the same way as used in the proof above, we can prove

$$\beta_\lambda^a = r_\lambda^{a1}\alpha^1 \qquad 0 \leqslant \alpha^1 \neq 0$$

Hence we obtain an expression of the form (9) for β_λ^a. Otherwise, there exist some $\lambda_2 > 0$, some a_2 (say, $a_2 = 2$ without loss of generality) and a subsequence $\nu_n(1) \to \infty$ of ν_n such that

$$t_{\lambda_2\nu\nu_n(1)}^{22}\beta_{\nu\nu_n(1)}^2 \to \alpha^2 \neq 0 \qquad 0 \leqslant \alpha^2 \leqslant \beta_{\lambda_1}^2 \tag{16}$$

Now for an arbitrary $\lambda > 0$, (11) can be rewritten as

$$\beta_\lambda^a = \frac{t_{\lambda\nu}^{a1}}{t_{\lambda_1\nu}^{11}} t_{\lambda_1\nu}^{11}\beta_\nu^1 + \frac{t_{\lambda\nu}^{a2}}{t_{\lambda_2\nu}^{22}} t_{\lambda_2\nu}^{22}\beta_\nu^2$$

We can also select a subsequence $\nu_n(2) \to \infty$ of $\nu_n(1)$ (the subsequence may depend on λ) such that the two fractions in the above formula converge to non-negative numbers r_λ^{a1} and r_λ^{a2} as $\nu = \nu_n(2) \to \infty$, respectively. Then, by (15) and (16), letting $\nu = \nu_n(2) \to \infty$ in the above formula, we obtain

$$\beta_\lambda^a = r_\lambda^{a1}\alpha^1 + r_\lambda^{a2}\alpha^2$$

Since α^1 and α^2 are not zero, it follows that $\mathscr{R}_\lambda = (r_\lambda^{ab})$ is finite. Thus the form (9) follows again and the proof is terminated. QED

By Lemma 1, it suffices for us to consider only Q processes $\psi(\lambda)$ that have the form given by (1) and (4).

We introduce the notation:

$$h_\lambda^{ab} = \lambda[\alpha^a\phi(\lambda), X^b] = [\alpha^a, X^b - X^b(\lambda)] \uparrow h^{ab} = [\alpha^a, X^b] \qquad \lambda \uparrow \infty \tag{17}$$

$$\mathscr{H}_\lambda = (h_\lambda^{ab}) \uparrow \mathscr{H} = (h^{ab}) \qquad \lambda \uparrow \infty \tag{18}$$

The above relation follows from $X^a(\lambda) \downarrow 0 (\lambda \uparrow \infty)$ and (7.4), By (2.11.40)

$$(\nu - \lambda)[\alpha^a\phi(\lambda), X^b(\nu)] = h_\lambda^{ab} - h_\lambda^{ab} \tag{19}$$

We first consider a special case: $\mathscr{M}_\lambda = 0$, $\mathscr{H} < \infty$. That is,

$$[F(\lambda)] = \mathscr{R}_\lambda[\alpha\phi(\lambda)] \tag{20}$$

$$[\alpha^a, 1] < \infty \qquad \alpha = 1, 2 \tag{21}$$

Theorem 2. Assume that $F'(\lambda)$ and $F^2(\lambda)$ are linearly independent, and that $F^1(\lambda)$ and $F^2(\lambda)$ have the forms (20) and (21). Then for $\psi(\lambda)$ given by (1) to be a Q process, it is necessary and sufficient that it can be obtained as follows: Select non-negative row vectors α^1 and α^2, which are linearly independent and satisfy

$$[\alpha^a, 1] \leqslant 1 \qquad \alpha = 1, 2 \tag{22}$$

Then, set

$$\psi(\lambda) = \phi(\lambda) + [X(\lambda)]'(I - \mathscr{T}_\lambda)^{-1}[\alpha\phi(\lambda)] \tag{23}$$

where $\mathscr{T}_\lambda = \{[\alpha, X(\lambda)]\}$.

The process $\psi(\lambda)$ does not satisfy the system of forward equations. The vectors α^1 and α^2 (they satisfy (22)) are uniquely determined by this process. For the process $\psi(\lambda)$ to be honest it is necessary and sufficient that

$$[a^a, 1] = 1 \qquad a = 1, 2 \tag{24}$$

Proof. (a) Let the Q process $\psi(\lambda)$ be of the forms (20) and (21). Substituting (20) into (2), we obtain

$$\mathscr{R}_\lambda \mathscr{H}_\lambda [1] \leqslant [1] \tag{25}$$

Again substituting (20) into (3) and noting (2.10.9) and (19), we obtain

$$\mathscr{R}_\lambda [a\phi(v)] = \mathscr{R}_v[\alpha\phi(v)] + \mathscr{R}_\lambda(\mathscr{H}_v - \mathscr{H}_\lambda)\mathscr{R}_v[\alpha\phi(v)] \tag{26}$$

Multiplying the above formula from the right by $(vI - Q)$, we obtain

$$\mathscr{R}_\lambda[\alpha] = \mathscr{R}_v[\alpha] + \mathscr{R}_\lambda(\mathscr{H}_v - \mathscr{H}_\lambda)\mathscr{R}_v[\alpha]$$

Now that $F^a(\lambda)$ $(a = 1, 2)$ are linearly independent, by (20) it follows that α^1 and α^2 are also linearly independent. Hence from the above formula, we obtain

$$\mathscr{R}_\lambda = [I - \mathscr{R}_\lambda(\mathscr{H}_\lambda - \mathscr{H}_v)]\mathscr{R}_v \tag{27}$$

Select a subsequence $\lambda \to \infty$ such that

$$\mathscr{R}_\lambda \to \mathscr{R} \geqslant 0$$

By (25), (27) and (22) we obtain

$$\mathscr{R}\mathscr{H}[1] \leqslant [1] \tag{28}$$

$$\mathscr{R} = [I - \mathscr{R}(\mathscr{H} - \mathscr{H}_v)]\mathscr{R}_v \tag{29}$$

Set $[\bar{a}] = \mathscr{R}[\alpha] \geqslant [0]$, and then (28) and (29) become

$$[\bar{\alpha}^a, 1] \leqslant 1 \qquad a = 1, 2 \tag{30}$$

$$\mathscr{R} = (I - \bar{\mathscr{T}}_\lambda)\mathscr{R}_\lambda \qquad \bar{\mathscr{T}}_\lambda = \{[\bar{\alpha}, X(\lambda)]\} \tag{31}$$

But by (30), we obtain $[\bar{\alpha}^a, X^1(\lambda) + X^2(\lambda)] < 1$ $(a = 1, 2)$. Therefore the inverse matrix $(I - \mathscr{T}_\lambda)^{-1}$ exists and is non-negative. From (31) it follows that

$$\mathscr{R}_\lambda = (I - \bar{\mathscr{T}}_\lambda)^{-1}\mathscr{R}$$

Substituting the above formula into (20), we obtain

$$[F(\lambda)] = \bar{\mathscr{R}}_\lambda[\bar{\alpha}\phi(\lambda)] \qquad \bar{\mathscr{R}}_\lambda = (I - \bar{\mathscr{T}}_\lambda)^{-1} \tag{32}$$

Thus it remains to prove that $\bar{\alpha}^1$ and $\bar{\alpha}^2$ are linerly independent. This can be seen from the above formula, because if $\bar{\alpha}^1$ and $\bar{\alpha}^2$ are linearly dependent, then $F^1(\lambda)$ and $F^2(\lambda)$ are also linearly dependent.

(b) Let α^1 and α^2 be non-negative and linearly independent, and satisfy (22).

In (a) we have already proved that the inverse matrix $\mathscr{R}_\lambda = (I - \mathscr{T}_\lambda)^{-1}$ exists and is non-negative.

The $F^a(\lambda)$ $(a = 1, 2)$ defined by $[F(\lambda)] = (I - \mathscr{T}_\lambda)^{-1}[\alpha\phi(\lambda)]$ are linearly independent. In fact, let

$$0 = [c]'[F(\lambda)] = [c]'(I - \mathscr{T}_\lambda)^{-1}[\alpha\phi(\lambda)]$$

Multiplying the above formula from the right by $(\lambda I - Q)$, we obtain

$$0 = [c]'(I - \mathscr{T}_\lambda)^{-1}[\alpha]$$

Since α^a $(a = 1, 2)$ are independent, thus $[c]'(I - \mathscr{T}_\lambda)^{-1} = [0]'$; hence

$$[c]' = [0]'.$$

By (22) we obtain $\mathscr{H}[1] \leqslant [1]$, that is, $(I - \mathscr{H})[1] \geqslant [0]$. Since $[I - \mathscr{T}_\lambda)[1] \geqslant \mathscr{H}_\lambda[1]$ and $(I - \mathscr{T}_\lambda)^{-1}$ is non-negative, therefore $[1] \geqslant (I - \mathscr{T}_\lambda)^{-1}\mathscr{H}_\lambda[1]$. Hence (25) is satisfied, so that (2) holds. By a direct verification, we know (27) is satisfied. Therefore (26) holds and, consequently, (3) holds.

(c) Since α^1 and α^2 are linearly independent, and since $u_j \equiv X_i^1(\lambda)F_j^1(\lambda) + X_i^2(\lambda)F_j^2(\lambda)$ satisfies

$$(\lambda u - uQ)_j = [X_i(\lambda)]'[\alpha_j] \neq 0$$

it follows that the Q process $\psi(\lambda)$ does not satisfy the system of forward equations.

For the equality to hold in (2), i.e. in (25), it is necessary and sufficient that $(I - \mathscr{T}_\lambda)^{-1}\mathscr{H}_\lambda[1] = [1]$. That is,

$$(\mathscr{H} - \mathscr{T}_\lambda)[1] = \mathscr{H}_\lambda[1] = (I - \mathscr{T}_\lambda)[1]$$

i.e. $\mathscr{H}[1] = [1]$, and (24) follows.

Assume that both α^a and $\bar{\alpha}^a$ $(a = 1, 2)$ satisfy (22) and, moreover, correspond to the same process, that is,

$$(I - \mathscr{T}_\lambda)^{-1}[\alpha\phi(\lambda)] = (I - \bar{\mathscr{T}}_\lambda)^{-1}[\bar{\alpha}\phi(\lambda)]$$

Multiplying the above formula from the right by $(\lambda I - Q)$, we obtain

$$(I - \mathscr{T}_\lambda)^{-1}[\alpha] = (I - \bar{\mathscr{T}}_\lambda)^{-1}[\alpha]$$

Since $X^a(\lambda) \downarrow 0$ $(\lambda \uparrow \infty)$ and because of (22), we obtain $[\alpha] = [\bar{\alpha}]$ by letting $\lambda \uparrow \infty$ in the above formula.

Thus the proof of the theorem is completed. QED

We now consider the general case.

Theorem 3. Let $F^a(\lambda)$ $(a = 1, 2)$ be linearly independent. For $\psi(\lambda)$ given by (1) and (4) to be a Q process it is necessary and sufficient that it can be obtained as follows: Select two non-negative row vectors $\bar{\alpha}^a$ $(a = 1, 2)$ such that $\bar{\alpha}^a\phi(\lambda) \in l$,

and choose two non-negative matrices

$$\varphi = \begin{pmatrix} 0 & \bar{s}^{12} \\ \bar{s}^{21} & 0 \end{pmatrix} \qquad \text{and} \qquad \mathcal{M} = \begin{pmatrix} \bar{M}^{11} & 0 \\ 0 & \bar{M}^{22} \end{pmatrix}$$

Moreover, they possess the following properties:

(i) If r_a is exit, then $\bar{M}^{aa} = 0$.

(ii) $\bar{M}^{aa} > 0$ $(a = 1, 2)$; or $\bar{M}^{aa} = 0$, $\bar{M}^{bb} > 0$ $(b \neq a)$, $\bar{\alpha}^a \neq 0$; or $\bar{M}^{aa} = 0$ $(a = 1, 2)$, α^1 and α^2 are linearly independent.

(iii) $\bar{h}^{ab} < \infty$ $(a \neq b)$.

(iv) $\bar{s}^{12} \leqslant 1$, $\bar{s}^{21} \leqslant 1$, $\bar{s}^{ab} \geqslant \bar{h}^{ab} + \bar{M}^{aa}/(r_2 - r_1)(a \neq b)$.

Set

$$\begin{aligned} \mathcal{R}_\lambda &= (I - \bar{\mathcal{S}} + \bar{\mathcal{H}}_\lambda + \bar{\mathcal{M}}\,\mathcal{U}_\lambda)^{-1} \\ \mathcal{M}_\lambda &= (I - \bar{\mathcal{S}} + \bar{\mathcal{H}}_\lambda + \bar{\mathcal{M}}\,\mathcal{U}_\lambda)^{-1}\bar{\mathcal{M}} \end{aligned} \tag{33}$$

where $\bar{\mathcal{H}}_\lambda \uparrow \bar{\mathcal{H}}$ $(\lambda \uparrow \infty)$, which are determined by (17) and (18) for $\bar{\alpha}^a$ $(a = 1, 2)$; by (7.14) we obtain

$$0 < \mathcal{U}_\lambda = (U_\lambda^{ab}) \uparrow \mathcal{U} = \begin{pmatrix} +\infty & 1/(r_2 - r_1) \\ 1/(r_2 - r_1) & +\infty \end{pmatrix} \qquad \lambda \uparrow \infty \tag{34}$$

Finally $\psi(\lambda)$ is determined by (1), (4) and (33).

For the process $\psi(\lambda)$ to be honest it is necessary and sufficient that $\bar{s}^{12} = \bar{s}^{21} = 1$. For this process to satisfy the system of forward equations it is necessary and sufficient that $[\bar{\alpha}] = [0]$. For the process $\psi(\lambda)$ to have the forms (1), (20) and (21) it is necessary and sufficient that $\bar{M}^{11} = \bar{M}^{22} = 0$, $\bar{h}^{aa} < \infty$ $(a = 1, 2)$.

Proof. The proof is to be reached in several steps.

(a) Suppose that the Q process $\psi(\lambda)$ has the forms of (1) and (4) and is such that $F^a(\lambda)$ $(a = 1, 2)$ are linearly independent.

Substituting (4) into (2), we obtain

$$\mathcal{S}_\lambda[1] \leqslant [1] \qquad \mathcal{S}_\lambda \equiv \mathcal{R}_\lambda\mathcal{H}_\lambda + \mathcal{M}_\lambda\mathcal{U}_\lambda \tag{35}$$

Substituting (4) into (3), owing to (2.10.9), (7.15), (19), and to the properties of $X^a(\lambda)$ being an exit family, we obtain

$$\begin{aligned} &\mathcal{R}_\lambda[\alpha\phi(v)] + \mathcal{M}_\lambda[X(v)\mu] \\ &\quad = \mathcal{R}_v[\alpha\phi(v)] + \mathcal{M}_v[X(v)\mu] + (\mathcal{R}_\lambda\mathcal{H}_v + \mathcal{M}_\lambda\mathcal{U}_v - \mathcal{S}_\lambda)\{\mathcal{R}_v[\alpha\phi(v)] \\ &\qquad + \mathcal{M}_v[X(v)\mu]\} \end{aligned} \tag{36}$$

Multiplying the above formula from the right by $(vI - Q)$, we have

$$\begin{aligned} \mathcal{R}_\lambda[\alpha] &= \mathcal{R}_v[\alpha] + (\mathcal{R}_\lambda\mathcal{H}_v + \mathcal{M}_\lambda\mathcal{U}_v - \mathcal{S}_\lambda)\mathcal{R}_v[\alpha] \\ &= (I - \mathcal{S}_\lambda + \mathcal{R}_\lambda\mathcal{H}_v + \mathcal{M}_\lambda\mathcal{U}_v)\mathcal{R}_v[\alpha] \end{aligned} \tag{37}$$

Substituting it into (36), we obtain

$$\mathcal{M}_\lambda = (I - \mathcal{S}_\lambda + \mathcal{R}_\lambda\mathcal{H}_v + \mathcal{M}_\lambda\mathcal{U}_v)\mathcal{M}_v \tag{38}$$

Set $\delta_\lambda^a = 1 - s_\lambda^{aa}$, then $\delta_\lambda^a > 0$. In fact, $\delta_\lambda^a \geqslant 0$ by (35). If $\delta_\lambda^a = 0$, then $s_\lambda^{aa} = 1$. By (35),

$$s_\lambda^{ab} = \sum_{t=1}^{2} (r_\lambda^{at}h_\lambda^{tb} + M_\lambda^{at}U_\lambda^{tb}) = 0 \qquad (b \neq a)$$

Therefore,

$$r_\lambda^{at}\alpha^t = M_\lambda^{at} = 0 \qquad (t = 1, 2)$$

Hence

$$s_\lambda^{aa} = \sum_{t=1}^{2} (r_\lambda^{at}h_\lambda^{ta} + M_\lambda^{at}U_\lambda^{ta}) = 0$$

This is in contradiction with $s_\lambda^{aa} = 1$.

Dividing the ath of (35), (37) and (38) by δ_λ^a, we obtain

$$\bar{\mathcal{S}}_\lambda[1] \leqslant [1] \qquad \bar{\mathcal{S}}_\lambda \equiv \begin{pmatrix} 0 & \bar{s}_\lambda^{12} \\ \bar{s}_\lambda^{21} & 0 \end{pmatrix} \tag{39}$$

$$\bar{\mathcal{R}}_\lambda[\alpha] = (I - \bar{\mathcal{S}}_\lambda + \bar{\mathcal{R}}_\lambda\mathcal{H}_v + \bar{\mathcal{M}}_\lambda\mathcal{U}_v)\mathcal{R}_v[\alpha] \tag{40}$$

$$\bar{\mathcal{M}}_\lambda = (I - \bar{\mathcal{S}}_\lambda + \bar{\mathcal{R}}_\lambda\mathcal{H}_v + \bar{\mathcal{M}}_\lambda\mathcal{U}_v)\mathcal{M}_v \tag{41}$$

where

$$\bar{s}_\lambda^{aa} = 0 \qquad (a = 1, 2) \tag{42}$$

$$\bar{s}_\lambda^{ab} = \sum_{t=1}^{2} \bar{r}_\lambda^{at}h_\lambda^{tb} + \sum_{t=1}^{2} \bar{M}_\lambda^{at}U_\lambda^{tb} \qquad (a \neq b) \tag{43}$$

$$\bar{r}_\lambda^{ab} = \frac{r_\lambda^{ab}}{\delta_\lambda^a} \qquad \bar{M}_\lambda^{ab} = \frac{M_\lambda^{ab}}{\delta_\lambda^a} \tag{44}$$

Choose a subsequence $\lambda \to \infty$ such that

$$\bar{\mathcal{S}}_\lambda \to \bar{\mathcal{S}} \qquad \bar{\mathcal{R}}_\lambda \to \bar{\mathcal{R}} \qquad \bar{\mathcal{M}}_\lambda \to \bar{\mathcal{M}} \tag{45}$$

Then $\bar{\mathcal{S}} = (\bar{s}^{ab})$, $\bar{\mathcal{R}} = (\bar{r}^{ab})$ and $\bar{\mathcal{M}} = (\bar{M}^{ab})$ are non-negative, and by (39)–(44) we have[1]

$$\bar{s}^{aa} = 0 (a = 1, 2) \qquad \bar{s}^{12} \leqslant 1 \qquad \bar{s}^{21} \leqslant 1 \tag{46}$$

$$\bar{s}^{ab} \geqslant \sum_{t=1}^{2} \bar{r}^{at}h^{tb} + \sum_{t=1}^{2} \bar{M}^{at}U^{tb} \qquad (a \neq b) \tag{47}$$

$$\bar{\mathcal{R}}[\alpha] = (I - \bar{\mathcal{S}} + \bar{\mathcal{R}}\mathcal{H}_v + \bar{\mathcal{M}}\,\mathcal{U}_v)\mathcal{R}_v[\alpha] \tag{48}$$

$$\bar{\mathcal{M}} = (I - \bar{\mathcal{S}} + \bar{\mathcal{R}}\mathcal{H}_v + \bar{\mathcal{M}}\,\mathcal{U}_v)\mathcal{M}_v \tag{49}$$

[1] Let $0.\infty = 0$.

By (47) we obtain

$$\bar{M}^{12} = \bar{M}^{21} = 0 \tag{50}$$

If $h^{ib} = \infty$, then

$$\bar{r}^{ai} = 0 \qquad (a \neq b) \tag{51}$$

By Lemma 1, it follows that (i) of this theorem holds. Now set $[\bar{\alpha}] = \bar{\mathscr{R}}[\alpha]$; then $\bar{\alpha}^a \geqslant 0$ and $\bar{\alpha}^a \phi(\lambda) \in l$ $(a = 1, 2)$. Equations (50), (46) and (47) become (iv). Hence we obtain (iii). Equations (48) and (49) become

$$[\bar{\alpha}] = (I - \bar{\mathscr{S}} + \bar{\mathscr{H}}_v + \bar{\mathscr{M}}\mathscr{U}_v)\mathscr{R}_v[\alpha] \tag{52}$$

$$\bar{\mathscr{M}} = (I - \bar{\mathscr{S}} + \bar{\mathscr{H}}_v + \bar{\mathscr{M}}\mathscr{U}_v)\mathscr{M}_v \tag{53}$$

Next we prove

$$\bar{\alpha}^a \neq 0 \qquad \text{if } \bar{M}^{aa} = 0 \tag{54}$$

For otherwise, we may as well assume $\bar{M}^{11} = 0$, $\bar{\alpha}^1 = 0$. Then by (52) and (53) we have

$$r_\lambda^{11}\alpha^1 + r_\lambda^{12}\alpha^2 = \bar{s}^{12}(r_\lambda^{21}\alpha^1 + r_\lambda^{22}\alpha^2)$$
$$M_\lambda^{1a} = \bar{s}^{12} M_\lambda^{2a} \qquad (a = 1, 2)$$

Therefore, $F^1(\lambda) = \bar{s}^{12} F^2(\lambda)$. This is in contradiction with the linear independence of $F^1(\lambda)$ and $F^2(\lambda)$.

We proceed to prove that the inverse matrix

$$\mathscr{Z}_v^{-1} \equiv (I - \bar{\mathscr{S}} + \bar{\mathscr{H}}_v + \bar{\mathscr{M}}\mathscr{U}_v)^{-1} \tag{55}$$

exists and is non-negative. Hence by (52) and (53) we have

$$\mathscr{R}_v[\alpha] = \mathscr{Z}_v^{-1}[\bar{\alpha}] \qquad \mathscr{M}_v = \mathscr{Z}_v^{-1}\bar{\mathscr{M}}$$

Therefore,

$$\left.\begin{aligned}[F(\lambda)] &= \mathscr{Z}_\lambda^{-1}\bar{\mathscr{R}}[\alpha\phi(\lambda)] + \mathscr{Z}_\lambda^{-1}\bar{\mathscr{M}}[X(\lambda)\mu] \\ [F(\lambda)] &= \mathscr{Z}_\lambda^{-1}[\bar{\alpha}\phi(\lambda)] + \mathscr{Z}_\lambda^{-1}\bar{\mathscr{M}}[X(\lambda)\mu]\end{aligned}\right. \tag{56}$$

That is, $\psi(\lambda)$ is determined by (1), (4) and (3).

By (iv), (54) and (34), we obtain

$$1 > \bar{s}^{ab} - (\bar{h}^{ab} - \bar{M}^{aa}U_v^{ab}) \geqslant \bar{s}^{ab} - (\bar{h}^{ab} - \bar{M}^{aa}U^{ab}) \geqslant 0 \quad (a \neq b)$$

Hence $I - \bar{\mathscr{S}} + \bar{\mathscr{H}}_v + \bar{\mathscr{M}}\mathscr{U}_v$ has the form

$$\begin{pmatrix} 1 + l^{11} & -l^{12} \\ -l^{21} & 1 + l^{22} \end{pmatrix} \qquad \begin{aligned} 1 + l^{11} > l_{12} \geqslant 0 \\ 1 + l^{22} > l_{21} \geqslant 0 \end{aligned}$$

Consequently, its determinant is $\Delta > 0$, and its inverse matrix $\mathscr{Z}_v^{-1}$ is

$$\frac{1}{\Delta}\begin{pmatrix} 1 + l^{22} & l^{21} \\ l^{12} & 1 + l^{11} \end{pmatrix} \geqslant 0$$

(b) We proceed to prove that for $F^a(\lambda)$ $(a = 1, 2)$ to be linearly independent the necessary and sufficient condition is (ii).

Obviously, the linear independence of $F^a(\lambda)$ $(a = 1, 2)$ is equivalent to that of $[\bar{\alpha}\phi(\lambda)] + \bar{\mathscr{M}}[X(\lambda)\mu]$. Let

$$[c]'\{[\bar{\alpha}\phi(\lambda)] + \bar{\mathscr{M}}[X(\lambda)\mu]\} = 0 \tag{57}$$

Then multiplying the above expression from the right by $(\lambda I - Q)$, we obtain

$$[c]'[\bar{\alpha}] = 0 \qquad [c]'\bar{\mathscr{M}} = [0]' \tag{58}$$

Conversely, if (58) holds, then (57) certainly holds. In order that (58) is equivalent to $[c] = [0]$, it is necessary and sufficient that (ii) holds.

(c) Suppose that $[\bar{\alpha}], \bar{\mathscr{S}}$ and $\bar{\mathscr{M}}$ satisfy the condition of this theorem.

Because (ii) implies (54), it has been pointed out in (a) that the inverse matrix (55) exists and is non-negative. Therefore, $\psi(\lambda)$ can be determined by (1), (4) (where α^a is replaced by $\bar{\alpha}^a$, $a = 1, 2$) and (33). Now we proceed to prove (2) and (3); that is, (35) and (36) where $[\alpha]$ is replaced by $[\bar{\alpha}]$ hold.

Since

$$(\mathscr{R}_\lambda\bar{\mathscr{H}}_\lambda + \mathscr{M}_\lambda\mathscr{U}_\lambda)[1] = \mathscr{Z}_\lambda^{-1}(\bar{\mathscr{H}}_\lambda + \bar{\mathscr{M}}\mathscr{U}_\lambda)[1] = \mathscr{Z}_\lambda^{-1}\{\mathscr{Z}_\lambda - (I - \bar{\mathscr{S}})\}[1]$$
$$= [1] - \mathscr{Z}_\lambda^{-1}(I - \bar{\mathscr{S}})[1]$$

by (iv) and as $\mathscr{Z}_\lambda^{-1} \geqslant 0$, it follows that (35) where $[\bar{\alpha}]$ is replaced by $[\alpha]$ holds.

Secondly,

$$I - \mathscr{R}_\lambda\bar{\mathscr{H}}_\lambda + \mathscr{M}_\lambda\mathscr{U}_\lambda + \mathscr{R}_\lambda\bar{\mathscr{H}}_v + \mathscr{M}_\lambda\mathscr{U}_v = \mathscr{Z}_\lambda^{-1}(\mathscr{Z}_\lambda - \bar{\mathscr{H}}_\lambda - \bar{\mathscr{M}}\mathscr{U}_\lambda + \bar{\mathscr{H}}_v + \bar{\mathscr{M}}\mathscr{U}_v)$$
$$= \mathscr{Z}_\lambda^{-1}(I - \bar{\mathscr{S}} + \bar{\mathscr{H}}_v + \bar{\mathscr{M}}\mathscr{U}_v) = \mathscr{Z}_\lambda^{-1}\mathscr{Z}_v \tag{59}$$

According, (37) and (38), where α^a and $\mathscr{H}_v$ are replaced by $\bar{\alpha}^a$ and $\bar{\mathscr{H}}_v$ respectively, hold, and it follows that (36) where $[\alpha]$ is replaced by $[\bar{\alpha}]$ holds.

(d) Now we proceed to prove that for (56) to have the form given by (21) it is necessary and sufficient that $\bar{M}^{11} = \bar{M}^{22} = 0$, $\bar{h}^{aa} < \infty$ $(a = 1, 2)$.

Suppose that (56) has the form given by (20) and (21). By Theorem 2 that is,

$$\mathscr{Z}_\lambda^{-1}\{[\bar{\alpha}\phi(\lambda)] + \bar{\mathscr{M}}[X(\lambda)\mu]\} = (I - \mathscr{T}_\lambda)^{-1}[\alpha\phi(\lambda)] \tag{60}$$

Multiplying the above formula from the right by $(\lambda I - Q)$ we obtain

$$\mathscr{Z}_\lambda^{-1}[\bar{\alpha}] = (I - \mathscr{T}_\lambda)^{-1}[\alpha] \tag{61}$$

Hence $\mathscr{Z}_\lambda^{-1}\bar{\mathscr{M}} = 0, \bar{\mathscr{M}} = 0$, so that $\bar{M}^{11} = \bar{M}^{22} = 0$. By (61) it follows that $[\bar{\alpha}] = \mathscr{Z}_\lambda(I - \mathscr{T}_\lambda)^{-1}[\alpha]$. Since $[\alpha^a, 1] \leqslant 1$, it follows that $\bar{h}^{aa} < \infty$ $(a = 1, 2)$. Conversely, if $\bar{M}^{11} = \bar{M}^{22} = 0$ and $\bar{h}^{aa} < \infty$ $(a = 1, 2)$ then $[F(\lambda)]$ in (56) obviously has the form given by (20) and (21).

(e) The necessary and sufficient condition for the process to be honest or to satisfy the system of forward equations is obvious. Thus the proof is concluded.

QED

5.11 THE CONDITION THAT $\alpha\phi(\lambda)\in l$

Suppose $\alpha \geqslant 0$. By (7.3) it follows that $\alpha\phi(\lambda)\in l$ is equivalent to

$$\sum_i \alpha_i[1 - X_i^1(\lambda) - X_i^2(\lambda)] < \infty \qquad \lambda > 0 \tag{1}$$

In this section, we shall give the necessary and sufficient condition under which we decide whether $\alpha\phi(\lambda)\in l$ directly from Q. Obviously, it suffices to consider $\sum_{i\geqslant 0}$ and $\sum_{i\leqslant 0}$ of (1). We shall only consider $\sum_{i\geqslant 0}$, for the case $\sum_{i\leqslant 0}$ is completely similar.

Lemma 1. We have

$$u_{1i}(\lambda) = u_{2i}(\lambda)\sum_{j\geqslant i}\frac{z_{j+1}-z_j}{u_{2j}(\lambda)u_{2,j+1}(\lambda)} \tag{2}$$

for $u_1(\lambda)$ and $u_2(\lambda)$ in Theorem 5.9.

Proof. Let the quantity determined by the right-hand side of (2) be $v_i(\lambda)$. We prove that $v(\lambda)$ is a decreasing solution of equation (5.1) and (5.14) holds, where $v(\lambda)$ is replaced by $u_1(\lambda)$.

First since $u_2(\lambda)$ and $u_2^+(\lambda)$ are increasing,

$$\begin{aligned} 0 < v_i(\lambda) &= u_{2i}(\lambda)\sum_{j\geqslant i}\frac{z_{j+1}-z_j}{u_{2j}(\lambda)u_{2,j+1}(\lambda)} = u_{2i}(\lambda)\sum_{j\geqslant i}\frac{1}{u_{2j}^+(\lambda)}\left(\frac{1}{u_{2j}(\lambda)} - \frac{1}{u_{2,j+1}(\lambda)}\right) \\ &< \frac{u_{2i}(\lambda)}{u_{2i}^+(\lambda)}\sum_{j\geqslant i}\left(\frac{1}{u_{2j}(\lambda)} - \frac{1}{u_{2,j+1}(\lambda)}\right) = \frac{u_{2i}(\lambda)}{u_{2i}^+(\lambda)}\left(\frac{1}{u_{2i}(\lambda)} - \frac{1}{u_2(r_2,\lambda)}\right) \\ &\leqslant \frac{1}{u_{2i}^+(\lambda)} < \infty \end{aligned} \tag{3}$$

Therefore, the series (2) is convergent.

Secondly,

$$v_i^+(\lambda) = u_{2i}^+(\lambda)\sum_{j\geqslant i}\frac{z_{j+1}-z_j}{u_{2j}(\lambda)u_{2,j+1}(\lambda)} - \frac{1}{u_{2i}(\lambda)} \tag{4}$$

From (3) we see that

$$v_i^+(\lambda) < \frac{1}{u_{2i}(\lambda)} - \frac{1}{u_{2i}(\lambda)} = 0$$

Hence $v(\lambda)$ is decreasing. By (2) and (4) it follows that (5.14), where $v(\lambda)$ is replaced by $u_1(\lambda)$, holds. By (4) we obtain

$$D_\mu v_i^+(\lambda) = D_\mu u_{2i}^+(\lambda)\sum_{j\geqslant i}\frac{z_{j+1}-z_j}{u_{2j}(\lambda)u_{2,j+1}(\lambda)} = \lambda u_{2i}(\lambda)\sum_{j\geqslant i}\frac{z_{j+1}-z_j}{u_{2j}(\lambda)u_{2,j+1}(\lambda)} = \lambda v_i(\lambda)$$

that is, $v(\lambda)$ is a strictly decreasing positive solution of equation (5.1).

If r_2 is irregular, by Theorem 5.7, the decreasing solution of equation (5.1) satisfying (5.14) is unique. Hence $v(\lambda) = u_1(\lambda)$.

If r_2 is regular, according to Theorem 5.7

$$v_i(\lambda) = K[v_i - \theta s_i] \qquad (K \text{ is a constant})$$

By (2), we have $0 = v(r_2,\lambda) = K[v(r_2) - \theta s(r_2)]$, so that

$$\theta = \frac{v(r_2)}{s(r_2)} = \theta$$

Hence $v(\lambda) = Ku_1(\lambda)$. Therefore, $v^+(\lambda) = Ku_1^+(\lambda)$. By (4) and the first expession of (6.8), we have

$$-\frac{1}{u_2(r_2,\lambda)} = v^+(r_2,\lambda) = Ku_1^+(r_2,\lambda) = -\frac{K}{u_2(r_2,\lambda)}$$

So $K = 1$, and thus, $v(\lambda) = u_1(\lambda)$. The proof is completed. QED

Lemma 2. Let r_2 be regular or exit, then

$$\lim_{z_n\to r_2}\frac{u_{1n}(\lambda)}{r_2 - z_n} = \frac{1}{u_2(r_2,\lambda)} \tag{5}$$

Proof. By (2)

$$\begin{aligned} u_{1n}(\lambda) &< \frac{u_{2n}(\lambda)}{[u_{2n}(\lambda)]^2}\sum_{j\geqslant n}(z_{j+1}-z_j) = \frac{r_2 - z_n}{u_{2n}(\lambda)} \\ u_{1n}(\lambda) &> \frac{u_{2n}(\lambda)}{[u_2(r_2,\lambda)]^2}\sum_{j\geqslant n}(z_{j+1}-z_j) = \frac{u_{2n}(\lambda)(r_2 - z_n)}{[u_2(r_2,\lambda)]^2} \end{aligned} \tag{6}$$

That is,

$$\frac{u_{2n}(\lambda)}{[u_2(r_2,\lambda)]^2} < \frac{u_{1n}(\lambda)}{r_2 - z_n} < \frac{1}{u_{2n}(\lambda)}$$

From this, we obtain (5), and the proof is completed. QED

Theorem 3. Let r_2 be regular, then

$$\sum_{i\geqslant 0}\alpha_i[1 - X_i^1(\lambda) - X_i^2(\lambda)] < \infty \tag{8}$$

if and only if

$$\sum_{i\geqslant 0}\alpha_i(r_2 - z_i) < \infty \tag{9}$$

or equivalently

$$\sum_{i\geqslant 0}\alpha_i N_i<\infty \tag{10}$$

where

$$N_i=\sum_{j\geqslant i}(z_{j+1}-z_j)\sum_{k=0}^{j}\mu_k=\left(\sum_{k=0}^{i}\mu_k\right)(r_2-z_i)+\sum_{k\geqslant i+1}(r_2-z_k)\mu_k \tag{11}$$

Proof. Since

$$1-X_i^1(\lambda)-X_i^2(\lambda)=\lambda u_{1i}(\lambda)\sum_{j\leqslant i}u_{2j}(\lambda)\mu_j+\lambda u_{2i}(\lambda)\cdot\sum_{j>i}u_{1j}(\lambda)\mu_j \tag{12}$$

and when r_2 is regular we have $\sum_{j\geqslant 0}\mu_j<\infty$, from (5). Letting

$$0<\frac{u_{2i}(\lambda)}{r_2-z_i}\sum_{j>i}u_{1j}(\lambda)\mu_j<\frac{u_{2i}(\lambda)u_{1i}(\lambda)}{r_2-z_i}\sum_{j>i}\mu_j\to 1\times 0=0$$

Therefore

$$\lim_{i\to+\infty}\frac{1-X_i^1(\lambda)-X_i^2(\lambda)}{r_2-z_i}=\lim_{i\to+\infty}\frac{\lambda u_{1i}(\lambda)}{r_2-z_i}\sum_{j\leqslant i}u_{2j}(\lambda)\mu_j=\lambda[X^2(\lambda)\mu,1]$$

If r_2 is regular, then $0<\lambda[X^2(\lambda)\mu,1]<\infty$. From this it follows that (8) and (9) are equivalent. Secondly, (10) clearly implies (9). By (11)

$$N_i<(r_2-z_i)\sum_{j\geqslant 0}\mu_j$$

Therefore, if r_2 is regular, then (9) also implies (10), and the proof is concluded.

Theorem 4. Suppose that r_2 is exit, then (8) and (10) are equivalent, and (8) or (10) implies (9).

Proof. By (6), (7) and (11), if $j\geqslant i\geqslant 0$

$$u_{1j}(\lambda)<\frac{r_2-z_j}{u_{2j}(\lambda)}\leqslant\frac{r_2-z_j}{u_{2i}(\lambda)}\leqslant\frac{r_2-z_i}{u_{20}(\lambda)} \tag{13}$$

$$u_{1j}(\lambda)>\frac{u_{2j}(\lambda)}{[u_2(r_2,\lambda)]^2}(r_2-z_j)\geqslant\frac{u_{2i}(\lambda)}{[u_2(r_2,\lambda)]^2}(r_2-z_j) \tag{14}$$

$$\frac{u_{1j}(\lambda)}{N_j}<\frac{r_2-z_j}{u_{2j}(\lambda)N_j}\leqslant\frac{\mu_0(r_2-z_j)}{\mu_0u_{20}(\lambda)N_j}<\frac{1}{u_{20}(\lambda)\mu_0} \tag{15}$$

If $i\geqslant 0$

$$u_{1i}(\lambda)\sum_{j=0}^{i}u_{2j}(\lambda)\mu_j\leqslant\frac{u_{1i}(\lambda)u_{2i}(\lambda)}{r_2-z_i}\left(\sum_{j=0}^{i}\mu_j\right)(r_2-z_i) \tag{16}$$

$$u_{1i}(\lambda)\sum_{j=0}^{i}u_{2j}(\lambda)\mu_j\geqslant\frac{u_{1i}(\lambda)u_{20}(\lambda)}{r_2-z_i}\left(\sum_{j=0}^{i}\mu_j\right)(r_2-z_i) \tag{17}$$

Hence by (12), (15), (16) and (13)

$$\begin{aligned}&\frac{1-X_i^1(\lambda)-X_i^2(\lambda)}{N_i}\\&\leqslant\frac{\lambda}{u_{20}(\lambda)\mu_0}\sum_{i<0}u_{2j}(\lambda)\mu_j+\lambda\frac{u_{1i}(\lambda)u_{2i}(\lambda)}{r_2-z_i}\frac{\left(\sum_{j=0}^{i}\mu_j\right)(r_2-z_i)}{N_i}+\lambda\frac{u_{2i}(\lambda)}{N_i}\sum_{j>i}\frac{(r_2-z_j)\mu_j}{u_{20}(\lambda)}\\&\leqslant\frac{\lambda}{u_{20}(\lambda)\mu_0}\sum_{j<0}u_{2j}(\lambda)\mu_j+\lambda\frac{u_{1i}(\lambda)u_{2i}(\lambda)}{r_2-z_i}+\lambda\frac{u_{2i}(\lambda)}{u_{20}(\lambda)}\end{aligned}$$

By (5)

$$\overline{\lim_{i\to+\infty}}\frac{1-X_i^1(\lambda)-X_i^2(\lambda)}{N_i}\leqslant\frac{\lambda}{u_{20}(\lambda)\mu_0}\sum_{j<0}u_{2j}(\lambda)\mu_j+\lambda+\lambda\frac{u_2(r_2,\lambda)}{u_{20}(\lambda)}<\infty \tag{18}$$

Similarly, by (12), (17) and (14)

$$\begin{aligned}&\frac{1-X_i^1(\lambda)-X_i^2(\lambda)}{N_i}\\&>\lambda\frac{u_{1i}(\lambda)}{N_i}\sum_{j=0}^{i}u_{2j}(\lambda)\mu_j+\lambda\frac{u_{2i}(\lambda)}{N_i}\sum_{j>i}u_{1j}(\lambda)\mu_j\\&>\lambda\frac{u_{i1}(\lambda)u_{20}(\lambda)}{r_2-z_i}\frac{\sum_{j=0}^{i}\mu_j(r_2-z_i)}{N_i}+\frac{\lambda u_{2i}(\lambda)}{N_i}\sum_{j>i}\frac{u_{2i}(\lambda)}{[u_2(r_2,\lambda)]^2}(r_2-z_j)\mu_j\\&>\frac{\lambda u_{2i}(\lambda)u_{20}(\lambda)}{[u_2(r_2,\lambda)]^2}\frac{\sum_{j=0}^{i}\mu_j(r_2-z_i)}{N_i}+\frac{\lambda u_{2i}(\lambda)u_{20}(\lambda)}{[u_2(r_2,\lambda)]^2}\frac{\sum_{j>i}(r_2-z_j)\mu_j}{N_j}\\&=\lambda X_i^2(\lambda)X_0^2(\lambda)\end{aligned}$$

and so

$$\begin{aligned}&\underline{\lim_{i\to+\infty}}\frac{1-X_i^1(\lambda)-X_i^2(\lambda)}{N_i}\\&\geqslant\lambda X_0^2(\lambda)>0\end{aligned} \tag{19}$$

Theorem 4 follows from (18) and (19), and the proof is terminated. QED

Theorem 5. If r_2 is entrance or natural, then (8) holds if and only if

$$\sum_{i\geqslant 0}\alpha_i<\infty \tag{20}$$

Proof. The sufficiency is obvious. The necessity follows from

$$\lim_{i\to+\infty}[1-X_i^1(\lambda)]=1-X^1(r_2,\lambda)>0 \qquad \text{QED}$$

CHAPTER 6

Birth–Death Processes

6.1 INTRODUCTION

Let $E = \{0, 1, 2, 3, \ldots\}$. The matrix Q has the form

$$Q = \begin{pmatrix} -(a_0 + b_0) & b_0 & 0 & 0 & \cdots \\ a_1 & -(a_1 + b_1) & b_1 & 0 & \cdots \\ 0 & a_2 & -(a_2 + b_2) & b_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \tag{1}$$

where $a_0 \geqslant 0, b_0 > 0, a_i > 0, b_i > 0 (i > 0)$. We call the Q process of the form (1) a birth–death process. In this chapter, Q will always have the form (1), and the Q process will always be understood to refer to a birth–death process.

We point out that, for a matrix Q of the form (1), the relation

$$q_i = \sum_{j \neq i} q_{ij} \tag{2}$$

is satisfied when $a_0 = 0$, and is not satisfied for $i = 0$ when $a_0 > 0$. That is, Q is conservative when $a_0 = 0$ and is single non-conservative when $a_0 > 0$. The dimension of the solution space $\mathscr{M}_\lambda$ is $m^+ \leqslant 1$ when $a_0 \geqslant 0$. Therefore construction of the birth–death process in the case $a_0 = 0$ or in the case $a_0 > 0$ and $m^+ = 0$ is solved by Theorem 3.2.1 and Theorem 3.3.3. It is slightly more complex if $a_0 > 0$ and $m^+ = 1$, because in this case it is possible that there exist Q processes that satisfy the system of backward or forward equations, and also Q processes that satisfy neither the system of forward equations nor that of backward equations. Birth–death processes are regarded as diffusions in Feller (1957b, 1971) and all possible birth–death processes that satisfy simultaneously the system of backward and forward equations are found in the case $a_0 \geqslant 0$. The author (Xiang-qun Yang, 1965a) has constructed all possible birth-death processes.

This chapter is concerned with the construction theory of birth–death processes, which is very similar to the construction theory of bilateral birth–death processes dealt with in the previous chapter. For this reason we shall make simple statements or supply brief and simple verifications on some occasions.

6.2 CLASSIFICATION OF BOUNDARY POINT AND SECOND-ORDER DIFFERENCE OPERATOR

For the matrix Q of (1.1), we call

$$\begin{aligned} &z_0 = 1/a_0 \,(\text{if } a_0 > 0) \qquad z_0 = 0 \,(\text{if } a_0 = 0) \\ &z_1 = z_0 + 1/b_0 \\ &\vdots \\ &z_n = z_0 + \frac{1}{b_0} + \cdots + \frac{a_1 a_2 \cdots a_{n-1}}{b_0 b_1 b_2 \cdots b_{n-1}} \qquad (n = 2, 3, \ldots) \end{aligned} \tag{1}$$

the natural scale; we call

$$z = \lim_{n \to +\infty} z_n \tag{2}$$

the boundary point; and we call

$$\mu_0 = 1 \qquad \mu_n = \frac{b_0 b_1 \cdots b_{n-1}}{a_1 \cdots a_{n-1} a_n} \qquad (n > 1) \tag{3}$$

the canonical measure.

The boundary point z may be classified by means of the natural scale and the canonical measure. We say that the boundary point z is

(i) regular if $z < \infty, \sum_i \mu_i < \infty$;

(ii) exit if it is not regular and $\sum_i (z - z_i)\mu_i < \infty$;

(iii) entrance if it is not regular and $\sum_i z_i \mu_i < \infty$;

(iv) natural for all other cases.

We also introduce the characteristic numbers as follows:

$$\begin{aligned} &m_0 = 1/b_0 = (z_1 - z_0)\mu_0 \\ &m_i = \frac{1}{b_i} + \sum_{k=0}^{i-1} \frac{a_i a_{i-1} \cdots a_{i-k}}{b_i b_{i-1} \cdots b_{i-k} b_{i-k-1}} = (z_{i+1} - z_i) \sum_{k=0}^{i} \mu_k \qquad i > 0 \end{aligned} \tag{4}$$

$$\begin{aligned} &e_0 = z_0 \sum_{k=0}^{\infty} \mu_k \\ &e_i = \frac{1}{a_i} + \sum_{k=0}^{\infty} \frac{b_i b_{i+1} \cdots b_{i+k}}{a_i a_{i+1} \cdots a_{i+k} a_{i+k+1}} = (z_i - z_{i-1}) \sum_{k=i}^{\infty} \mu_k \qquad i > 0 \end{aligned} \tag{5}$$

$$N_i = \sum_{j=i}^{\infty} m_j = (z - z_i) \sum_{j=0}^{i} \mu_j + \sum_{j=i+1}^{\infty} (z - z_j)\mu_j \tag{6}$$

$$R = \sum_{j=0}^{\infty} m_j = \sum_{j=0}^{\infty} (z - z_j)\mu_j$$
$$S = \sum_{j=0}^{\infty} e_j = \sum_{j=0}^{\infty} z_j\mu_j \tag{7}$$

Theorem 1. The boundary point z is

(i) regular if and only if $R < \infty, S < \infty$;
(ii) exit if and only if $R < \infty, S = \infty$;
(iii) entrance if and only if $R = \infty, S < \infty$;
(iv) natural if and only if $R = \infty, S = \infty$.

Proof. It is similar to Theorem 5.3.1.

We can introduce the second-order difference operator by (5.4.1), but in the case $i = 0$. We shall make appropriate modifications.

Suppose that u is a column vector on E. We define u^+ as follows:

$$u_i^+ = \frac{u_{i+1} - u_i}{z_{i+1} - z_i} \qquad (i \geqslant 0) \tag{8}$$

For convenience, we will adopt the following convention from now on:

$$u_{-1}^+ = a_0 u_0 \qquad u_{-1} = 0 \tag{9}$$

Let u be a column vector on $\{-1\} \cup E$. We define $D_\mu u$ as follows:

$$D_\mu u_i = \frac{u_i - u_{i-1}}{\mu_i} \qquad (i \geqslant 0) \tag{10}$$

Theorem 2. If u is a column vector on E, then

$$Qu = D_\mu u^+ \tag{11}$$

That is,

$$a_i u_{i-1} - (a_i + b_i)u_i + b_i u_{i+1} = D_\mu u_i^+ \qquad i \geqslant 0 \tag{12}$$

Proof. It is similar to Theorem 5.4.1, but we must notice convention (9).

Theorem 3. Theorem 5.4.2, together with its corollary, and Lemma 5.4.3 are also true, and even their notation need not be changed, if only we understand E as referring to the set of non-negative integers and Q is the matrix of (1.1).

Lemma 4. The solutions of equations

$$\begin{aligned} &-(a_0 + b_0)u_0 + b_0 u_1 = -f_0 \\ &a_i u_{i-1} - (a_i + b_i)u_i + b_i u_{i+1} = -f_i \qquad 0 < i < n \\ &u_n = f_n \end{aligned} \tag{13}$$

are

$$\begin{aligned} u_i = &\frac{z_n - z_i}{a_0(z_n - z_0) + 1} f_0 + \frac{a_0(z_i - z_0) + 1}{a_0(z_n - z_0) + 1} f_n \\ &+ \frac{z_n - z_i}{a_0(z_n - z_0) + 1} \sum_{j=1}^{i-1} [a_0(z_j - z_0) + 1] f_j \mu_j \\ &+ \frac{a_0(z_i - z_0) + 1}{a_0(z_n - z_0) + 1} \sum_{j=i}^{n-1} (z_n - z_j) f_j \mu_j \end{aligned} \tag{14}$$

Proof. By (5.4.12)

$$\begin{aligned} u_i = &\frac{z_n - z_i}{z_n - z_0} u_0 + \frac{z_i - z_0}{z_n - z_0} f_n + \frac{z_n - z_i}{z_n - z_0} \sum_{j=1}^{i-1} (z_j - z_0) f_j \mu_j \\ &+ \frac{z_i - z_0}{z_n - z_0} \sum_{j=i}^{n-1} (z_n - z_j) f_j \mu_j \qquad 0 < i < n \end{aligned} \tag{15}$$

In particular,

$$u_1 = \frac{z_n - z_1}{z_n - z_0} u_0 + \frac{z_1 - z_0}{z_n - z_0} f_n + \frac{z_1 - z_0}{z_n - z_0} \sum_{j=1}^{n-1} (z_n - z_j) f_j \mu_j \tag{16}$$

So by (13)

$$u_0 = \frac{b_0}{a_0 + b_0} u_1 + \frac{f_0}{a_0 + b_0} = \frac{u_1}{a_0(z_1 - z_0) + 1} + \frac{(z_1 - z_0) f_0}{a_0(z_1 - z_0) + 1} \tag{17}$$

By (16) and (17) we derive

$$u_0 = \frac{z_n - z_0}{a_0(z_n - z_0) + 1} f_0 + \frac{1}{a_0(z_n - z_0) + 1} f_n + \frac{1}{a_0(z_n - z_0) + 1} \sum_{j=1}^{n-1} (z_n - z_j) f_j \mu_j$$

Substituting the above expression into (15) we obtain (14), and the proof is completed. QED

Lemma 5. The solution of the equation

$$D_\mu u^+ = f \tag{18}$$

namely the system of equations

$$\begin{aligned} &-(a_0 + b_0)u_0 + b_0 u_1 = f_0 \\ &a_i u_{i-1} - (a_i + b_i)u_i + b_i u_{i+1} = f_i \qquad (i > 0) \end{aligned} \tag{19}$$

is

$$u_i = [a_0(z_i - z_0) + 1]u_0 + \sum_{j=0}^{i-1} (z_i - z_j) f_j \mu_j \tag{20}$$

Proof. By (19) we have

$$u_0^+ = a_0 u_0 + f_0 \mu_0$$
$$u_i^+ - u_{i-1}^+ = f_i \mu_i \qquad i>0 \tag{21}$$

Therefore

$$u_i^+ = u_0^+ + \sum_{j=1}^{i} (u_j^+ - u_{j-1}^+) = a_0 u_0 + f_0 \mu_0 + \sum_{j=1}^{i} f_j \mu_j = a_0 u_0 + \sum_{j=0}^{i} f_j \mu_j$$

$$u_i = u_0 + \sum_{j=0}^{i-1} (u_{j+1} - u_j) = u_0 + \sum_{j=0}^{i-1} u_j^+ (z_{j+1} - z_j)$$

$$= u_0 + \sum_{j=0}^{i-1} \left(a_0 u_0 + \sum_{k=0}^{i} f_k \mu_k \right) (z_{j+1} - z_j)$$

$$= u_0 + a_0(z_i - z_0) u_0 + \sum_{j=0}^{i-1} (z_{j+1} - z_j) \sum_{k=0}^{j} f_k \mu_k$$

$$= [a_0(z_i - z_0) + 1] u_0 + \sum_{j=0}^{i-1} (z_i - z_j) f_j \mu_j \qquad \text{QED}$$

Corollary

The solution of the equation

$$Qu = 0 \tag{22}$$

is

$$u_i = [a_0(z_i - z_0) + 1] u_0 \qquad i \geqslant 0 \tag{23}$$

6.3 SOLUTION OF THE EQUATION $\lambda u - D_\mu u^+ = 0$

Theorem 1. For every $\lambda > 0$, the solution $u(\lambda)$ of the equation

$$\lambda u - D_\mu u^+ = 0 \tag{1}$$

with condition $u_0 = 1$ exists and is unique, and has the following properties:

(i) Both $u(\lambda)$ and $u^+(\lambda)$ are strictly increasing;
(ii) $u(z, \lambda) \equiv \lim_{i\to\infty} u_i(\lambda) < \infty$ if and only if z is exit or regular;
(iii) $u(\lambda)\mu \in l$, i.e. $u^+(z, \lambda) \equiv \lim_{i\to\infty} u_i^+(\lambda) < \infty$ if and only if z is regular or entrance.

Proof. By (2.20) it follows that $u_i(\lambda)$ necessarily satisfies

$$u_i(\lambda) = 1 + a_0(z_i - z_0) + \sum_{j=0}^{i-1} (z_i - z_j) u_j(\lambda) \mu_j \tag{2}$$

From this we can determine $u_1(\lambda), u_2(\lambda), \ldots$, and hence $u(\lambda)$ exists and is unique. By (2) it follows that $u(\lambda)$ is strictly increasing.

Secondly,

$$u_i^+(\lambda) = a_0 + \sum_{j=0}^{i} D_\mu u_j^+(\lambda) \mu_j = a_0 + \lambda \sum_{j=0}^{i} u_j(\lambda) \mu_j \tag{3}$$

So $u^+(\lambda)$ is strictly increasing.

The proof of (ii) and (iii) is similar to Lemma 5.4.5, and the proof is completed.

Theorem 2. Suppose that $u(\lambda)$ is the solution of (1) in Theorem 1; set

$$v_i(\lambda) = u_i(\lambda) \sum_{j=i}^{\infty} \frac{z_{j+} - z_j}{u_j(\lambda) u_{j+1}(\lambda)} \tag{4}$$

Then $v(\lambda)$ is strictly decreasing while $v^+(\lambda)$ is strictly increasing, and

$$\lambda v_i(\lambda) - D_\mu v_i^+(\lambda) = \begin{cases} 0 & \text{if } i > 0 \\ 1 & \text{if } i = 0 \end{cases} \tag{5}$$

$$u_i^+(\lambda) v_i(\lambda) - v_i^+(\lambda) u_i(\lambda) = 1 \qquad i \geqslant 0 \tag{6}$$

$$v^+(z, \lambda) \equiv \lim_{i\to\infty} v_i^+(\lambda) = -\frac{1}{u(z, \lambda)} \tag{7}$$

(The right-hand side is zero when $u(z, \lambda) = \infty$.)

Proof. As we did in Lemma 5.11.1, we can obtain the convergence of the series in (4) in a way similar to (5.11.3). Similarly to (5.11.4) we have

$$v_i^+(\lambda) = u_i^+(\lambda) \sum_{j\geq i} \frac{z_{j+1} - z_j}{u_j(\lambda) u_{j+1}(\lambda)} - \frac{1}{u_i(\lambda)} \tag{8}$$

$$v_i^+(\lambda) < \frac{1}{u_i(\lambda)} - \frac{1}{u_i(\lambda)} = 0 \tag{9}$$

Hence $v(\lambda)$ is strictly decreasing. From (8) we obtain (6). And again by (8)

$$-\frac{1}{u_i(\lambda)} \leqslant v_i^+(\lambda) \leqslant \sum_{j=i}^{\infty} u_j^+(\lambda) \frac{z_{j+1} - z_j}{u_j(\lambda) u_{j+1}(\lambda)} - \frac{1}{u_i(\lambda)}$$

$$= \sum_{j=i}^{\infty} \left(\frac{1}{u_j(\lambda)} - \frac{1}{u_{j+1}(\lambda)} \right) - \frac{1}{u_i(\lambda)}$$

$$= -\frac{1}{u(z, \lambda)} \tag{10}$$

From this follows (7). By (8) and (2.10) we obtain

$$D_\mu v_i^+(\lambda) = D_\mu u_i^+(\lambda) \sum_{j\geq i} \frac{z_{j+1} - z_j}{u_j(\lambda) u_{j+1}(\lambda)} = \lambda v_i(\lambda)$$

when $i > 0$, and we have proved the first line of (5).

To prove the second line of (5) note that by definition (4) of $v(\lambda)$ we have

$$v_1(\lambda) = v_0(\lambda)u_1(\lambda) - \frac{z_1 - z_0}{u_0(\lambda)} = v_0(\lambda)u_1(\lambda) - b_0^{-1} \tag{11}$$

As $u(\lambda)$ is the solution of (1), so

$$b_0 u_1(\lambda) = (\lambda + a_0 + b_0)u_0(\lambda) = \lambda + a_0 + b_0 \tag{12}$$

Substituting the above expression into (11) we get the second line of (5), and the proof is terminated. QED

6.4 CONSTRUCTION OF THE MINIMAL SOLUTION

For the solution $u(\lambda)$ of (3.1) and $v(\lambda)$ determined by (3.4),

$$\phi_{ij}(\lambda) = \begin{cases} u_i(\lambda)v_j(\lambda)\mu_j & \text{if } j \geq i \\ v_i(\lambda)u_j(\lambda)\mu_j & \text{if } j \leq i \end{cases} \tag{1}$$

Similarly to (5.6.1)–(5.6.5) we have

$$\mu_i \phi_{ij}(\lambda) = \mu_j \phi_{ji}(\lambda) \tag{2}$$

If f is a column vector and g is a row vector, then

$$[\phi(\lambda)f]_i = \sum_j \phi_{ij}(\lambda) f_j = v_i(\lambda) \sum_{j=0}^{i} u_j(\lambda) f_j \mu_j + u_i(\lambda) \sum_{j=i+1}^{\infty} v_j(\lambda) f_j \mu_j \tag{3}$$

$$[g\phi(\lambda)]_i = \sum_i g_i \phi_{ij}(\lambda) = v_j(\lambda)\mu_j \sum_{i=0}^{j} g_i u_i(\lambda) + u_j(\lambda)\mu_j \sum_{i=j+1}^{\infty} g_i v_i(\lambda) \tag{4}$$

If $g = f\mu$ then

$$g\phi(\lambda) = (\phi(\lambda)f)\mu \tag{5}$$

Theorem 1. The following holds[1]:

$$\lambda \sum_j \phi_{ij}(\lambda) = 1 - a_0 v_i(\lambda) - \frac{u_i(\lambda)}{u(z, \lambda)} \tag{6}$$

Proof. By (3) and recalling convention (2.9),

$$\lambda \sum_j \phi_{ij}(\lambda) = v_i(\lambda) \sum_{j=0}^{i} [u_j^+(\lambda) - u_{j-1}^+(\lambda)] + u_i(\lambda) \sum_{j=i+1}^{\infty} [v_j^+(\lambda) - v_{j-1}^+\lambda)]$$

$$= v_i(\lambda)u_i^+(\lambda) - u_i(\lambda)v_i^+(\lambda) - a_0 v_i(\lambda) + u_i(\lambda)v^+(z, \lambda)$$

On account of (3.6) and (3.7) the above expression is precisely (6). QED

[1] Suppose $0/00 = 0$.

Theorem 2. If $f \in m, g \in l$ then $\phi(\lambda)f \in m, g\phi(\lambda) \in l$ and

$$\lambda[\phi(\lambda)f] - Q[\phi(\lambda)f] = f \qquad \lambda > 0 \tag{7}$$

$$\lambda[g\phi(\lambda)] - [g\phi(\lambda)]Q = g \qquad \lambda > 0 \tag{8}$$

Proof. We only prove (7). In the case $i > 0$ the proof is similar to Theorem 5.6.2. But in the case $i = 0$ we must take care. By (3) we obtain

$$[\phi(\lambda)f]_i^+ = v_i^+(\lambda) \sum_{j=0}^{i} u_j(\lambda) f_j \mu_j + u_i^+(\lambda) \sum_{j=i+1}^{\infty} v_j(\lambda) f_j \mu_j \tag{9}$$

$$[\phi(\lambda)f]_0^+ = v_0^+(\lambda) f_0 + u_0^+(\lambda) \sum_{j=1}^{\infty} v_j(\lambda) f_j \mu_j \tag{10}$$

Recall the convention (2.9),

$$[\phi(\lambda)f]_{-1}^+ = a_0[\phi(\lambda)f]_0 = a_0 v_0(\lambda) f_0 + a_0 u_0(\lambda) \sum_{j=1}^{\infty} v_i(\lambda) f_j \mu_j$$

$$= v_{-1}^+(\lambda) f_0 + u_{-1}^+(\lambda) \sum_{j=1}^{\infty} v_j(\lambda) f_j \mu_j \tag{11}$$

By (10), (11) and (3.5)

$$D_\mu[\phi(\lambda)f]_0^+ = D_\mu v_0^+(\lambda) f_0 + D_\mu u_0^+(\lambda) \sum_{j=1}^{\infty} v_j(\lambda) f_j \mu_j$$

$$= [\lambda v_0(\lambda) - 1] f_0 + \lambda u_0(\lambda) \sum_{j=1}^{\infty} v_j(\lambda) f_j \mu_j$$

$$= [\phi(\lambda)f]_0 - f_0$$

QED

Lemma 3. Suppose that $f \in m$, and z is regular or exit. Then

$$[\phi(\lambda)f](z) = \lim_{i \to \infty} [\phi(\lambda)f]_i = 0 \tag{12}$$

Proof. Because if z is regular or exit, then $z < \infty$ and $u(z, \lambda) < \infty$. By the definition (3.4) of $v(\lambda)$ we have $\lim_{i \to \infty} v_i(\lambda) = v(z, \lambda) = 0$. By (6) we obtain $[\lambda\phi(\lambda)\mathbf{1}](z) = 0$. From this follows (12). QED

Theorem 4. $\phi(\lambda)$ is the minimal Q process. For $\phi(\lambda)$ to be honest it is necessary and sufficient that $a_0 = 0$ and z is entrance or natural.

If z is natural, then $\phi(\lambda)$ is the unique Q process satisfying the system of either forward or backward equations.

If z is entrance, then $\phi(\lambda)$ is the unique Q process satisfying the system of backward equations.

If z is exit, then $\phi(\lambda)$ is the unique Q process satisfying the system of forward equations.

Proof. From Theorem 1 the norm condition of $\phi(\lambda)$ follows. By Theorem 2 we obtain the B condition and F condition for $\phi(\lambda)$. To prove the resolvent equation for $\phi(\lambda)$ let $f \in m$ and $F(\lambda) = \phi(\lambda)f$. Then $F(\lambda) - F(\nu) + (\lambda - \nu)\phi(\lambda)F(\nu) \in m$ is the solution of equation (3.1). Consequently,

$$F(\lambda) - F(\nu) + (\lambda - \nu)\phi(\lambda)F(\nu) = cu(\lambda) \tag{13}$$

where c is a constant. If z is regular or exit, by Lemma 3 we get $cu(z, \lambda) = 0$ and $c = 0$ as $i \to \infty$ in the above expression. If z is entrance or natural, then because the left-hand side of (13) is bounded and $u(\lambda)$ is unbounded, we get $c = 0$. Hence we always have $c = 0$. Take $f_i = \delta_{ij}$ and obtain the resolvent equation of $\phi(\lambda)$. Thus, $\phi(\lambda)$ is the Q process satisfying the system of backward and forward equations.

Suppose that $\psi(\lambda)$ is an arbitrary Q process. By the backward inequality (2.8.10) and Theorem 2.7.3 it follows that for fixed $j, u_i \equiv \psi_{ij}(\lambda) - \phi_{ij}(\lambda)$ satisfy the following equation:

$$\lambda u_i - \sum_k q_{ik} u_k = \begin{cases} c_1 \geqslant 0 & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases} \tag{14}$$

Hence $u_i - c_1 v_i(\lambda)$ is a solution of equation (3.1), so that $u_i - c_1 v_i(\lambda) = c_2 u_i(\lambda)$; that is,

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + c_1 v_i(\lambda) + c_2 u_i(\lambda) \tag{15}$$

where c_1 and c_2 are constants which are independent of i, and $c_1 \geqslant 0$.

If z is exit or regular, by Lemma 3 and $v(z, \lambda) = 0$, which is proved in Lemma 3, we obtain $c_2 u(z, \lambda) \geqslant 0$ as $i \to \infty$ in (15) and, hence, $c_2 \geqslant 0$. If z is entrance or natural, as the left-hand side of (15) is bounded, and only $u(\lambda)$ is unbounded in the right-hand side of (15) it follows that $c_2 = 0$. Thus, we always have $c_2 \geqslant 0$, that is $\psi(\lambda) \geqslant \phi(\lambda)$. We have proved the minimality of $\phi(\lambda)$.

If $\psi(\lambda)$ satisfies the system of backward equations, then $u_i = \psi_{ij}(\lambda) - \phi_{ij}(\lambda)$ is a solution of equation (3.1), so that

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + cu_i(\lambda) \tag{16}$$

where the constant $c \geqslant 0$ is independent of i. If z is entrance or natural, we can prove $c = 0$ in the same way as was used in the proof of the minimality of $\phi(\lambda)$; hence $\phi(\lambda)$ is the unique Q process satisfying the system of backward equations.

If $\psi(\lambda)$ satisfies the system of forward equations, then $v_j = \psi_{ij}(\lambda) - \phi_{ij}(\lambda)$ is a solution of the equation

$$\lambda v - vQ = 0 \tag{17}$$

whereas $u(\lambda)\mu$ is the unique linearly independent solution of (17), so that

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + cu_j(\lambda)\mu_j \tag{18}$$

where c is a constant independent of j. If z is exit or natural, then by the norm condition of $\psi(\lambda)$ and $\phi(\lambda)$ we obtain $c = 0$. Hence $\phi(\lambda)$ is the unique Q process satisfying the system of forward equations, and the proof is concluded.

Remark

If $a_0 = 0$ then $c_1 = 0$ in (14). Therefore by the proof of this theorem it can be seen that any Q process $\psi(\lambda)$ has the following form:

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + a_0 v_i(\lambda) F_j^1(\lambda) + \frac{u_i(\lambda)}{u(z, \lambda)} F_j^2(\lambda) \tag{19}$$

where $F^a(\lambda) \geqslant 0$. Here $\psi(\lambda)$ satisfies the system of backward equations if and only if $F^1(\lambda) = 0$; and $\psi(\lambda)$ satisfies the system of forward equations if and only if $\psi(\lambda)$ has the form (18).

6.5 SEVERAL LEMMAS

From now on, we will simply write

$$X_i^1(\lambda) = a_0 v_i(\lambda) \qquad X_i^2(\lambda) = \frac{u_i(\lambda)}{u(z, \lambda)} \tag{1}$$

$$X_i^1 = \frac{a_0(z - z_i)}{a_0(z - z_0) + 1} \qquad X_i^2 = \frac{a_0(z_i - z_0) + 1}{a_0(z - z_0) + 1} \tag{2}$$

Obviously, $X^1 + X^2 = 1$, and (4.6) becomes

$$\lambda\phi(\lambda)\mathbf{1} = 1 - X^1(\lambda) - X^2(\lambda) \tag{3}$$

Lemma 1. $X^a(\lambda)\,(a = 1, 2)$ are two exit families and

$$X^1(\lambda) \downarrow 0 \qquad \lambda X_i^1(\lambda) \to \begin{cases} 0 & \text{if } i > 0 \\ a_0 & \text{if } i = 0 \end{cases} \quad (\lambda \uparrow \infty) \tag{4}$$

$$X_2(\lambda) \downarrow 0 \qquad \lambda X_2(\lambda) \to 0 \qquad (\lambda \uparrow \infty) \tag{5}$$

$$\lambda\phi(\lambda)X^a = X^a - X^a(\lambda) \qquad a = 1, 2 \tag{6}$$

Proof. It suffices to prove (6). When $a_0 = 0$, (6) is obvious for $a = 1$. For $a = 2$, (6) follows from (3). In what follows, we suppose $a_0 > 0$.

If $z = \infty$, then $X^2 = X^2(\lambda) = 0, X^1 = 1$, by (3) we also get (6).

If $z < \infty$, then

$$
\begin{aligned}
&\lambda \sum_j \phi_{ij}(\lambda)(z - z_j) \\
&= v_i(\lambda) \sum_{j=0}^{i} \lambda u_j(\lambda)\mu_j \sum_{k=j}^{\infty} (z_{k+1} - z_k) \\
&\quad + u_i(\lambda) \sum_{j=i+1}^{\infty} \lambda v_j(\lambda)\mu_j \sum_{k=j}^{\infty} (z_{k+1} - z_k) \\
&= v_i(\lambda)\left(\sum_{k=0}^{i} (z_{k+1} - z_k) \sum_{j=0}^{k} \lambda u_j(\lambda)\mu_j \right. \\
&\quad \left. + \sum_{k=i+1}^{\infty} (z_{k+1} - z_k) \sum_{j=0}^{i} \lambda u_j(\lambda)\mu_j \right) \\
&\quad + u_i(\lambda) \sum_{k=i+1}^{\infty} (z_{k+1} - z_k) \sum_{j=i+1}^{k} \lambda v_j(\lambda)\mu_j \\
&= v_i(\lambda)\left(\sum_{k=0}^{i} (z_{k+1} - z_k)[u_k^+(\lambda) - u_{-1}^+(\lambda)] \right. \\
&\quad \left. + \sum_{k=i+1}^{\infty} (z_{k+1} - z_k)[u_i^+(\lambda) - u_{-1}^+(\lambda)] \right) \\
&\quad + u_i(\lambda) \sum_{k=i+1}^{\infty} (z_{k+1} - z_k)[v_k^+(\lambda) - v_i^+(\lambda)] \\
&= v_i(\lambda)\left(\sum_{k=0}^{i} [u_{k+1}(\lambda) - u_k(\lambda)] + u_i^+(\lambda)(z - z_{i+1}) \right. \\
&\quad \left. - u_{-1}^+(\lambda)(z - z_0) \right) \\
&\quad + u_i(\lambda)\left(\sum_{k=i+1}^{\infty} [v_{k+1}(\lambda) - v_k(\lambda)] - v_i^+(\lambda)(z - z_{i+1}) \right) \\
&= v_i(\lambda)[u_i(\lambda) - u_0(\lambda) + u_i^+(\lambda)(z - z_i) - a_0 u_0(\lambda)(z - z_0)] \\
&\quad + u_i(\lambda)[v(z, \lambda) - v_i(\lambda) - v_i^+(\lambda)(z - z_i)] \\
&= [u_i^+(\lambda)v_i(\lambda) - u_i(\lambda)v_i^+(\lambda)](z - z_i) - [a_0(z - z_0) + 1]v_i(\lambda) \\
&= (z - z_i) - [a_0(z - z_0) + 1]v_i(\lambda)
\end{aligned} \tag{7}
$$

Multiplying both sides by $a_0/[a_0(z - z_0) + 1]$ we obtain (6) for $a = 1$. From this and (3), (6) follows for $a = 2$ and this completes the proof. QED

Lemma 2. (i) Let X^0 and $\bar{X}$ be the maximal passive solution and the maximal

exit solution in Definition 2.11.3, respectively. Then when z is regular or exit, $X^0 = 0, \bar{X} = X^2$; when z is entrance or natural, $X^0 = X^2, \bar{X} = 0$.

(ii) The standard image of $X^1(\lambda)$ is X^1. If z is regular or exit, then the standard image of $X^2(\lambda)$ is X^2.

Proof. By the corollary to Lemma 2.5 it follows that the unique linearly independent solution of equation (2.11.18) is X^2. But both X^0 and $\bar{X}$ are solutions of (2.11.18); hence $X^0 = cX^2, \bar{X} = \bar{c}X^2$ (c and $\bar{c}$ are constants).

Suppose z is exit or regular, then $X^2 \neq 0$. By (6)

$$
\begin{aligned}
\lambda\phi(\lambda)X^0 &= c[X^2 - X^2(\lambda)] = X^0 - cX^2(\lambda) \\
\lambda\phi(\lambda)\bar{X} &= \bar{c}[X^2 - X^2(\lambda)] = \bar{X} - \bar{c}X^2(\lambda)
\end{aligned} \tag{8}
$$

Comparing (2.11.17) with (2.11.19), we get $c = 0$, and $\bar{c} = 1$.

Suppose z is entrance or natural. If $a_0 z = \infty$, then $X^2 = 0$ and hence $X^0 = X^2, \bar{X} = 0$. If $a_0 z < \infty$ then $X^2 \neq 0$. And because $X^2(\lambda) = 0$ the solution space $\mathscr{M}_\lambda^+(1)$ is a null space, so the standard image of $\bar{X}(\lambda) = 0$ is $\bar{X} = 0$ while (6) becomes

$$\lambda\phi(\lambda)X^2 = X^2$$

By the maximality of X^0 in Lemma 2.11.2 we obtain $X^2 \leqslant X^0 = cX^2$, so that $c \geqslant 1$. But obviously, $X_i^0 = cX_i^2 \leqslant 1$, and we obtain $c \leqslant 1$ by setting $r \to \infty$. Hence $c = 1, X^0 = X^2$.

By (6) for the standard image $\bar{X}^2$ of $X^2(\lambda)$ we have $\bar{X}^2 \leqslant X^2$. As is the case with Lemma 5.7.5, $u = X^2 - \bar{X}^2$ is the solution of equation (2.11.18) satisfying (5.7.21). By the maximality of X^0 we obtain $X^2 - \bar{X}^2 \leqslant X^0$. When z is regular or exit $X^0 = 0, X^2 = \bar{X}^2$. That is, the standard image of $X^2(\lambda)$ is X^2; hence the standard image of $X^1(\lambda)$ is X^1.

If z is entrance or natural, from (2.11.13) follows that the standard image of $X^1(\lambda)$ is $\bar{X}^1 = 1 - X^0 - \bar{X} = X^1$ and the proof is concluded. QED

Lemma 3. Set

$$
\bar{\eta}_i = \begin{cases} X_i^2\mu_i & \text{if } z \text{ is regular} \\ [a_0(z_i - z_0) + 1]\mu_i & \text{if } z \text{ is entrance} \end{cases} \tag{9}
$$

$$
\bar{\eta}_j(\lambda) = \begin{cases} X_i^2(\lambda)\mu_i & \text{if } z \text{ is regular} \\ a_0 u_i(\lambda)\mu_i/u^+(z, \lambda) & \text{if } z \text{ is entrance} \end{cases}
$$

Then $\bar{\eta}(\lambda) \in \mathscr{L}_\lambda^+$ $(\lambda > 0)$ is an entrance family, and

$$\lambda\bar{\eta}\phi(\lambda) = \bar{\eta} - \bar{\eta}(\lambda) \tag{10}$$

Proof. We need only prove (10). By (6) and (4.5) we derive (10) when z is regular.

If z is entrance, then $u(z,\lambda)=\infty, u^+(z,\lambda)<\infty$. By (3.7)

$$v^+(z,\lambda)=0 \tag{11}$$

Furthermore

$$v(z,\lambda)=\frac{1}{u^+(z,\lambda)} \tag{12}$$

In fact, notice that $u(\lambda)$ is increasing and $v(\lambda)$ is decreasing, so

$$0\leqslant -u_i(\lambda)v_i^+(\lambda)=u_i(\lambda)[v^+(z,\lambda)-v_i^+(\lambda)]=u_i(\lambda)\sum_{k=i+1}^{\infty}\lambda v_k(\lambda)\mu_k$$

$$\leqslant v_0(\lambda)\sum_{k=i+1}^{\infty}\lambda u_k(\lambda)\mu_k=v_0(\lambda)[u^+(z,\lambda)-u_i^+(\lambda)]\to 0 \qquad (i\to\infty)$$

From this and (3.6) follows (12).

Secondly, by noting (11) and (12),

$$\lambda\sum_k \phi_{ik}(\lambda)(z_k-z_0)$$

$$=v_i(\lambda)\sum_{k=0}^{i}\lambda\mu_k(\lambda)\mu_k\sum_{j=0}^{k-1}(z_{j+1}-z_j)+u_i(\lambda)\sum_{k=i+1}^{\infty}\lambda v_k(\lambda)\mu_k\sum_{j=0}^{k-1}(z_{j+1}-z_j)$$

$$=v_i(\lambda)\sum_{j=0}^{i-1}(z_{j+1}-z_j)\sum_{k=i+1}^{\infty}\lambda u_k(\lambda)\mu_k$$

$$+u_i(\lambda)\left(\sum_{j=0}^{i}(z_{j+1}-z_j)\sum_{k=i+1}^{\infty}\lambda v_k(\lambda)\mu_k+\sum_{i=j+1}^{\infty}(z_{j+1}-z_i)\sum_{k=i+1}^{\infty}\lambda v_k(\lambda)\mu_k\right)$$

$$=v_i(\lambda)\sum_{j=0}^{i-1}(z_{j+1}-z_j)[u_i^+(\lambda)-u_j^+(\lambda)]$$

$$+u_i(\lambda)\left(\sum_{j=0}^{i}(z_{j+1}-z_j)[v^+(z,\lambda)-v_i^+(\lambda)]\right.$$

$$\left.+\sum_{j=i+1}^{\infty}(z_{j+1}-z_j)[v^+(z,\lambda)-v_i^+(\lambda)]\right)$$

$$=v_i(\lambda)\{u_i^+(\lambda)(z_i-z_0)-[u_i(\lambda)-u_0(\lambda)]\}$$

$$+u_i(\lambda)\{-v_i^+(\lambda)(z_{j+1}-z_0)-[v(z,\lambda)-v_{i+1}(\lambda)]\}$$

$$=v_i(\lambda)[u_i^+(\lambda)(z_i-z_0)-u_i(\lambda)+1]$$

$$+u_i(\lambda)\left(-v_i^+(\lambda)(z_i-z_0)-\frac{1}{u^+(z,\lambda)}+v_i(\lambda)\right)$$

$$=[u_i^+(\lambda)v_i(\lambda)-u_i(\lambda)v_i^+(\lambda)](z_i-z_0)+v_i(\lambda)-\frac{u_i(\lambda)}{u^+(z,\lambda)}$$

$$=(z_i-z_0)-v_i(\lambda)-\frac{u_i(\lambda)}{u^+(z,\lambda)}$$

Hence by (4.5) and (6) follows (10), and the proof is terminated. QED

Lemma 4. For a family $\eta(\lambda)(\lambda>0)$ to be an entrance family it is necessary and sufficient that it has the following Riesz expression:

$$\eta(\lambda)=\alpha\phi(\lambda)+d\bar{\eta}(\lambda) \tag{13}$$

where row vector $\alpha\geqslant 0$ is such that $\alpha\phi(\lambda)\in l$, the constant $d\geqslant 0$, and $d=0$ if z is entrance or natural. Also

$$\bar{\eta}(\lambda)=\begin{cases}X^2(\lambda)\mu & \text{if } z \text{ is regular}\\ a_0u(\lambda)\mu/u^+(z,\lambda) & \text{if } z \text{ is entrance}\end{cases} \tag{14}$$

Proof. Since $\mathscr{L}_\lambda^+$ contains a non-null entrance family $\bar{\eta}(\lambda)$ if and only if z is regular or entrance, this lemma is a particular case of Lemma 2.11.3. QED

Lemma 5. If z is regular, then

$$U_\lambda^a=\lambda[X^2(\lambda)\mu,X^a]\uparrow U^a \qquad \lambda\uparrow\infty \tag{15}$$

where

$$U^1=\frac{a_0}{a_0(z-z_0)+1} \qquad U^2=+\infty \tag{16}$$

Proof. Now (2.11.40) becomes

$$\begin{aligned}\lambda[X^2(\lambda)\mu,X^a]-y[X^2(v)\mu,X^a]&=(\lambda-v)[X^2(\lambda)\mu,X^a(v)]\\&=(\lambda-v)[X^2(v)\mu,X^a(\lambda)] \qquad \lambda,v>0\end{aligned} \tag{17}$$

From this it follows that U_λ^a is monotone.

Next, by (6) and (4.9) we have

$$[\lambda\phi(\lambda)X^a]_i^+=\lambda v_i^+(\lambda)\sum_{j=0}^{i}u_j(\lambda)X_j^a\mu_j+\lambda u_i^+(\lambda)\sum_{j=i+1}^{\infty}v_j(\lambda)X_j^a\mu_j$$

$$=[X^a]_i^+-[X^a(\lambda)]_i^+$$

Noticing (3.7) and letting $i\to\infty$ in the above expression we obtain

$$-U_\lambda^a=[X^a]^+(z)-[X^a(\lambda)]^+(z)$$

From (2) we have

$$[X^1]^+(z) = -\frac{a_0}{a_0(z-z_0)+1}$$

$$[X^2](z) = \frac{a_0}{a_0(z-z_0)+1}$$

Therefore to prove this lemma, we need only prove

$$\lim_{\lambda\to\infty}[X^2(\lambda)]^+(z) = \infty \qquad \lim_{\lambda\to\infty}(X^1(\lambda)]^+(z) = 0 \tag{18}$$

This is similar to the proof of (5.7.18) and (5.7.19). Thus we complete the proof of this lemma. QED

6.6 CONSTRUCTION OF THE Q PROCESSES SATISFYING THE BACKWARD EQUATIONS

By Theorem 4.4 it follows that the Q process satisfying the system of backward equations is unique if z is entrance or natural. Hence we will suppose that z is exit or regular. In that case $z<\infty, u(z,\lambda)<\infty$.

By the remark of Theorem 4.4 it follows that the Q process satisfies the system of backward equations if and only if $\psi(\lambda)$ has the form

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + X_i^2(\lambda)F_j(\lambda) \tag{1}$$

where the row vector $F(\lambda)\geqslant 0$. Hence the construction problem of the Q processes satisfying the system of backward equations has already been contained in Theorem 3.2.1. Certainly, in the case of birth–death processes we will obtain a more specific and simpler form.

In this case, in Theorem 3.2.1, $H=\{0\}, \bar\eta(\lambda)$ becomes $dX^2(\lambda)\mu$, while the constant $d\geqslant 0$, and $d=0$ if z is exit, where

$$\bar V^1 = dU^1 = \frac{da_0}{a_0(z-z_0)+1}$$

Note that $X^0=0, \bar X = X^2$ follows from Lemma 5.2. Therefore Theorem 3.2.1 takes the following form.

Theorem 1. Let z be regular or exit. For $\psi(\lambda)$ to be a Q process satisfying the system of backward equations it is necessary and sufficient that either $\psi(\lambda)=\phi(\lambda)$ or $\psi(\lambda)$ can be obtained as follows: Select a row vector $a\geqslant 0$ such that $\alpha\phi(\lambda)\in l$, take a constant $d\geqslant 0$, and $d=0$ if z is exit. Moreover

$$\eta(\lambda) = \alpha\phi(\lambda) + dX^2(\lambda)\mu \neq 0 \tag{2}$$

Take a constant c so that

$$c \geqslant [\alpha, X^1] + \frac{da_0}{a_0(z-z_0)+1} \tag{3}$$

is satisfied, where $X^a(a=1,2)$ are determined by (5.2). Finally set

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + X_i^2(\lambda)\frac{\sum_k \alpha_k\phi_{kj}(\lambda) + dX_j^2(\lambda)\mu_j}{c+[\alpha, X^2 - X^2(\lambda)] + d\lambda[X^2(\lambda)\mu, X^2]} \tag{4}$$

For the process $\psi(\lambda)$ to be honest it is necessary and sufficient that $a_0=0$, and $c=0$. For the process $\psi(\lambda)$ to satisfy the system of forward equations it is necessary and sufficient that $\alpha=0$.

6.7 CONSTRUCTION OF THE Q PROCESSES SATISFYING THE FORWARD EQUATIONS

Because of Theorem 4.4, we need only consider the case that z is regular or entrance. Since the Q process $\psi(\lambda)$ satisfies the system of forward equations if and only if the $\psi(\lambda)$ has the form (4.18), that is

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + F_i(\lambda)\bar\eta_j(\lambda) \tag{1}$$

where $\bar\eta(\lambda)$ is determined by (5.14), the problem of construction of the Q processes satisfying the system of forward equations is solved in Theorem 3.4.1.

First let us suppose that z is regular. In this case, $\bar\eta(\lambda)=X^2(\lambda)\mu$. According to the notation in Theorem 3.4.1, either $\bar\xi(\lambda)=0$ or $\bar\xi(\lambda)=X^2(\lambda)$. If $\bar\xi(\lambda)=0$, then

$$\bar W_\lambda = \lambda[\bar\eta(\lambda), \bar X] = U_\lambda^2 \uparrow U^2 = +\infty$$

This is in contradiction with (3.4.3). Therefore it necessarily follows that $\bar\xi(\lambda)=X^2(\lambda)$, and hence $\delta>0$. According to the same consideration, the k in (3.4.3) must be δ^{-1}. By (3.4.4) we obtain $\beta^0\leqslant\delta$. In addition,

$$\bar V^0 = U^1 = \frac{a_0}{a_0(z-z_0)+1} \qquad X^0 = 0$$

Therefore if we replace β^0 by the constant β, then Theorem 3.4.1 takes the following form.

Theorem 1. Let z regular. For $\psi(\lambda)$ to be a Q process satisfying the system of forward equations it is necessary and sufficient that either $\psi(\lambda)=\phi(\lambda)$ or $\psi(\lambda)$ is obtained as follows: Choose a non-negative constant β and a positive constant $\delta, \beta\leqslant\delta$, and a constant c that satisfies

$$\frac{(\delta-\beta)a_0}{a_0(z-z_0)+1} \leqslant c \tag{2}$$

Set

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \frac{[\beta X_i^1(\lambda) + \delta X_i^2(\lambda)]X_j^2(\lambda)\mu_j}{c + \lambda[X^2(\lambda)\mu, \beta X^1 + \delta X^2]} \tag{3}$$

For the process $\psi(\lambda)$ to be honest it is necessary and sufficient that $a_0 > 0, \beta = \delta, c = 0$ or $a_0 = 0, c = 0$. For the process $\psi(\lambda)$ to satisfy the system of backward equations it is necessary and sufficient and $a_0 = 0$ or $a_0 > 0, \beta = 0$.

When z is entrance, the construction of the Q processes satisfying the system of forward equations, of course, can be deduced from Theorem 3.4.1. We shall, however, proceed as follows.

We assume that z is entrance or natural, whence $u(z, \lambda) = \infty$. If $a_0 = 0$, then the minimal solution is honest, and the Q process is unique. Consequently we further assume that $a_0 > 0$. Thus, (5.3) becomes

$$\lambda\phi(\lambda)\mathbf{1} = \mathbf{1} - X^1(\lambda)$$

By (4.19), an arbitrary Q process has the form

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + X_i^1(\lambda)F_j(\lambda) \tag{4}$$

Hence, in this case, the construction of the Q process is a special case of Theorem 3.3.3.

On account of the decreasing property of $v(\lambda)$ it follows that

$$\sup_i X_i^1(\lambda) = a_0 v_0(\lambda) < 1 \tag{5}$$

Therefore, if z is natural, then is null entrance, by Theorem 3.3.3 the Q process is unique.

If z is entrance, by Theorem 3.3.3 it follows that every non-minimal Q process is obtained by Theorem 3.3.1. According to the notation in Theorem 3.3.1, in this case

$$\bar{n}(\lambda) = d\frac{u(\lambda)\mu}{u^+(z, \lambda)}$$

and constant $d \geqslant 0$, whereas

$$Y_\lambda = \lambda[\bar{n}(\lambda), \mathbf{1}] = \frac{d}{u^+(z, \lambda)}\sum_i \lambda u_i(\lambda)\mu_i$$

$$= \frac{d}{u^+(z, \lambda)}[u^+(z, \lambda) - u^+_{-1}(\lambda)] = d - \frac{da_0}{u^+(z, \lambda)}$$

Therefore $Y \leqslant d < \infty$. By (3.3.5), if $\alpha \neq 0$ then it must follow that $[\alpha, 1] = \infty$. By (5) and Lemma 3.3.2 there does not exit such a row vector α. Therefore it is certain that $\alpha = 0$. Hence $d > 0$. Thus the Q process of Theorem 3.3.2 satisfies the system of forward equations. If we write $\bar{c}$ for $(c - \bar{\sigma}^0/d)$ in Theorem 3.3.1,

$$c + d\lambda[\bar{n}(\lambda), \mathbf{1} - X^0] = d(\bar{c} + \lambda[\bar{n}(\lambda), \mathbf{1}])$$

If we still write also c for $\bar{c}$, then we obtain the form of Theorem 3.3.3 as follows.

Theorem 2. If z is natural, $a_0 \geqslant 0$, then the Q process is unique. Assume that z is entrance. Then the Q process is unique if $a_0 = 0$; if $a_0 > 0$ every Q process $\psi(\lambda)$ satisfies the system of forward equations and, moreover, either $\psi(\lambda) = \phi(\lambda)$ or

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \frac{X_i^1(\lambda)\bar{n}_j(\lambda)}{c + \lambda[\bar{n}(\lambda), \mathbf{1}]} \tag{6}$$

where constant $c \geqslant 0$, $\bar{n}(\lambda) = u(\lambda)\mu/u^+(z, \lambda)$. The process $\psi(\lambda)$ is honest if and only if $c = 0$.

6.8 CONSTRUCTION OF THE Q PROCESSES SATISFYING NEITHER BACKWARD NOR FORWARD EQUATIONS

When z is entrance or natural it has already been investigated in the preceding section. We assume that z is regular or exit in this section. Since when $a_0 = 0$ any Q process satisfies the system of backward equations, we assume further that $a_0 > 0$.

By (4.19), each Q process $\psi(\lambda)$ has the form

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + X_i^1(\lambda)F_j^1(\lambda) + X_i^2(\lambda)F_j^2(\lambda) \tag{1}$$

where $F^a(\lambda) \geqslant 0$. By Theorem 2.12.1 we have $\lambda[F^1(\lambda), 1] \leqslant 1$.

We shall determine $F^a(\lambda)(a = 1, 2)$ such that $\psi(\lambda)$ determined by (1) satisfies the norm condition, the resolvent equation are the Q condition.

If the norm condition holds for $\psi(\lambda)$, by (53), that is

$$X^1(\lambda)\lambda[F^1(\lambda), \mathbf{1}] + X^2(\lambda)\lambda[F^2(\lambda), \mathbf{1}] \leqslant X^1(\lambda) + X^2(\lambda)$$

holds. Letting $i \to \infty$ and noting that $X^1(z, \lambda) = 0$, $X^2(z, \lambda) = 1$ we obtain $\lambda[F^2(\lambda), \mathbf{1}] \leqslant 1$. Therefore, the norm condition is equivalent to

$$F^a(\lambda) \geqslant 0 \qquad \lambda[F^a(\lambda), \mathbf{1}] \leqslant 1 \qquad a = 1, 2 \tag{2}$$

As in bilateral birth–death processes, the resolvent equation for $\psi(\lambda)$ is also equivalent to

$$F^a(\lambda)A(\lambda, \nu) = F^a(\nu) + (\nu - \lambda)\sum_{b=1}^{a}[F^a(\lambda), X^b(\nu)]F^b(\nu) \qquad (a = 1, 2, \lambda, \nu > 0) \tag{3}$$

(a) Now suppose that $F'(\nu), F^2(\nu)$ are linearly dependent for some $\nu > 0$. Therefore provided we make some proper modifications the discussion in section 5.9 is still valid. We shall state them in the form of the following lemma.

Lemma 1. Suppose that z is regular or exit, and $F^a(\lambda)$ $(a = 1, 2)$ are linearly dependent for some (hence for all) $\lambda > 0$.

For $\psi(\lambda)$ of (1) to satisfy the norm condition and the resolvent equation it is necessary and sufficient that either $\psi(\lambda) = \phi(\lambda)$ or $\psi(\lambda)$ is obtained as follows: Choose constants $d_a \geqslant 0$ $(a = 1, 2)$, $d_1 + d_2 > 0$, $p \geqslant 0$ ($p = 0$ if z is exit). Select the row vector $\alpha \geqslant 0$ so that $\alpha\phi(\lambda) \in l$ and

$$\eta(\lambda) = \alpha\phi(\lambda) + pX^2(\lambda)\mu \neq 0 \tag{4}$$

If $d_1 > d_2$ it is also required that

$$[\alpha, X^2] < \infty \qquad p = 0 \tag{5}$$

Choose a constant c satisfying

$$\begin{aligned} &c \geqslant 0 && \text{if } d_1 = d_2 \\ &c \geqslant (d_1 - d_2)W_2 && \text{if } d_1 > d_2 \\ &c \geqslant (d_2 - d_1)W_1 && \text{if } d_1 < d_2 \end{aligned} \tag{6}$$

where

$$\begin{aligned} W_1 &= [\alpha, X^1] + \frac{pa_0}{a_0(z - z_0) + 1} \\ W_2 &= [\alpha, X^2] + pU^2. \end{aligned} \tag{7}$$

Finally, set

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \frac{[d_1 X_i^1(\lambda) + d_2 X_i^2(\lambda)]\eta_j(\lambda)}{c + \lambda[\eta(\lambda), d_1 X^1 + d_2 X^2]} \tag{8}$$

Remark

Upon comparison with Theorem 5.9.1, this lemma does not require that

$$W_1 = \lim_{\lambda \to \infty} \lambda[\eta(\lambda), X^1] < \infty$$

since by Lemma 2.11.4 it follows that W_1 is finite.

To prove that $\psi(\lambda)$ of Lemma 1 is a Q process, we must also verify the Q condition, that is

$$\lim_{\lambda \to \infty} \lambda X_i^a(\lambda)\lambda F_j^a(\lambda) = 0 \qquad a = 1, 2 \tag{9}$$

By (8) and (2.11.26), we have

$$\lim_{\lambda \to \infty} \lambda F_j^a(\lambda) = \frac{d_a \alpha_j}{c + d_1 W_1 + d_2 W_2} \tag{10}$$

Therefore, by (5.4) and (5.5) it follows that (9) holds except for the case $a = 1$, $i = 0$. For the case $a = 1$ and $i = 0$, (9) becomes

$$\frac{a_0 d_1 \alpha}{c + d_1 W_1 + d_2 W_2} = 0 \tag{11}$$

Now assume $d_1 > d_2 \geqslant 0$. By (5) and (6) we obtain $W_2 = [\alpha, X^2] < \infty$, $p = 0$. By (4) we derive $\alpha \neq 0$. The above remark has already pointed out that $W_1 < \infty$. Therefore, (11) cannot possibly hold. Therefore, it must be that $d_2 \geqslant d_1 \geqslant 0$. Thus (6) becomes

$$\begin{aligned} &c \geqslant 0 && \text{if } d_1 = d_2 \\ &c \geqslant (d_2 - d_1)\left([\alpha, X^1] + \frac{pa_0}{a_0(z - z_0) + 1}\right) && \text{if } d_1 < d_2 \end{aligned} \tag{12}$$

For (11) to hold it is necessary and sufficient that either $d_1\alpha = 0$ or $W_2 = \infty$. Since $[\alpha, X^1] \leqslant W_1 < \infty$ this necessary and sufficient condition becomes

$$d_1\alpha = 0, \text{ or } p > 0, \text{ or } p = 0 \text{ and } [\alpha, 1] = \infty \tag{13}$$

We have thus obtained the following theorem.

Theorem 2. Let z be regular or exit, $a_0 > 0$. Let $F^a(\lambda)$ $(a = 1, 2)$ be linearly dependent. For $\psi(\lambda)$ given by (1) to be a Q process it is necessary and sufficient that either $\psi(\lambda) = \phi(\lambda)$ or $\psi(\lambda)$ is obtained as follows: Take a row vector $\alpha \geqslant 0$ so that $\alpha\phi(\lambda) \in l$ and select constants $d_2 \geqslant d_1 \geqslant 0$, $d_2 > 0$, $p \geqslant 0$ (with $p = 0$ if z is exit) and c satisfying (4), (12) and (13). Set

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \frac{[d_1 X_i^1(\lambda) + d_2 X_i^2(\lambda)][\sum_k \alpha_k \phi_{kj}(\lambda) + pX_j^2(\lambda)\mu_j]}{c + \sum_{b=1}^2 d_b\{[\alpha, X^b - X^b(\lambda)] + p\lambda[X^2(\lambda)\mu, X^b]\}} \tag{14}$$

This process is honest if and only if $d_1 = d_2$, $c = 0$. A necessary and sufficient condition for this process to satisfy neither forward nor backward equations is $d_1 > 0$, $\alpha \neq 0$.

The proof of the last conclusion is as follows: Multiplying both sides of (1) by $(\lambda I - Q)$ from the left-hand side, we find that the B condition is equivalent to

$$0 = (\lambda I - Q)[X_i^1(\lambda)F_j^1(\lambda) + X_i^2(\lambda)F_j^2(\lambda)] = \begin{cases} 0 & \text{if } i > 0 \\ a_0 F_j^1(\lambda) & \text{if } i = 0 \end{cases}$$

Multiplying both sides of (1) by $(\lambda I - Q)$ from the right-hand side it follows that the F condition is equivalent to

$$\begin{aligned} 0 &= [X_i^1(\lambda)F_j^1(\lambda) + X_i^2(\lambda)F_j^2(\lambda)](\lambda I - Q) \\ &= \frac{[d_1 X_i^1(\lambda) + d_2 X_i^2(\lambda)]a_j}{c + \lambda[\eta(\lambda), d_1 X^1 + d_2 X^2]} \end{aligned}$$

But $X^1(\lambda)$, $X^2(\lambda)$ are linearly independent and, moreover, $d_1 + d_2 > 0$. Therefore, $d_1 X^1(\lambda) + d_2 X^2(\lambda) \neq 0$.

(b) We assume in (a) that $F^a(\lambda)$ $(a = 1, 2)$ are linearly dependent. We now assume that $F^a(\lambda)$ $(a = 1, 2)$ are linearly independent for some (hence for all) $\lambda > 0$.

In this case we can follow the notation and discussion in section 5.10. In particular, Lemma 5.10.1 is still valid after some obvious modifications.

Lemma 3. Let $\psi(\lambda)$ be a Q process possessing form (1). Then there exist row vectors $\alpha^a \geqslant 0$ $(a = 1, 2)$ such that $\alpha^a \phi(\lambda) \in l$ and a second-order square matrix $\mathscr{R}_\lambda = (r_\lambda^{ab}) \geqslant 0$ and quantities $p_\lambda^a \geqslant 0$ $(a = 1, 2, \lambda > 0)$ ($[p_\lambda] = [0]$ if z is exit) such that

$$[F(\lambda)] = \mathscr{R}_\lambda[\alpha\phi(\lambda)] + [p_\lambda]X^2(\lambda)\mu \tag{15}$$

We introduce the notations

$$h_\lambda^{ab} = \lambda[\alpha^a \phi(\lambda), X^b] = [\alpha^a, X^b - X^b(\lambda)] \uparrow h^{ab} = [\alpha^a, X^b] \qquad \lambda \uparrow \tag{16}$$

$$\mathscr{H}_\lambda = (h_\lambda^{ab}) \uparrow \mathscr{H} = (h^{ab}) \qquad \lambda \uparrow \infty \tag{17}$$

Consider a special case: $[p_\lambda] = [0], \mathscr{H} < \infty$, that is

$$[F(\lambda)] = \mathscr{R}_\lambda[\alpha\phi(\lambda)]$$

$$[\alpha^a, 1] < \infty \qquad a = 1, 2 \tag{19}$$

Theorem 4. Let z be regular or exit, $a_0 > 0$. It is impossible that a Q process $\psi(\lambda)$ having the form (1) satisfies conditions (18) and (19), and moreover $F^a(\lambda)$ $(a = 1, 2)$ are linearly independent.

Proof. Suppose that $\psi(\lambda)$ is a Q process having the form (1) given by (18) and (19), and moreover that $F^a(\lambda)$ $(a = 1, 2)$ are linearly independent.

As we have done in the proof of Theorem 5.10.1, by the norm condition and the resolvent equation for $\psi(\lambda)$ it follows that $[F(\lambda)]$ has the form (5.10.33), that is,

$$[F(\lambda)] = (I - \bar{\mathscr{T}}_\lambda)^{-1}[\bar{\alpha}\phi(\lambda)] \tag{20}$$

where row vectors $\bar{\alpha}^a \geqslant 0$ so that $[\bar{\alpha}^a, 1] \leqslant 1, \bar{\mathscr{T}}_\lambda = \{[\bar{\alpha}, X(\lambda)]\}$.

The $\psi(\lambda)$ must also satisfy the Q condition, that is

$$0 = \lim_{\lambda \to \infty} \lambda[X(\lambda)]'\lambda[F(\lambda)] = \lim_{\lambda \to \infty} \lambda[X(\lambda)]'(I - \bar{\mathscr{T}}_\lambda)^{-1}[\lambda\bar{\alpha}\phi(\lambda)] \tag{21}$$

Since by (5.4), (5.5) and the dominated convergence theorem

$$\lim_{\lambda \to \infty} (I - \bar{\mathscr{T}}^\lambda)^{-1} = \lim_{\lambda \to \infty} \sum_{n=0}^{\infty} \bar{\mathscr{T}}_\lambda^n = I$$

Therefore, (21) holds for $i > 0$. For $i = 0$, (21) becomes

$$0 = \begin{pmatrix} a_0 \\ 0 \end{pmatrix}[\bar{\alpha}] = a_0 \bar{\alpha}^1$$

Thus $\bar{\alpha}^1 = 0$. By (20) it follows that $F^a(\lambda)$ $(a = 1, 2)$ are linearly dependent, and the proof is completed.

Remark

From this proof it can be seen that $\psi(\lambda)$ determined by (20) and (1) is a $\bar{Q}$ process although it is not a Q process, where $\bar{q}_{ij} = q_{ij}$ $(i > 0, j \in E)$, $\bar{q}_{0j} = q_{0j} + a_0 \bar{\alpha}_l^1$.

We now consider the general case of (15).

Lemma 5. Let z be exit or regular, $a_0 > 0$. Suppose that $F^a(\lambda)$ $(a = 1, 2)$ are linearly independent. For $\psi(\lambda)$ of the form (1) to satisfy the norm conditions and the resolvent equation it is necessary and sufficient that $\psi(\lambda)$ can be obtained as follows: Select two non-negative row vectors $\bar{\alpha}^a$ $(a = 1, 2)$ such that $\bar{\alpha}^a \phi(\lambda) \in l$. Take a non-negative matrix

$$\bar{\mathscr{S}} = \begin{pmatrix} 0 & \bar{s}^{12} \\ \bar{s}^{21} & 0 \end{pmatrix}$$

and take a constant $\bar{p}^2 \geqslant 0$ (with $\bar{p}^2 = 0$ if z is exit), and moreover, the following properties are satisfied:

(i) either $\bar{p}^2 > 0$, or $\bar{p}^2 = 0$ and $\bar{\alpha}^1$ and $\bar{\alpha}^2$ furthermore are linearly independent.

(ii) $\bar{h}^{ab} < \infty$ $(a \neq b)$.

(iii) $\bar{s}^{12} \leqslant 1, \bar{s}^{21} \leqslant 1, \bar{s}^{ab} \geqslant \bar{h}^{ab} + \dfrac{\bar{p}^2 a_0}{a_0(z - z_0) + 1}$ $(a \neq b)$.

Finally, set

$$[\bar{p}] = \begin{pmatrix} 0 \\ \bar{p}^2 \end{pmatrix}$$

and

$$\mathscr{R}_\lambda = (I - \bar{\mathscr{S}} + \bar{\mathscr{H}}_\lambda + [\bar{p}][U_\lambda]')^{-1}$$

$$[p_\lambda] = (I - \bar{\mathscr{S}} + \bar{\mathscr{H}}_\lambda + [\bar{p}][U_\lambda]')^{-1}[\bar{p}] \tag{22}$$

whereas $\psi(\lambda)$ is determined by (1), (15) and (22). Here $\bar{\mathscr{H}}_\lambda = (\bar{h}_\lambda^{ab}) \uparrow \bar{\mathscr{H}} = (\bar{h}^{ab})$ $(\lambda \uparrow \infty)$ are determined by (16) and (17) for $\bar{\alpha}^a$. By (5.15) and (5.16)

$$[U_\lambda]' = (U_\lambda^1, U_\lambda^2) \uparrow [U]' = \left(\frac{a_0}{a_0(z - z_0) + 1}, +\infty\right). \tag{23}$$

For $\psi(\lambda)$ to have the forms (1), (18) and (19) it is necessary and sufficient that $\bar{p}^2 = 0$, $\bar{h}^{22} < \infty$.

Proof. To repeat the proof of Theorem 5.10.3, it suffices to make evident modifications. We must also notice that by Lemma 2.11.4, if $\bar{\alpha}^1 \geqslant 0$ so that $\bar{\alpha}^1\phi(\lambda)\in l$, then $\bar{h}^{11} = \lim_{\lambda\to\infty}\bar{h}_1^\lambda < \infty$ and the proof is completed. QED

We discuss the condition that $\psi(\lambda)$ of Lemma 5 is a Q process. By Theorem 4, certainly

$$\bar{p}^2 > 0 \qquad \text{or} \qquad \bar{h}^{22} = \infty \tag{24}$$

To prove that $\psi(\lambda)$ is a Q process, we must also verify the Q condition, that is

$$\lim_{\lambda\to\infty} [\lambda X(\lambda)]'\mathscr{X}_\lambda^{-1}\lambda([\bar{\alpha}\phi(\lambda)] + [\bar{p}]X^2(\lambda)\mu) = 0 \tag{25}$$

where

$$\mathscr{X}_\lambda = (I - \bar{\mathscr{S}} + \bar{\mathscr{H}}_\lambda + [\bar{p}][U_\lambda]') = \begin{pmatrix} 1 + \bar{h}_\lambda^{11} & -\bar{s}^{12} + \bar{h}_\lambda^{12} \\ -\bar{s}^{21} + \bar{h}_\lambda^{21} + \bar{p}^2U_\lambda^1 & 1 + \bar{h}_\lambda^{22} + \bar{p}^2U_\lambda^2 \end{pmatrix}$$

By (24) and Lemma 5(iii), we obtain $\lim_{\lambda\to\infty}\det\mathscr{X}_\lambda = +\infty$. Notice that $\bar{h}^{11} < \infty, \bar{h}^{ab} < \infty$ $(a \neq b)$, and it follows that

$$\lim_{\lambda\to\infty}\mathscr{X}_\lambda^{-1} = \lim_{\lambda\to\infty}\frac{1}{\det\mathscr{X}_\lambda}\begin{pmatrix} 1 + \bar{h}_\lambda^{22} + \bar{p}^2U_\lambda^2 & \bar{s}^{21} - \bar{h}^{21} - \bar{p}^2U_\lambda^1 \\ \bar{s}^{12} - \bar{h}_\lambda^{12} & 1 + \bar{h}_\lambda^{11} \end{pmatrix}$$
$$= \begin{pmatrix} (1 + \bar{h}^{11})^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

Owing to (5.4) and (5.5), (25) holds for $i > 0$. For $i = 0$ the limit on the left-hand side of (25) is

$$\begin{pmatrix} a_0 \\ 0 \end{pmatrix}'\begin{pmatrix} (1 + \bar{h}^{11})^{-1} & 0 \\ 0 & 0 \end{pmatrix}[\bar{\alpha}] = \frac{a_0\bar{\alpha}^1}{1 + \bar{h}^{11}}$$

Therefore, (25) holds if and only if $\bar{\alpha}^1 = 0$. Thus by (22) and (15)

$$[F(\lambda)] = \mathscr{X}_\lambda^{-1}\begin{pmatrix} 0 \\ \bar{\alpha}^2\phi(\lambda) + \bar{p}^2X^2(\lambda)\mu \end{pmatrix}$$

Therefore $F^a(\lambda)$ $(a = 1, 2)$ are linearly dependent. This is contradictory to the linear independence of $F^a(\lambda)$ $(a = 1, 2)$; thus the Q condition (25) cannot possibly hold. QED

Combining this with Theorem 4, we have obtained the following theorem.

Theorem 6. If z is regular or exit, $a_0 > 0$, then there does not exist a Q process $\psi(\lambda)$ which has the form (1) so that $F^1(\lambda)$ and $F^2(\lambda)$ are linearly independent.

Remark

From the preceding part of the proof of Theorem 6 it can be seen that $\psi(\lambda)$ which satisfies the condition (24) in Lemma 5 is a $\bar{Q}$ process although it is not a Q process, where

$$\bar{q}_{ij} = q_{ij} \qquad (i > 0, j\in E) \qquad \bar{q}_{0j} = q_{0j} + \frac{\alpha_0\bar{\alpha}_j^1}{1 + \bar{h}^{11}}$$

6.9 THE CONDITION THAT $\alpha\phi(\lambda)\in l$

Suppose $\alpha \geqslant 0$. By (5.3) it follows that $\alpha\phi(\lambda)\in l$ is equivalent to

$$\sum_{i=0}^{\infty}\alpha_i[1 - X_i^1(\lambda) - X_i^2(\lambda)] < \infty \tag{1}$$

In this section, we shall give the condition under which $\alpha\phi(\lambda)\in l$ is directly determined from Q

Lemma 1. Assume that $a_0 \geqslant 0$, z is regular or exit. Then

$$\lim_{i\to\infty}\frac{v_i(\lambda)}{z - z_i} = \frac{1}{u(z, \lambda)} \tag{2}$$

Proof. Using (3.4), we can give the proof by following that of Lemma 5.11.2. QED

Lemma 2. Suppose $a_0 \geqslant 0$, z is regular. Then

$$\lim_{i\to\infty}\frac{1 - X_i^1(\lambda) - X_i^2(\lambda)}{z - z_i} = \lambda[X^2(\lambda)\mu, 1] \tag{3}$$

Proof. The proof is similar to Theorem 5.11.3. QED

Theorem 3. If $a_0 \geqslant 0$ and z is regular, then the necessary and sufficient condition under which $\alpha\phi(\lambda)\in l$, that is, (1) holds, is that

$$\sum_{i=0}^{\infty}\alpha_i(z - z_i) < \infty \tag{4}$$

which is equivalent to

$$\sum_{i=0}^{\infty}\alpha_iN_i < \infty \tag{5}$$

where N_i is determined by (2.6).

Proof. Applying Lemma 2, this proof can be provided by following that of Theorem 5.11.3. QED

Theorem 4. If $a_0 \geqslant 0$ and z is exit, then the necessary and sufficient condition for $\alpha\phi(\lambda) \in l$ is that (5) holds.

Proof. This proof is similar to Theorem 5.11.4. QED

Theorem 5. If $a_0 \geqslant 0$ and z is entrance or natural, then the necessary and sufficient condition for $\alpha\phi(\lambda) \in l$ is that

$$\sum_{i=0}^{\infty} \alpha_i < \infty \tag{6}$$

Proof. The necessity follows from

$$1 - X_i^1(\lambda) \geqslant 1 - X_0^1(\lambda) > 0 \qquad (i \geqslant 0)$$

The sufficiency is obvious. QED

PART III MARTIN BOUNDARY AND ITS APPLICATION IN THE CONSTRUCTION THEORY

CHAPTER 7

Martin Boundary and Q Processes

7.1 INTRODUCTION

When the Q process is not unique, we must compactify the state space E so as to solve the problem of construction of the Q process. That is, we must adjoin some 'boundary points' to E so that certain requirements can be satisfied. In other words, we need to introduce and study the boundary of the Q process. The simplest compactification of E is one-point compactification. For instance, the unilateral birth–death process is done in just this manner, and moreover the one-point compactification is enough. However, in general it is not sufficient to perform the one-point compactification. For example, for the bilateral birth–death process, it is obvious that we must compactify E by two 'points'.

In Feller (1956) a kind of boundary is introduced, which is called a Feller boundary. In his article, Feller (1957a) introduces the exit and passive boundaries for conservative Q, and then by applying these boundaries, he constructs all the Q processess that satisfy the systems of backward and forward equations simultaneously in the case that both the exit and entrance boundaries are finite. The method used by Feller is a purely analytical one.

In Doob (1959) the Martin boundary of the Markov chain is introduced, its characteristic being that the Martin boundary is closely connected with the path of the Markov chain (henceforth the Markov chain will be abbreviated as MC). Doob proves the convergence of the MC's path in the Martin boundary. In Dynkin (1969), Hunt (1960) and Watanabe (1960a, b), the theory of the Martin boundary of MC is further investigated and developed. In Kunita (1962), the Martin boundary is applied to a study of the instantaneous return process. But all the discussions in the articles mentioned above are conducted so that some restrictions are imposed on the Markov chain (e.g. if the MC is required to be non-recurrent, there must be a 'centre' at least). Though these restrictions are inessential, they need to be removed. In Hunt (1960) the introduction of standard measure can exclude the restriction that a 'centre' is needed. The following

formula in Dynkin (1969, section 9)

$$K(i,j)=\frac{G(i,j)}{\sum_s r_s G(s,j)}=\frac{f_{ij}}{\sum_s r_s f_{sj}}$$

provides the way to rule out the restriction of the non-recurrence of the MC. Zhen-ting Hou (1974) derives the Martin boundary without imposing any restriction on the MC.

In this chapter we first introduce the Martin boundary of general MC's, whose detailed description basically follows Dynkin (1969). Then we shall conduct extensive discussions in close association with the Q process. We apply the standard image introduced in Feller (1957a). It is similar to Definition 2.11.2, but narrower. We shall present and discuss in detail the λ-image. Following the method in Kunita (1962), we derive the Martin exit and passive boundaries of the Q matrix without any conditions attached, and moreover, by applying these boundaries, we further describe the general analytical expressions of the Q processes treated in section 2.12.

The content of this chapter is absolutely necessary and fundamental either to the construction of the Q process by means of probability methods or to the probabilistic interpretation of the analytical method.

7.2 MARKOV CHAINS

We are given that $(\Omega,\mathscr{F},P)$ is a complete probability space[1], on which β is defined to be a random variable with values of non-negative integers and '∞'. We say that $X_T(\omega)=\{x_n(\omega)^2, n\leqslant\beta\}$ ($\omega\in\Omega$, n is a non-negative integer) is a homogeneous MC or simply MC defined on the probability space $(\Omega,\mathscr{F},P)$[3] and taking values in the denumerable index set E. If for any integer $n\geqslant 2$, the non-negative integers $0\leqslant t_1<t_2<\cdots<t_{n+1}$ and any $i_1,i_2,\ldots,i_{n+1}\in E$, provided $P\{x(t_a)=i_a,\ 1\leqslant a\leqslant n\}>0$, then we have

$$P\{x(t_{n+1})=i_{n+1}|x(t_a)=i_a,\ 1\leqslant a\leqslant n\}=P\{x(t_{n+1})=i_{n+1}|x(t_n)=i_n\} \tag{1}$$

and moreover the value on the right-hand side is dependent only on $t_{n+1}-t_n$, and not on t_n.

Write

$$p_{ij}^n=P\{x(n)=j|x(0)=i\} \tag{2}$$

We call matrix (p_{ij}^n) the n-step transition probability matrix of the chain and simply write $p_{ij}=p_{ij}^1$. The matrix, $P=(p_{ij})$ is non-negative and its row sums are not more than one. Every such matrix P may become a one-step transition probability matrix for some Markov chain X_T. Therefore we also say the matrix P is a Markov chain. Note that

$$p_{ij}^n=(P^n)_{ij} \tag{3}$$

We call

$$d_i=1-\sum_j p_{ij} \qquad i\in E \tag{4}$$

the stopping quantity of the chain. We call $H=\{i|d_i>0\}$ the set of stopping states. If H is an empty set, we say that the chain is honest; otherwise it is stopping. Evidently, for the chain to be honest, the necessary and sufficient condition is

$$P_i\{\beta=\infty\}=1 \qquad i\in E \tag{5}$$

Here and afterwards we write $P_i\{\cdot\}=P\{\cdot|x_0=i\}$, $E_i\{\cdot\}$ represents the mathematical expectation taken with respect to P_i. For instance,

$$E_i\{f,\Lambda\}=E_i fI_\Lambda=\int_\Lambda f\,\mathrm{d}p_i \tag{6}$$

Here I_Λ denotes the indicator function of the set Λ.

Set

$$\eta_i^*=\inf\{n|1\leqslant n\leqslant\beta, x_n=i\} \tag{7}$$

$$\eta_i=\inf\{n|0\leqslant n\leqslant\beta, x_n=i\} \tag{8}$$

We set $\inf\varnothing=\infty$, where $\varnothing$ is the empty set. We write

$$\begin{aligned} &f_{ij}^{n*}=P_i\{\eta_j^*=n\} && n\geqslant 1\\ &f_{ij}^n=P_i\{\eta_j=n\} && n\geqslant 0\\ &f_{ij}^*=\sum_{n=1}^\infty f_{ij}^{n*} && f_{ij}=\sum_{n=0}^\infty f_{ij}^n \end{aligned} \tag{9}$$

We call f_{ij}^{n*} (or f_{ij}^n) the probability that the MC first visits the state j from the state i at the nth step, starting from the first (or zeroth) step.

And we call f_{ij}^* (or f_{ij}) the probability that the MC reaches j from i at the nth step for some $n\geqslant 1$ (some $n\geqslant 0$). Obviously

$$\begin{aligned} &f_{ij}^0=0 \qquad f_{ij}^{n*}=f_{ij}^n\ (n\geqslant 1) \qquad f_{ij}^*=f_{ij} \qquad \text{if } i\neq j\\ &f_{ii}=f_{ii}^0=1 \qquad f_{ii}^n=0 \qquad (n\geqslant 1)\\ &f_{ij}^*=\sum_k p_{ik}f_{kj} \qquad f_{ik}f_{kj}\leqslant f_{ij} \end{aligned} \tag{10}$$

[1] From now on when considering the sub-Borel field of $\mathscr{F}$ we always mean it to be completed according to P and will not make any more statement about it.
[2] We take $x_n(\omega)\equiv x(n,\omega)$, and shall leave out ω if it is not emphasized hereafter.
[3] If $(\Omega,\mathscr{F},P)$ in the definition of the MC is relaxed into a finite measure space, X_T is likewise called a Markov chain.

If $f_{ij}^* > 0$ ($i \neq j$), we say that the chain can lead to j from i, and write $i \Rightarrow j$. Set $i \Rightarrow i$. If $i \Rightarrow j, j \Rightarrow i$, we say that i and j communicate and write $i \Leftrightarrow j$. Assume $C \subset E$; if any two states in C communicate, and furthermore $p_{ik} = 0$ ($i \in C$) for any $k \in E - C$, then we call C an irreducible class. If $f_i^* \equiv f_{ii}^* = 1$, then we say that the state i is recurrent.

Theorem 1. Write

$$G(i,j) = \sum_{n=0}^{\infty} p_{ij}^n \tag{11}$$

Then

$$G(i,j) = f_{ij} G(j,j) \qquad G(i,i) = \frac{1}{1 - f_i^*} \tag{12}$$

Especially, for i to be recurrent, the necessary and sufficient condition is $G(i,i) = \infty$.

Proof. Write

$$\delta_j(i) = \delta_{ij} \tag{13}$$

Then

$$p_{ij}^n = E_i \delta_j(x_n) \qquad G(i,j) = E_i \sum_{n=0}^{\beta} \delta_j(x_n) \tag{14}$$

Write

$$A_m = (x_0 \neq j, x_1 \neq j, \ldots, x_{m-1} \neq j, x_m = j)$$

Then

$$P_i(A_m) = f_{ij}^m \qquad P_i(A_m, x_{m+k} = j) = f_{ij}^m p_{jj}^k$$

Therefore

$$\begin{aligned} G(i,j) &= E_i \sum_{n=0}^{\beta} \delta_j(x_n) \\ &= E_i \sum_{m=0}^{\infty} I_{A_m} \sum_{n=m}^{\beta} \delta_j(x_n) \\ &= \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} E_i I_{A_m} \delta_j(x_n) \\ &= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} P_i(A_m, x_{m+k} = j) \\ &= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} f_{ij}^m p_{jj}^k = f_{ij} G(j,j) \end{aligned}$$

The proof of the second formula in (12) can be found in Zi-kun Wang and Xiang-qun Yang (1988, section 2.2, Theorem 1). QED

Remark

If j is non-recurrent, then $G(i,j) < \infty$; thus $\sum_{n=0}^{\beta} \delta_j(x_n) < \infty$ P_i almost surely; that is, with probability 1, the number of times that the chain visits j is finite.

Theorem 2 (Decomposition theorem of state space). The state space E of the MC has the following unique decomposition:

$$E = E_0 \cup \left(\bigcup_{a \in \mathscr{A}} E_a \right) \tag{15}$$

where E_0 is composed of all the non-recurrent states. $\mathscr{A}$ is an empty set or finite set or denumerably infinite set, and moreover $0 \notin \mathscr{A}$. Every E_a ($a \in \mathscr{A}$) is an irreducible recurrent class.

The verification can be seen in Zi-kun Wang and Xiang-qun Yang (1988, section 2.3, Theorem 1). We point out that every stopping state must be non-recurrent, that is,

$$H \equiv \left\{ i: \sum_j p_{ij} < 1 \right\} \subset E_0 \tag{16}$$

The chain $X_r = \{x_n, n \leqslant \beta\}$ moves as follows: If x_0 is in some E_a ($a \in \mathscr{A}$), then $\beta = \infty$, and the chain will move in E_a forever, and moreover visits every state in E_a infinitely many times. If $x_0 \in E_0$, then either $\beta < \infty$, and at this moment there must exist $x_\beta \in H$; or $\beta = \infty$, and in this case the chain either moves in E_0 forever, or gets into some recurrent class E_a at some step, and then moves in E_a forever. As for further consideration of E_0, we have the Blackwell decomposition theorem below.

Let

$$\Omega_\infty = (\beta = \infty) \qquad \Omega_F = (\beta < \infty) \tag{17}$$

Provided $A \subset E$, set

$$\begin{aligned} \bar{\mathscr{L}}(A) &= \Omega_\infty \cap \limsup_{n \to \infty} \{x_n \in A\} \\ \underline{\mathscr{L}}(A) &= \Omega_\infty \cap \liminf_{n \to \infty} \{x_n \in A\} \end{aligned} \tag{18}$$

If $P_i\{\bar{\mathscr{L}}(A)\} = 0$ ($i \in E$), then we call A a transient set; if $P_i\{\underline{\mathscr{L}}(A)\} > 0$ for some

$i \in E$, then we say that A is a sojourn set. Suppose that A is a sojourn set, and moreover, $P_i\{\bar{\mathscr{L}}(A)\} = P_i\{\underline{\mathscr{L}}(A)\}$ $(i \in E)$, then we say that A is an almost closed set. When A is a transient set or an almost closed set, we write $\mathscr{L}(A)$ for any one set that is equal to $\underline{\mathscr{L}}(A)$ with the exception of a P_i-null probability set (for all i).

Assume A to be an almost closed set. If for any $B \subset A$, either B or $A - B$ is a transient set, then we call A an atomic almost closed set. If for any $B \subset A, B$ is not an atomic almost closed set, then we call A a completely non-atomic almost closed set.

The following is the Blackwell decomposition theorem.

Theorem 3 (Chung, 1967, I, 17, Theorem 4). The state space E has the following decomposition:

$$E = A_0 + \left(\bigcup_{a \in \mathscr{A}} A_a\right) \tag{19}$$

where A_0 is a completely non-atomic almost closed set, and so can be dropped. $\mathscr{A}$ is an empty set or a finite set or a denumerably infinite set, and moreover $0 \notin \mathscr{A}$. A_a $(a \in \mathscr{A})$ is an atomic almost closed set. Furthermore

$$P\{\mathscr{L}(A_0)\} + \sum_{a \in \mathscr{A}} P\{\mathscr{L}(A_a)\} = P(\Omega_\infty) \tag{20}$$

If we rule out the difference of a transient set, the decomposition (19) is unique.

Note that in the decomposition (15) every irreducible recurrent class is an atomic almost closed set.

7.3 MARTIN BOUNDARY THEORY

The study of the ultimate behaviour of the path of the MC, $X_T = \{x_n, n \leqslant \beta\}$, that is the behaviour of x_n, when $n \uparrow \beta$, gives rise to the Martin boundary theory of the MC. Applying the boundary theory, we can also describe all the excessive and harmonic functions relative to the chain. Assume that the one-step transition matrix of X_T is $P = (p_{ij})$. If the stopping state set H is not empty, we can select $\Delta \notin E$, and let

$$p_{\Delta\Delta} = 1 \qquad p_{\Delta i} = 0 \qquad p_{i\Delta} = 1 - \sum_j p_{ij} \qquad i \in E \tag{1}$$

$$x_n = \Delta \qquad \text{if } \beta < \infty, n > \beta \tag{2}$$

Then $\{x_n, n \geqslant 0\}$ is an MC, whose one-step transition matrix is $(p_{ij}, i, j \in E \cup \{\Delta\})$. From now on it is to be understood always like this.

7.3.1 Excessive function and excessive measure

Assume u to be a function on E[1]. The function Pu is defined as follows:

$$(Pu)_i = \sum_j p_{ij} u_j$$

Assume that v is a measure on E. The measure vP is defined like this:

$$(vP)_j = \sum_i v_i p_{ij}$$

Definition 1. A non-negative (including $+\infty$) function u is called P-excessive, if $Pu \leqslant u$; it is said to be P-harmonic if $Pu = u$. We write the class composed of bounded harmonic functions as $\mathscr{M}^+$, and denote the class of harmonic functions bounded by k by $\mathscr{M}^+(k)$.

Evidently, if u is excessive, then $P^n u \leqslant u$ for all n.

Definition 2. A non-zero excessive (or harmonic) function u is said to be minimal if $u = u^1 + u^2$ and moreover u^1 and u^2 are excessive (or harmonic) functions. Then $u = c^a u^a$ $(a = 1, 2)$, c^a is a constant. The non-zero harmonic function u is called completely non-minimal if, for any non-zero harmonic function $v \leqslant u, v$ is not minimal.

Definition 3. A (non-negative) measure μ is called P-excessive, if $\mu P \leqslant \mu$; it is said to be harmonic if $\mu P = \mu$.

Assume that u is a P-excessive function on E and furthermore let $u(\Delta) = 0$. Then $(u_i, i \in E \cup \{\Delta\})$ is likewise excessive with respect to $(p_{ij}, i, j \in E \cup \{\Delta\})$. Applying the Markov property of X_T and the excessive property of u:

$$E[u(x_{n+1}) | x_0, x_1, \ldots, x_n] = E[u(x_{n+1}) | x_n] = \sum_j P_{x_n j} u_j \leqslant u(x_n) \tag{3}$$

Hence $\{u(x_n), n \geqslant 0\}$ is a supermartingale while $\{-u(x_n), n \geqslant 0\}$ is a sub-martingale, and thus

$$Eu(x_0) \geqslant Eu(x_1) \geqslant Eu(x_2) \geqslant \cdots \tag{4}$$

Assume $0 < a < b$. Let U_N be the number of times that the sequence $u(x_0), \ldots, u(x_N)$ goes down across the interval $[a, b]$, that is the number of times that the sequence passes from the right side of the interval $[a, b]$ to its left side, and hence the number of times that $-u(x_0), \ldots, -u(x_N)$ passes from the left of the interval $[-b, -a]$ to its right. According to Zi-kun Wang (1965a, section 1.4,

[1] Hereafter u_i will also be written as $u(i)$; similarly p_{ij} will be denoted by $p(i, j)$ and so on.

Lemma 3), or Doob (1953, Ch. VII, Theorem 3.3),

$$EU_N \leqslant \frac{E[-u(x_N)-(-b)]^+}{(-a)-(-b)}$$

$$= \frac{E[u(x_N)-b]^-}{b-a}$$

$$\leqslant \frac{Eu(x_N)}{b-a}$$

$$\leqslant \frac{Eu(x_0)}{b-a} \tag{5}$$

Let v be the number of times that $u(x_0), u(x_1), \ldots, u(x_n), \ldots$ moves down across the interval $[a,b]$. Then $v_N \uparrow v$. Accordingly

$$Ev \leqslant \frac{Eu(x_0)}{b-a} \tag{6}$$

In particular,

$$E_i v \leqslant \frac{u(i)}{b-a} \tag{7}$$

Hence if $u(i) < \infty$, then $E_i v < \infty$, and therefore P_i almost surely[1] $v < \infty$; that is, the number of times that $\{u(x_n), n \geqslant 0\}$ goes down across $[a,b]$ is finite. Use the following fact: if the number of times that a sequence goes down across any interval $[a,b]$ with rational ends is finite, then the sequence must have a finite or infinite limit. Therefore, if $u(i) < \infty$, then there is a limit

$$\xi = \lim_{n\to\infty} u(x_n) \qquad P_i \text{ almost surely} \tag{8}$$

But

$$E_i u(x_n) = \sum_j p_{ij}^* u(j) \leqslant u(i)$$

so that according to Fatou's lemma,

$$E_i \xi \leqslant u(i)$$

Consequently we have P_i almost surely $\xi < \infty$. So we have proved the theorem below.

Theorem 1. Assume u to be an excessive function, $u(i) < \infty$. Then there almost surely exists a finite limit $\lim_{n\to\infty} u(x_n)$ on Ω_∞.

[1] If P is a probability, 'P almost surely' means 'with P probability one'.

Theorem 2. Assume that the states of the MC communicate. Then the chain is recurrent if and only if all its excessive functions are constants.

Proof. Suppose that u is excessive. If for some j we have $u(j) = \infty$, then for any i, according to the fact that i and j communicate, there must exist n such that $p_{ij}^n > 0$, thus

$$u_i \geqslant \sum_k p_{ik}^n u(k) \geqslant p_{ij}^n u(j) = \infty$$

Hence $u \equiv \infty$. And therefore we only need to consider the finite excessive function u.

Let us assume X_T to be recurrent. By Theorem 1 there exists a finite limit

$$\lim_{n\to\infty} u(x_n) = \xi \qquad P_i \text{ almost surely}$$

According to the recurrence property of X_T and the fact that the states communicate, for an arbitrary j we have

$$P_i(x_n = j \text{ for infinitely many } n) = 1$$

so that

$$P_i(\xi = u(j)) = 1$$

Since j is arbitrary,

$$P_i(\xi = u(k)) = 1$$

for the state k. It follows that $u(j) = u(k)$, that is, $u \equiv$ constant.

Suppose that all the excessive functions of X_T are constants. Fix k and write $u_i = f_{ik}$. Then in accordance with the notation in (2.7),

$$\sum_j p_{ij} u_j = p_i(\eta_k^* < \infty) = f_{ik}^* \leqslant f_{ik} = u_i \tag{9}$$

This means that u is excessive, therefore $u =$ constant. But it is obvious that $u_k = f_{kk} = 1$; hence $u \equiv 1$, that is, $f_{ik} \equiv 1$ for all i and k. Taking $i \neq j$ arbitrarily, by (2.10),

$$1 = f_{ij} = \sum_m p_{im} f_{mj} = \sum_m p_{im}$$

$$f_{ii}^* = \sum_j p_{ij} f_{ji} = \sum_j p_{ij} = 1$$

that is, i is recurrent, so the chain is recurrent, and the proof is completed. QED

The following corollary does not require the condition that the states communicate.

Corollary

When j is fixed, f_{ij} as a function of i is an excessive function.

Proof. From (9) the corollary follows. QED

Corollary

Assume u to be an excessive function; then u takes the constant value in each recurrent class.

Theorem 3. For any measure $\gamma, \gamma G$ is an excessive measure. Especially when i is fixed, $G(i,j)$ is an excessive measure.

Proof. Since

$$G = \sum_{n=0}^{\infty} p^n$$

it follows that

$$GP = \sum_{n=1}^{\infty} p^n \leqslant G \qquad (\gamma G)P \leqslant \gamma G$$

QED

Corollary

Suppose that there is a measure γ such that $\sum_i \gamma_i f_{ij} > 0$ $(j \in E)$, and h is an excessive function. If h is γ-integrable, then h is a finite function; if $\sum_i \gamma_i h_i = 0$, then $h = 0$.

Proof. Since $P^n h \leqslant h$, it follows that

$$\gamma_i p_{ij}^n h \leqslant \sum_i \gamma_i \sum_j p_{ij}^n h_j \leqslant \sum_i \gamma_i h_i$$

Because of the assumption with respect to γ there exist i and n such that $\gamma_i p_{ij}^n > 0$ for every j. Therefore if $\sum_i \gamma_i h_i < \infty$, then $h_j < \infty$; if $\sum_i \gamma_i h_i = 0$, then $h_j = 0$. The proof is concluded. QED

7.3.2 Density function of an excessive measure

Suppose that the MC X_T has an initial distribution γ[1]. In order to emphasize the dependence on γ we had better write the measure P as P_γ and the mathematical expectation E as E_γ. So the mathematical expectation of the number of times that the chain X_T is in the state j is

$$\eta_j = \sum_i \gamma_i G(i,j)$$

The measure

$$\eta = \gamma G \tag{10}$$

is a P-excessive measure.

Suppose D is a finite subset of E; let τ_D (or simply τ) be the last exit time from D of the chain X_T, that is,

$$\tau = \sup\{n: 0 \leqslant n \leqslant \beta, x_n \in D\} \tag{11}$$

If for all n $(0 \leqslant n < \beta)$, we have $x_n \notin D$, then we leave τ undefined.

Write

$$L_D(i) = P_i(\tau = 0) = P_i(x_0 \in D, x_n \notin D \text{ for } 1 \leqslant n \leqslant \beta) \tag{12}$$

so that $L_D(i) = 0$ for $i \notin D$; when $i \in D$, $L_D(i)$ is the probability that the chain leaves D at the first step and will never come back to D. Thus when $i \in D$ and moreover when i is recurrent, $L_D(i) = 0$.

Evidently, for any $j \in E$

$$P_i(x_\tau = j) = \sum_{m=0}^{\infty} P_i(\tau = m, x_m = j)$$

$$= \sum_{m=0}^{\infty} p_{ij}^m L_D(j) = G(i,j) L_D(j) \tag{13}$$

$$\sum_j G(i,j) L_D(j) = \sum_j P_i(x_\tau = j) \leqslant 1 \tag{14}$$

From (13) it follows that

$$P_\gamma(x_\tau = j) = \eta(j) L_D(j) \qquad \text{where } \eta = \gamma G \tag{15}$$

Suppose n is a non-negative integer. When $\tau < n$, or $\tau = \infty$, or τ is undefined, we shall set $x_\tau = \Delta$. Assume $i_0, i_1, \ldots, i_n \in E$; moreover write

$$R(i_n, \ldots, i_0) = P(i_n, i_{n-1}) \cdots P(i_1, i_0)$$

Then it follows that

$$P_i(x_\tau = i_0, x_{\tau-1} = i_1, \ldots, x_{\tau-n} = i_n) = \sum_{m=n}^{\infty} P_i(\tau = m, x_m = i_0, x_{m-1} = i_1, \ldots, x_{m-n} = i_n)$$

$$= \sum_{m=n}^{\infty} P^{m-n}(i, i_n) R(i_n, \ldots, i_0) L_D(i_0)$$

$$= G(i, i_n) R(i_n, \ldots, i_0) L_D(i_0)$$

[1] The total measure of γ may not necessarily be 1.

Multiplying both sides of the above formula by γ_i and summing over i we obtain

$$P_\gamma(x_\tau = i_0, \ldots, x_{\tau-n} = i_n) = \eta(i_n)R(i_n, \ldots, i_0)L_D(i_0) \tag{16}$$

When $n = 0$, the formula above becomes (15).

Lemma 4. Suppose every state of the chain X_T is non-recurrent and let u be a non-negative function on E; set $u(\Delta) = 0$ also. If $u\eta = (u_j\eta_j, j \in E)$ is an excessive measure, then with respect to the measure P_γ, $\{u(x_{\tau-n}), n \geqslant 0\}$ is a supermartingale relative to $\{x_{\tau-n}, n \geqslant 0\}$. That is,

$$E_\gamma[u(x_{\tau-n}) | x_\tau, x_{\tau-1}, \ldots, x_{\tau-(n-1)}] \leqslant u(x_{\tau-(n-1)}) \tag{17}$$

Proof. Since $u\eta$ is an excessive measure it follows that

$$\sum_{i_n} u(i_n)\eta(i_n)P(i_n, i_{n-1}) \leqslant u(i_{n-1})\eta(i_{n-1})$$

Thus

$$\sum_{i_n} \eta(i_n)R(i_n, \ldots, i_0)L_D(i_0)u(i_n) \leqslant \eta(i_{n-1})R(i_{n-1}, \ldots, i_0)L_D(i_0)u(i_{n-1})$$

Noting (16) we obtain (17) from the formula above. The proof is terminated.
QED

The function u in Lemma 4 is the density of the excessive measure $\mu \equiv u\eta$ with regard to η.

Let K_γ denote all non-negative functions u satisfying the following condition:

$$S \equiv \sup\{E_\gamma u(x_{\tau_D}): \text{finite } D \subset E\} < \infty \tag{18}$$

Note that when $\tau_D = \infty$ or τ_D is undefined, $x_{\tau_D} = \Delta$, and furthermore $u(\Delta) = 0$.

Theorem 5. Suppose that every state of the chain X_T is non-recurrent, and assume that $u \in K_\gamma$ and that $u\eta$ is an excessive measure. Then on Ω_∞, there exists P_γ almost surely a finite limit

$$\lim_{n\to\infty} u(x_n)$$

Proof. Let v_N be the number of times that $u(x_\tau), \ldots, u(x_{\tau-N})$ passes down across $[a, b]$, that is, the number of times that $u(x_{\tau-N}), \ldots, u(x_\tau)$ moves up across $[a, b]$. By (5)

$$E_\gamma v_N \leqslant \frac{E_\gamma u(x_\tau)}{b-a}$$

Let v_D denote the number of times that $u(x_0), \ldots, u(x_\tau)$ goes up across $[a, b]$. By the non-recurrence property when D is finite, for P_γ almost certainly $\tau < \infty$; hence $v_N \uparrow v_D$ $(N \to \infty)$. Therefore

$$E_\gamma v_D \leqslant \frac{E_\gamma u(x_{\tau_D})}{b-a}$$

Since $u \in K_\gamma$ it follows that

$$E_\gamma v_D \leqslant S/(b-a)$$

Let v represent the number of times that $u(x_0), u(x_1), \ldots, u(x_n), \ldots$ passes up across $[a, b]$, then $v_D \uparrow v$ $(D \uparrow E)$, and therefore

$$E_\gamma v \leqslant S/(b-a) \tag{19}$$

Consequently $P_\gamma(v < \infty) = 1$, thereby there is P_γ almost certainly a finite or infinite limit $\xi = \lim_{n\to\infty} u(x_n)$. It remains only to prove that P_γ almost surely holds $\xi < \infty$.

Suppose $v(c)$ is the number of times that $u(x_0), \ldots, u(x_n)$ moves up across $[c, 2c]$. Let

$$\varliminf_{c\to\infty} v(c) = \bar{v}$$

Evidently

$$(\xi = \infty) \subseteq (\bar{v} \geqslant 1)$$

so that

$$P_\gamma(\xi = \infty) \leqslant P_\gamma(\bar{v} \geqslant 1) \leqslant E_\gamma \bar{v}$$

By (19) and Fatou's lemma we obtain

$$P_\gamma(\xi = \infty) \leqslant E_\gamma \varliminf_{c\to\infty} v(c) \leqslant \varliminf_{c\to\infty} E_\gamma v(c)$$

$$\leqslant \varliminf_{c\to\infty} \frac{S}{c} = 0$$

and the theorem is proved.

7.3.3 Martin kernel

Definition 4. We call the measure $\gamma = (\gamma_i, i \in E)$ the standard measure of the chain X_T or P if

$$\sum_i \gamma_i < \infty \qquad 0 < \sum_i \gamma_i f_{ij} \tag{20}$$

From now on when considering a Martin boundary we always fix a standard measure γ. By Section 1.4, the measure generated by the initial distribution γ and the matrix $P=(P_{ij})$ is represented by P_γ. Since $P_\gamma=\sum_i\gamma_iP_i$, it follows that if $\gamma_i>0$ ($i\in E$), then $P_\gamma(\Lambda)=0$ if and only if $P_i(\Lambda)=0$ for all i. Obviously,

$$0<A_j\equiv\sum_i\gamma_if_{ij}<\infty \tag{21}$$

Definition 5. We call

$$K(i,j)=\frac{f_{ij}}{A_j}=\frac{f_{ij}}{\sum_s\gamma_sf_{sj}} \tag{22}$$

the Martin kernel of the chain

By Theorem 1 we have

$$A_iG(j,j)=(\gamma G)_j \qquad (j\in E_0)$$

Therefore

$$K(i,j)=\frac{G(i,j)}{\eta(j)} \qquad \text{if } j\in E_0 \tag{23}$$

where $\eta=\gamma G$ and E_0 is the set of all non-recurrent states.

By the first corollary to Theorem 2 and Theorem 3, when j is given, $K(\cdot,j)$ is an excessive function.

According to the definition and the last formula of (2.10) we have

$$K(i,j)\leqslant\frac{1}{A_i} \qquad K(i,j)\leqslant\frac{1}{A_j} \quad \sum_i\gamma_iK(i,j)=1 \tag{24}$$

Theorem 6. Suppose μ is a totally finite measure. Then on Ω_∞ there P_γ-almost surely exists a finite limit

$$\lim_{n\to\infty}\sum_i\mu(i)K(i,x_n) \tag{25}$$

In particular, for any i, on Ω_∞, there P_γ-almost surely exists a finite limit

$$\lim_{n\to\infty}K(i,x_n) \tag{26}$$

Proof. By (24) we have

$$\sum_i\mu(i)K(i,j)\leqslant\sum_i\mu(i)A_j^{-1}<\infty \tag{27}$$

Suppose that E_a is an irreducible recurrent class. When the class property of E_a is taken into account, the states in E_a may be considered to be equal. For instance, every state in E_a is equal to one state ξ_a. When $j\in E_a$, f_{ij} is independent of j and is denoted by $f_i(\xi_a)$. Thus $K(i,j)$ is also independent of $j\in E_a$, and is denoted by $K(i,\xi_a)$. Therefore when the chain gets into some irreducibly recurrent class E_a at a certain step, for all sufficiently large n,

$$\sum_i\mu(i)K(i,x_n)=\sum_i\mu(i)K(i,\xi_a)<\infty$$

Consequently the limit (25) exists and is finite.

What remains to be proved is: when $x_n\in E_0$ for all $n<\beta=\infty$, the limit (25) exists and is finite. In this case we only need to consider the non-recurrent chain on E_0, $\tilde X_T=\{x_n,n\leqslant\tilde\beta\}$, where $\tilde\beta=\sup\{n:x_n\in E_0\}$. The initial distribution of $\tilde X_T$ is $(\gamma_i,i\in E_0)$; the one-step transition matrix is $\tilde P=(p_{ij},i,j\in E_0)$; and as for $\tilde P$, we have the corresponding $\tilde f_{ij}=f_{ij}$, $\tilde K(i,j)=K(i,j)$ $(i,j\in E_0)$. Moreover,

$$\sum_i\mu(i)K(i,j)=\sum_{i\in E_0}\mu(i)\tilde K(i,j) \qquad j\in E_0$$

Accordingly we might as well suppose that every state of the chain $P=(p_{ij},i,j\in E)$ is non-recurrent; then we shall prove that the limit (25) exists and is finite.

Letting $u(j)=\sum_i\mu(i)K(i,j)$, and noting (23), we have $u\eta=\mu G$, while μG is an excessive measure. Now we proceed to prove $u\in K\gamma$. Actually by (15),

$$\begin{aligned}E_\gamma u(x_\tau)&=\sum_j P_\gamma(x_\tau=j)u(j)\\&=\sum_j u(j)\eta(j)L_D(j)\\&=\sum_{i,j}\mu(i)G(i,j)L_D(j)\end{aligned}$$

By (14)

$$E_\gamma u(x_\tau)\leqslant\sum_i\mu(i)<\infty$$

Thus

$$\sup\{E_\gamma u(x_{\tau_D})\colon \text{finite } D\subset E\}\leqslant\sum_i\mu(i)<\infty$$

That is $u\in K_\gamma$. The conclusion of the theorem is immediately reached by an application of Theorem 5, and the proof is completed. QED

7.3.4 Martin boundary

In the theory of the Martin boundary we usually regard every state in the irreducibly recurrent class E_a as one and the same state ξ_a, and we still

write

$$E = E_0 \cup \{\xi_a, a\in\mathscr{A}\} \tag{28}$$

We arrange arbitrarily the order of the E in (28): $E=\{e_1,e_2,\ldots\}$, and write $N(e_m)=m$. Let

$$d(i,j)=|2^{-N(i)}-2^{-N(j)}|+\sum_s |K(s,i)-K(s,j)|A_s 2^{-N(s)} \tag{29}$$

where A_j is defined by (21). Then d is a metric in E and induces a discrete topology from E; moreover $d(i,j)\leqslant 3$ $(i,j\in E)$. Completing E according to the metric d we obtain the complete metric space E^*.

Definition 6. We call $\partial E = E^* - E_0$ the Martin boundary of the chain P. The Borel field generated by the open sets in E^* is written as $\mathscr{E}^*$. An element in $\mathscr{E}^*$ is called a Borel set in E^*. And an $\mathscr{E}^*$-measurable function defined on a Borel set Γ is called a measurable function on Γ.

Evidently,

$$\partial E = (\partial E)_1 \cup (\partial E)_2 \qquad (\partial E)_1 = \{\xi_0, a\in\mathscr{A}\} \qquad (\partial E)_2 = \partial E - (\partial E)_1 \tag{29a}$$

Following the definition of the metric d, we evidently have the following.

Theorem 7. An infinite sequence $\{j_n\}$ in E is a fundamental sequence in the metric space E if and only if

(i) $\lim_{n\to\infty} N(j_n)$ exists (finite or infinite);

(ii) for every $i\in E$, $\{K(i,j_n)\}$ is a real Cauchy's fundamental sequence.

By Theorem 7 we can extend the domain E of definition for $N(i)$ to E^*; therefore

$$\begin{aligned} N(\xi)&<\infty \qquad (\xi\in(\partial E)_1)\\ N(\xi)&=\infty \qquad (\xi\in(\partial E)_2)\end{aligned} \tag{30}$$

For every i, $K(i,j)$ as a function of j can be continuously extended to E^*, that is

$$K(i,\xi)=\lim_{j\xrightarrow{d}\xi} K(i,j)$$

By (24) and Fatou's lemma, $K(\cdot,\xi)$ is an excessive function, and

$$K(i,\xi)\leqslant\frac{1}{A_i} \qquad \sum_i \gamma_i K(i,\xi)\leqslant 1 \qquad \xi\in E^* \tag{31}$$

Thus (24) is a special case of (31), and the second formula above is an equality for $\xi\in E_0\cup(\partial E)_1$.

Thus (29) can also be extended to $i\in E^*$ and $j\in E^*$; hence Theorem 7 is still correct if E in Theorem 7 is changed to E^* and $\{j_n\}$ is an infinite sequence in E^*.

Theorem 8. E^* is a sequentially compact space.

Proof. By Theorem 2, §3, Chapter 1 in Guan (1988), for the metric space the concept of sequential compactness and that of compactness are identical. Therefore it suffices to prove the sequential compactness.

Suppose $\{\xi_n\}$ is an infinite sequence; let $\xi_n\in E^*$. Then we can surely select a subsequence of $\{\xi_n\}$, which is also denoted by $\{\xi_n\}$, such that $\lim_{n\to\infty} N(\xi_n)$ exists (finite or infinite). By (31) for every $i\in E$ we have

$$K(i,\xi_n)\leqslant 1/A_i$$

Applying the diagonal process, we can select a subsequence $\{\xi_{nn}\}$ of $\{\xi_n\}$ such that for every $i\in E$, $\{K(i,\xi_{nn})\}$ are all fundamental sequences of real numbers. According to the paragraph preceding Theorem 8, the sequence $\{\xi_{nn}\}$ must converge in E^* to some point $\xi\in E^*$. This is just the sequential compactness. The proof is terminated. QED

Theorem 9. For P_γ-almost all $\omega\in\Omega$ either

$$x_\beta\in H \qquad \text{if } \beta<\infty \tag{32}$$

or there exists a limit

$$d-\lim_{n\to\infty} x_n = x_\infty\in\partial E \qquad \text{if } \beta=\infty \tag{33}$$

Here

$$H=\left\{i:\sum_i p_{ij}<1\right\}$$

is the stopping state set.

Proof. On Ω_F, $x_\beta\in H$ is obvious. On Ω_∞, as Theorem 6 has already pointed out, there P_γ almost surely exists a finite limit

$$\lim_{n\to\infty} K(i,x_n) \qquad \text{for all } i\in E$$

Thus there exists a limit

$$d-\lim_{n\to\infty} x_n = x_\infty\in E^*$$

If the chain gets into some recurrent class E_a, then $x_\infty=\xi_a\in\partial E$. Otherwise the chain will move in E_0 forever. As the number of times that the chain stays

at any non-recurrent state $i \in E_0$ is always finite, it follows that $x_\infty \notin E_0$, and hence $x_\infty \in E^* - E_0 = \partial E$. The proof is concluded. QED

7.3.5 Distribution of ultimate states

By Theorem 9 the ultimate state x_β is defined P_γ-almost surely. We write the distribution of x_β as follows:

$$\mu(\Gamma) \equiv \mu_1(\Gamma) = P_\gamma(x_\beta \in \Gamma) \qquad \Gamma \in \mathscr{E}^* \tag{34}$$

Then the mass of μ is distributed on $H \cup \partial E$.

Theorem 10. Suppose that u is a continuous function or a non-negative Borel function on E^* Then

$$E_i u(x_\beta) = \int_{H \cup \partial E} K(i, \xi) u(\xi) \mu(\mathrm{d}\xi) \tag{35}$$

$$E_i u(x_\beta) 1_{\beta < \infty} = \sum_j G(i, j) u_j \left(1 - \sum_s p_{js}\right) \tag{36}$$

$$E_i u(x_\beta) 1_{\beta = \infty} = \int_{\partial E} K(i, \xi) u(\xi) \mu(\mathrm{d}\xi) \tag{37}$$

In particular,

$$P_i(x_\beta \in \Gamma) = \int_\Gamma K(i, \xi) \mu(\mathrm{d}\xi) \qquad \Gamma \subset H \cup \partial E \tag{38}$$

$$\begin{aligned} P_i(x_\beta = j) &= P_i(x_\beta = j, \beta < \infty) \\ &= G(i, j)\left(1 - \sum_s p_{js}\right) \qquad j \in E_0 \end{aligned} \tag{39}$$

$$\begin{aligned} P_i(x_\beta \in \Gamma) &= P_i(x_\beta \in \Gamma, \beta = \infty) \\ &= \int_\Gamma K(i, \xi) \mu(\mathrm{d}\xi) \qquad \Gamma \subset \partial E \end{aligned} \tag{40}$$

$$\mu(j) \equiv P_\gamma(x_\beta = j) = \eta(j)\left(1 - \sum_s p_{js}\right) \qquad j \in E_0 \tag{41}$$

Proof. When $\xi_0 \in (\partial E)_1$

$$P_i(x_\beta = \xi_a) = f_i(\xi_a)$$

$$\mu(\xi_a) \equiv P_\gamma(x_\beta = \xi_a) = \sum_s \gamma_s f_s(\xi_a)$$

So

$$P_i(x_\beta = \xi) = K(i, \xi) \mu(\xi) \qquad \xi \in (\partial E)_1$$

Hence

$$E_i u(x_\beta) 1_{x_\beta \in (\partial E)_1} = \int_{(\partial E)_1} K(i, \xi) u(\xi) \mu(\mathrm{d}\xi) \tag{42}$$

Secondly, when $j \in E_0$,

$$\begin{aligned} P_i(x_\beta = j) &= P_i(x_\beta = j, \beta < \infty) \\ &= \sum_{n=0}^{\infty} P_i(x_n = j, n = \beta) = \sum_{n=0}^{\infty} p_{ij}^n \left(1 - \sum_s p_{js}\right) \end{aligned}$$

From this (39) follows. Then (36) and (41) follow from (39), and furthermore from (39) and (41) we obtain

$$E_i u(x_\beta) 1_{\beta < \infty} = \int_H K(i, \xi) u(\xi) \mu(\mathrm{d}\xi) \tag{43}$$

Thirdly, comparing (13), (15) and (22) we have

$$P_i(x_\tau = j) = K(i, j) P_\gamma(x_\tau = j) \qquad j \in E \tag{44}$$

Hence

$$\begin{aligned} E_i u(x_\tau) &= \sum_j u(j) P_i(x_\tau = j) = \sum_j u(j) K(i, j) P_\gamma(x_\tau = j) \\ &= E_\gamma K(i, x_\tau) u(x_\tau) \end{aligned} \tag{45}$$

Let $D \uparrow E$, then $\tau_D \uparrow \beta$, $x_{\tau_D} \to x_\beta$ when $x_\beta \in H \cup (\partial E)_2$; when $x_\beta \in (\partial E)_1$, if only D is sufficiently large, that is, when D and $(\partial E)_1$ have some non-empty intersection, $\tau_D = \infty$. According to the supposition $x_{\tau_D} = \Delta$, $u(\Delta) = 0$. Thus when u is a continuous function on E^*, noting the first formula of (31) we get

$$E_i u(x_\beta) 1_{x_\beta \in H \cup (\partial E)_1} = E_r K(i, x_\beta) u(x_\beta) 1_{x_\beta \in H \cup (\partial E)_1} = \int_{H \cup (\partial E)_2} K(i, \xi) u(\xi) \mu(\mathrm{d}\xi) \tag{46}$$

From (42) and (46) it follows that (35) is valid for the continuous function u on $H \cup \partial E$ and therefore it also holds for the non-negative Borel function u. By (35) and (43) we derive (37), and the proof is completed. QED

Theorem 11. Assuming that u is a continuous function on E^*, then

$$\int_{E^*} u(\xi) \mu(\mathrm{d}\xi) = E_r u(x_\beta) = \sum_j u(j) \eta(j) \left(1 - \sum_s p_{js}\right) + \lim_{n \to \infty} \sum_{i,j} r(i) p_{ij}^n u(j) \tag{47}$$

Proof. When u is a continuous function on E^*,

$$E_i u(x_\infty) = \lim_{n\to\infty} E_i u(x_n) = \lim_{n\to\infty} (p^n u)_i$$

By this and (39) we obtain (47). QED

7.3.6 *h*-Chain and the Martin representation of an excessive function

Suppose that h is a P-excessive function, and moreover is γ-integrable, that is, $\sum_i \gamma_i h_i < \infty$. On account of the corollary to Theorem 3, h is finite-valued. We write

$$E^h = \{i: h_i > 0\} \qquad p_{ij}^h = \frac{p_{ij}h_j}{h_i} \qquad i, j \in E^h \tag{48}$$

By the excessive property of h we can easily get

$$\begin{aligned} p_{ij} &= 0 \qquad i \notin E^h, j \in E^h \\ f_{ij} &= 0 \qquad i \notin E^h, j \in E^h \end{aligned} \tag{49}$$

whereas $P^h = (p_{ij}^h, i, j \in E^h)$ is a matrix that is non-negative and whose row sum is not more than 1. The MC $X_T = \{x_n, n \leqslant \beta\}$, whose one-step transition matrix is P^h, is called an h-chain. When $h \equiv 1$ or constant, the h-chain becomes an MC whose one-step transition matrix is P. Henceforth, all the characteristics of the h-chain will be preceded by h. For example,

$$G^h = \sum_{n=0}^{\infty} (P^h)^n$$

measure P_i^h and so on.

Clearly, the measure $\gamma_i^h = \gamma_i h_i$ $(i \in E^h)$ is a standard measure for the h-chain, and furthermore

$$P_{ij}^{hn} = \frac{P_{ij}^n h_j}{h_i} \qquad i, j \in E^h \tag{50}$$

Hence

$$G^h(i, j) = \frac{G(i, j)h_j}{h_i} \qquad i, j \in E^h \tag{51a}$$

$$\eta^h(j) = \eta(j)h_j \qquad j \in E^h \tag{51b}$$

where $\eta^h = \gamma^h G^h$, $\eta = \gamma G$.

By (51a) it can be seen that for an h-chain the state decomposition theorem in Theorem 2.2 takes the following form:

$$E^h = E_0^h \cup \left(\bigcup_{a \in \mathscr{A}} E_a^h \right) \tag{52a}$$

where

$$E_0^h = E^h \cap E_0 \qquad E_a^h = E^h \cap E_a \qquad a \in \mathscr{A} \tag{52b}$$

Let the stopping state set for the h-chain be denoted by H^h, that is,

$$H^h = \left\{ i: i \in E^h, \sum_{j \in E^h} p_{ij}^h < 1 \right\} = \left\{ i: i \in E^h, \sum_{j \in E} p_{ij} h_j < h_i \right\} \tag{52c}$$

Then we have

$$H^h \subset E_0^h \subset E_0 \tag{52d}$$

Suppose $i, j \in E^h$. By (49)

$$f_{ij}^{hn} = \sum{}' p_{ij_1}^h p_{j_1 j_2}^h \cdots p_{j_{n-1} j}^h = \frac{1}{h_i} \sum{}'' p_{ij_1} p_{j_1 j_2} \cdots p_{j_{n-1} j} h_j = \frac{f_{ij}^n h_j}{h_i} \tag{53}$$

where $\sum'$ represents summing $j_l \neq j$, $j_l \in E^h$, $1 \leqslant l \leqslant n-1$ while $\sum''$ indicates summing $j_l \neq j$, $j_l \in E$, $1 \leqslant l \leqslant n-1$. By (51b) we obtain

$$f_{ij}^h = \frac{f_{ij} h_j}{h_i} \qquad i, j \in E^h \tag{54}$$

Thus

$$A_j^h = \sum_{i \in E^h} \gamma_i^h f_{ij}^h = \sum_{i \in E} \gamma_i f_{ij} h_j = A_j h_j \qquad j \in E^h \tag{55}$$

Consequently the Martin kernel of P^h corresponding to γ^h is

$$K^h(i, j) = \frac{f_{ij}^h}{A_j^h} = \frac{K(i, j)}{h_i} \qquad i, j \in E^h \tag{56}$$

From the foregoing formula we can see that if $\{K^h(i, j_n)\}$ is a Cauchy sequence of real numbers, then so is $\{K(i, j_n)\}$. Therefore by Theorem 7 we can find that the Martin topology of the h-chain is identical with that of the 1-chain. Hence the Martin sequentially compact space E^{h*} of the h-chain can be looked on as a closed subspace of E^*, that is E^{h*} is the closure of E^h in E^*. And the Martin boundary ∂E^h of the h-chain is precisely the boundary of E^h *in* E^*, i.e.

$$\partial E^h = \partial E \cap E^{h*}$$

Suppose that $\{x_n, n \leqslant \beta\}$ is an h-chain; let the initial distribution be γ^h. We write μ_h for the distribution of the ultimate state x_β. The support of μ_h is contained in $H^h \cup \partial E^h$, but μ_h can be looked on as a measure on E^{h*} and more reasonably as a measure on E^*, that is

$$\mu_h(\Gamma) = p_{\gamma^h}^h (x_\beta \in \Gamma) \qquad \Gamma \in \mathscr{E}^* \tag{57}$$

And it follows that

$$\mu_h(E^*) = \sum_{i\in E^h} \gamma_i^h P_i^h(x_\beta \in E^*) = \sum_{j\in E^h} \gamma_i^h = \sum_i \gamma_i h_i \tag{58}$$

$$\mu_h(E^* - E^{h*}) = 0 \tag{59}$$

Theorem 12. Assume u to be a continuous function or a non-negative Borel function on E^*, then when $i\in E^h$,

$$E_i^h u(x_\beta) = \frac{1}{h_i}\int_{E^*} K(i,\xi)u(\xi)\mu_h(\mathrm{d}\xi) \tag{60}$$

$$E_i^h u(x_\beta)L_{\beta<\infty} = \frac{1}{h_i}\sum_j G(i,j)u(j)\left(h_j - \sum_s p_{js}h_s\right) \tag{61}$$

$$E_i^h u(x_\beta)L_{\beta=\infty} = \frac{1}{h_i}\int_{\partial E} K(i,\xi)u(\xi)\mu_h(\mathrm{d}\xi) \tag{62}$$

Especially, when $i\in E^h$,

$$P_i^h(x_\beta\in\Gamma) = \frac{1}{h_i}\int_\Gamma K(i,\xi)\mu_h(\mathrm{d}\xi) \tag{63}$$

$$\begin{aligned} P_i^h(x_\beta = j) &= P_i^h(x_\beta = j, \beta<\infty) \\ &= \frac{1}{h_i}G(i,j)\left(h_i - \sum_s p_{js}h_s\right) \qquad j\in E_0 \end{aligned} \tag{64}$$

$$\begin{aligned} P_i^h(x_\beta\in\Gamma) &= P_i(x_\beta\in\Gamma, \beta=\infty) \\ &= \frac{1}{h_i}\int_\Gamma K(i,\xi)\mu_h(\mathrm{d}\xi) \qquad \Gamma\subset\partial E \end{aligned} \tag{65}$$

$$\mu_h(j) = \eta(j)\left(h_j - \sum_s p_{js}h_s\right) \qquad j\in E_0 \tag{66}$$

If u is a continuous function on E^*, then

$$\int_{E^*} u(\xi)\mu_h(\mathrm{d}\xi) = \sum_j u_j\eta_j\left(h_j - \sum_s p_{js}h_s\right) + \lim_{n\to\infty}\sum_{i,j}\gamma_i p_{ij}^n h_j u_j \tag{67}$$

Proof. Applying the conclusions of Theorems 10 and 11 to the h-chain and noting (56) as well as $E_0^h\subset E_0$, we can prove the theorem. QED

Theorem 13. Suppose that h is a γ-integrable excessive function. Then

$$h_i = \int_{E^*} K(i,\xi)\mu_h\,\mathrm{d}\xi \tag{68a}$$

$$= \sum_j G(i,j)\left(h_j - \sum_s p_{js}h_s\right) + \int_{\partial E} K(i,\xi)\mu_h(\mathrm{d}\xi) \tag{68b}$$

Proof. Letting $u\equiv 1$ in (60) it follows that (68a) is valid for $i\in E^h$. Suppose $i\notin E^h$, that is $h_i = 0$. By (49) we have $K(i,j) = 0$ $(j\in E^h)$, so that $K(i,\xi) = 0$ $(\xi\in E^{h*})$. Noticing (59),

$$\int_{E^*} K(i,\xi)\mu_h(\mathrm{d}\xi) = \int_{E^{h*}} K(i,\xi)\mu_h(\mathrm{d}\xi) = 0$$

Therefore (68a) is also true for $i\notin E^h$.

Using (66) we obtain (68b) from (68a). The proof is terminated. QED

Remark

The measure μ_h is called the spectral measure of h whereas (68a) is said to be the Martin representation of the excessive function h.

7.3.7 Essential Martin boundary

Suppose $j\in E$, then $K(\cdot,j)$ is an excessive measure; hence the measure $\mu_{K(\cdot,j)}$ can be well defined. By (58) and (24),

$$\mu_{K(\cdot,j)}(E^*) = \sum_i r_i K(i,j) = 1 \tag{69}$$

Theorem 14. Suppose δ_j is the unit measure concentrated at j, then

$$\mu_{K(\cdot,j)} = \delta_j \qquad j\in E_0\cup(\partial E)_1 \tag{70}$$

Proof. Write $h = K(\cdot,j)$. When $j\in E_0$, then

$$(ph)_i = \sum_s p_{is}K(s,j) = \frac{G(i,j)-\delta_j(i)}{\eta(j)} = h_i - \frac{\delta_j(i)}{\eta(j)}$$

and so

$$h_j - (ph)_j = 1/\eta(j)$$

Thereupon by (66), $\mu_{K(\cdot,j)}(j) = 1$. Observing (69) we have (70).

When $j = \xi_a$ $(a\in\mathscr{A})$, we have

$$K(i,\xi_a) = f_i(\xi_a)/A(\xi_a)$$

and

$$\mu_h(\xi_a) = \sum{}'\gamma_i^h p_i^h(x_\beta = \xi_a) = \sum{}'\gamma_i^h f_i^h(\xi_a)$$

where $\sum'$ represents summation over E^h. Consider (54) and $h(\xi_a) = K(\xi_a, \xi_a) = 1/A(\xi_a)$ and then

$$\mu_h(\xi_a) = \sum' \gamma_i h_i \frac{f_i(\xi_a)h(\xi_a)}{h_i}$$

$$= \sum_i \gamma_i f_i(\xi_a) \cdot \frac{1}{A(\xi_a)} = 1 \tag{71}$$

Again applying (69) we obtain $\mu_{K(\cdot,\xi_a)} = \delta_{\xi_a}$. The proof is concluded. QED

Definition 7. Let

$$B = \{\xi : \xi \in \partial E, \mu_{K(\cdot,\xi)} = \delta_\xi\} \tag{72}$$

B is called the essential Martin boundary. If $\xi \in B$ and $\mu(\xi) > 0$, then ξ is said to be an atomic boundary point. The set B_1 composed of all the atomic boundary points is called the atomic boundary of the chain; $B_2 = B - B_1$ is called the non-atomic boundary of the chain. $B = B_1 \cup B_2$.

By Theorem 14, $\xi_a \in B$. Again since

$$\mu(\xi_a) = p_\gamma(x_\beta = \xi_a) = \sum_i \gamma_i p_i(x_\beta = \xi_a)$$

$$= \sum_i \gamma_i f_i(\xi_a) = A(\xi_a) > 0$$

it follows that every recurrent boundary point ξ_a is an atomic boundary point, that is,

$$(\partial E)_1 \subset B_1 \tag{73}$$

Theorem 15. Suppose $\xi \in B$, then $K(\cdot, \xi)$ is a harmonic function; moreover

$$\sum_i \gamma_i K(i, \xi) = 1 \qquad \xi \in B \tag{74}$$

Proof. Write $h = K(\cdot, \xi)$; since $\mu_h = \delta_\xi$ from (68b) we get $Gf = 0$, where $f = h - ph$. Accordingly,

$$\sum_j \eta(j) f(j) = \sum_j \gamma_j (Gf)_j = 0$$

and thus $f = 0$. That is, $K(\cdot, \xi)$ $(\xi \in B)$ is harmonic. Again by (58) and (3) when $\xi \in B$,

$$1 = \mu_{K(\cdot,\xi)} = \sum_i \gamma_i K(i, \xi) \leqslant 1$$

From what precedes, (74) follows. The proof is completed. QED

Theorem 16. B is a Borel subset of ∂E. For any γ-integrable excessive function h, we have

$$\mu_h(\partial E - B) = 0$$

Proof. As $\xi \in \partial E$, by (58) and (31)

$$\mu_{K(\cdot,\xi)}(E^*) = \sum_i \gamma_i K(i, \xi) \leqslant 1$$

so that

$$B = \{\xi : \xi \in \partial E, \mu_{K(\cdot,\xi)}(\xi) = 1\} \tag{75a}$$

For $\xi \in \partial E$, by (67),

$$\mu_{K(\cdot,\xi)}(\xi) = \lim_{m\to\infty} \int_{E^*} e^{-md(\zeta,\xi)} \mu_{K(\cdot,\xi)}(d\zeta)$$

$$= \lim_{m\to\infty} \sum_j e^{md(j,\xi)} \eta_i \left(K(j, \xi) - \sum_s p_{js} K(s, \xi) \right)$$

$$+ \lim_{m\to\infty} \lim_{n\to\infty} \sum_{i,j} \gamma_i p_{ij}^n K(j, \xi) e^{-md(i,j)} \tag{75b}$$

From this we know B is a Borel set.

Suppose $\xi \in \partial E$; simply write $K = K(\cdot, \xi)$. Assume φ and ψ are both continuous functions on E^*, $m \geqslant 0$, $n \geqslant 0$. Then

$$E_{rh}^h \varphi(x_n)\psi(x_{n+m}) = \sum_{i,j,s} \gamma_i h_i p_{ij}^{hn} \varphi(j) p_{js}^{hm} \psi(s)$$

$$= \sum_{i,j} \gamma_i p_{ij}^n h_j \varphi(j) E_j^h \psi(x_m)$$

Let $m \to \infty$; by an application of (62) we have

$$E_{rh}^h \varphi(x_n)\psi(x_\infty) = \sum_{i,j} \gamma_i p_{ij}^n h_j \varphi(j) \int_{\partial E} K^h(j, \xi)\psi(\xi)\mu_h(d\xi)$$

$$= \int_{\partial E} \sum_{i,j} \gamma_i K(i, \xi) p_{ij}^K(n) \varphi(j) \psi(\xi) \mu_h(d\xi)$$

$$= \int_{\partial E} E_{rk}^K \varphi(x_n)\psi(\xi)\mu_h(d\xi) \tag{76}$$

Considering

$$E_{rh}^K \varphi(x_\infty) = \int_{\partial E} \varphi(\zeta)\mu_K(d\zeta)$$

and letting $n \to \infty$ in (76), we obtain

$$\int_{\partial E} \varphi(\xi)\psi(\xi)\mu_h \, d\xi = \int_{\partial E}\left(\int_{\partial E} \varphi(\zeta)\mu_K \, d\zeta\right)\psi(\xi)\mu_h(d\xi)$$

Since the continuous function ψ can be arbitrarily selected, from the formula above it follows that

$$\varphi(\xi) = \int_{\partial E} \varphi(\zeta)\mu_{K(\cdot,\xi)}(d\zeta)$$

holds for μ_h-almost all $\xi \in \partial E$. In particular, taking φ as

$$\varphi_m(\zeta) = e^{-md(\zeta,\xi)} \qquad m = 1, 2, 3, \ldots$$

we get that

$$1 = \int_{\partial E} e^{-md(\zeta,\xi)}\mu_{K(\cdot,\xi)}(d\zeta) \qquad m = 1, 2, \ldots$$

holds for μ_h-almost all $\xi \in E$. Letting $m \to \infty$ in the foregoing formulae we have that

$$1 = \mu_{K(\cdot,\xi)}(\xi)$$

holds for μ_h almost all $\xi \in \partial E$. From (75) it follows immediately that $\mu_h(\partial E - B) = 0$. The proof is completed. QED

Corollary

Write the essential Martin boundary of an h-chain as B^h, and then

$$B^h = B \cap E^{h*}$$

Proof. Similar to (75),

$$B^h = \{\xi : \xi \in \partial E^h, \mu_{K^h(\cdot,\xi)}(\xi) = 1\}$$

However $\partial E^h = \partial E \cap E^{h*}$. When $\xi \in \partial E^h$ according to (75b) we may calculate $\mu_{K^h(\cdot,\xi)}(\xi)$:

$$\mu_{K^h(\cdot,\xi)}(\xi) = \lim_{m\to\infty} \sum_{j\in E^h} e^{md(j,\xi)}\eta_j^h\left(K^h(j,\xi) - \sum_{s\in E^h} p_{js}^h K^h(s,\xi)\right)$$
$$+ \lim_{m\to\infty}\lim_{n\to\infty} \sum_{i,j\in E^h} \gamma_i^h p_{ij}^{hn} K^h(j,\xi) e^{-md(i,j)}$$

Noting (49)–(52) and (56) we can see that the right-hand side of the formula above becomes the right-hand side of (75b); thus $\mu_{K(\cdot,\xi)}(\xi) = \mu_{K^h(\cdot,\xi)}(\xi)$. From this we can prove $B^h = B \cap E^{h*}$. The proof is completed. QED

Because of Theorem 16, Theorem 9 can be strengthened as follows:

Theorem 17. Suppose $X_T = \{x_n, n \leqslant \beta\}$ is a 1-chain. Then for P_γ-almost all $\omega \in \Omega$, either

$$x_\beta \in H \qquad \text{if } \beta < \infty \tag{77}$$

or there exists a limit

$$d - \lim_{n\to\infty} x_n = x_\infty \in B \qquad \text{if } \beta = \infty \tag{78}$$

Equations (60) and (62) can be strengthened like this:

$$E_i^h u(x_\beta) = \frac{1}{h_i}\int_{E_0\cup B} K(i,\xi)u(\xi)\mu_h(d\xi) \tag{79}$$

$$E_i^h u(x_\beta)1_{\beta=\infty} = \frac{1}{h_i}\int_B K(i,\xi)u(\xi)\mu_h(d\xi) \tag{80}$$

Especially

$$E_i u(x_\beta) = \int_{E_0\cup B} K(i,\xi)u(\xi)\mu(d\xi) \tag{81}$$

$$E_i u(x_\beta)1_{\beta=\infty} = \int_B K(i,\xi)u(\xi)\mu(d\xi) \tag{82}$$

7.3.8 Uniqueness of Martin representation

Theorem 18. Every γ-integrable excessive function h can be uniquely represented by

$$h_i = \int_{E_0\cup B} K(i,\xi)\lambda(d\xi) \tag{83}$$

where λ is a totally finite measure on the Borel set $E_0 \cup B$; therefore $\lambda = \mu_h$. Conversely, given any arbitrary totally finite measure λ on $E_0 \cup B$, (83) defines a γ-integrable excessive function. This function is harmonic if and only if $\lambda(E_0) = 0$.

Proof. Let $u = 1$ in (79). It follows that h has the representation (83), where $\lambda = \mu_h$ is a measure on $E_0 \cup B$, and the total finiteness of μ_h can be derived from (58).

Now we assume that h has the representation (83). We proceed to prove $\lambda = \mu_h$. For the continuous function u on E^*, applying h and $K(\cdot, \xi)$ to (67)

respectively and noting (83), we have

$$\int_{E^*} u(\xi)\mu_h(\mathrm{d}\xi) = \int_{E_0\cup B}\left(\int_{E^*} u(\xi)\mu_{K(\cdot,\zeta)}(\mathrm{d}\xi)\right)\lambda(\mathrm{d}\zeta)$$

But $\mu_{K(\cdot,\zeta)} = \delta_\zeta$ ($\zeta\in E_0\cup B$); hence

$$\int_{E^*} u(\xi)\mu_h(\mathrm{d}\xi) = \int_{E_0\cup B} u(\xi)\lambda(\mathrm{d}\xi)$$

Because the mass of μ_h all concentrates on $E_0\cup B$, on account of the arbitrariness of u we obtain $\lambda=\mu_h$ from the formula above.

Since $K(\cdot,\xi)$ is an excessive function, and moreover (31) is valid, it follows that (83) defines a γ-integrable excessive function. By Theorem 15, $K(\cdot,\xi)$ ($\xi\in B$) is harmonic; therefore if $\lambda(E_0)=0$, then the h in (83) is harmonic. Conversely, assume that h in (83) is harmonic. By (66) we obtain $\mu_h(j)=0$ for all $j\in E_0$, that is, $\lambda(E_0)=\mu_h(E_0)=0$. The proof is completed. QED

7.3.9 Minimal excessive function

Recall Definition 2 in 7.3.1 of the minimal excessive function. We have:

Theorem 19. The general form of the γ-integrable minimal excessive function is $CK(\cdot,\xi)$, $\xi\in E_0\cup B$, where C is a constant.

Proof. It can be seen from (68) that $\mu_{h_1+h_2}\simeq\mu_{h_1}+\mu_{h_2}$. Suppose $\xi\in E_0\cup B$ and $K(\cdot,\xi)=h_1+h_2$, and furthermore both h_1 and h_2 are excessive functions. Then $\mu_{h_1+h_2}=\mu_K(\cdot,\xi)=\delta_\xi$. Accordingly,

$$\mu_{h_1}(E^*-\xi)+\mu_{h_2}(E^*-\xi)=\delta_\xi(E^*-\xi)=0$$

and it follows that

$$\mu_{h_e}(E^*-\xi)=0\,(e=1,2)$$

According to the spectral representation (68) of h_e,

$$\begin{aligned}h_e(i) &= \int_{E^*} K(i,\zeta)\mu_{h_e}(\mathrm{d}\zeta)\\ &= K(i,\xi)\mu_{h_e}(\xi) = C_eK(i,\xi)\end{aligned}$$

where $C_e=\mu_{h_e}(\xi)$ is a constant. Therefore $K(\cdot,\xi)$ ($\xi\in E_0\cup B$) is a minimal excessive function.

Secondly, suppose h is a γ-integrable minimal excessive function, and let μ_h be its spectral measure. Also we may as well suppose $h\neq 0$. By (58) and the corollary to Theorem 3, $\mu_h(E_0\cup B)>0$. Hence there exists $\xi\in E_0\cup B$ such that μ_h has a positive measure in any neighbourhood of ξ. Let $U_n=\{\zeta:d(\zeta,\xi)<1/n\}$, $h_n=\int_{U_n}K(\cdot,\zeta)\mu_h\,\mathrm{d}\zeta$. Then both h_n and

$$h-h_n=\int_{(E_0\cup B)-U_n} K(\cdot,\zeta)\mu_h(\mathrm{d}\zeta)$$

are excessive functions. Owing to the minimality of h, $h_n=C_nh$ (C_n is a constant), so that

$$\sum_i \gamma_i h_n(i) = C_n\sum_i \gamma_i h(i)$$

Again on account of the uniqueness of the representation of h_n, we get

$$\mu_{hn}(\Gamma)=\mu_h(U_n\cap\Gamma)$$

Considering (58),

$$\mu_h(U_n)=\mu_{hn}(E^*)=\sum_i\gamma_ih_n(i)$$

$$\mu_h(E^*)=\sum_i\gamma_ih(i)$$

Thus

$$h_i=\frac{\mu_h(E^*)}{\mu_h(U_n)}\int_{U_n}K(i,\zeta)\mu_h(\delta\zeta)$$

Letting $n\to\infty$ we obtain $h_i=CK(i,\xi)$, where $C=\mu_h(E^*)$. The proof is terminated. QED

Theorem 20. The essential Martin boundary B is

$$B=\left\{\xi:\xi\in\partial E, K(\cdot,\xi)\text{ is a minimal harmonic function}, \sum_i\gamma_iK(i,\xi)=1\right\}\tag{84}$$

Proof. Write the set on the right-hand side of (84) as C. By Theorems 15 and 19, $B\subset C$. Suppose $\xi\in C$. Then $h\equiv K(\cdot,\xi)$ is γ-integrable. Write μ_h for its spectral measure. Evidently h has the representation

$$h_i=\int_{E_0\cup B}K(i,\zeta)\delta_\xi(\mathrm{d}\zeta)$$

By the uniqueness theorem, Theorem 18, $\mu_h=\delta_\xi$, that is $\xi\in B$, $C\subset B$. The proof is concluded. QED

7.3.10 Ultimate field and ultimate random variables

The infinite-dimensional function $f(j_0,j_1,j_2,\ldots)$ ($j_k\in E$, $k=0,1,2,\ldots$) is said to be invariant if for any $j_k\in E$, $k=0,1,2,\ldots$, $f(j_0,j_1,j_2,\ldots)=f(j_1,j_2,j_3,\ldots)$ holds.

Definition 8. A function Φ defined on Ω_∞ is called an ultimate random variable of the chain if there exists an invariant function f such that

$$\Phi(\omega) = f\{x_n(\omega), x_{n+1}(\omega), \ldots\} \qquad \omega \in \Omega_\infty$$

holds for $n = 0$ and therefore for all $n \geqslant 0$. We define an ultimate random variable Φ to take the value 0 on $\Omega_F = \Omega - \Omega_\infty$. We call the set $\Lambda \subset \Omega_\infty$ an ultimate set if the indicator I_Λ is an ultimate random variable. The Borel field on Ω_∞ composed of ultimate sets is denoted by $\mathscr{B}_\infty$ and is called the ultimate field.

Suppose $\Lambda_1, \Lambda_2 \in \mathscr{F}$. If $P_\gamma\{(\Lambda_1 - \Lambda_2) + (\Lambda_2 - \Lambda_1)\} = 0$, then we write $\Lambda_1 \simeq \Lambda_2$. If Φ_1 and Φ_2 are random variables and $P_\gamma(\Phi_1 \neq \Phi_2) = 0$, then we write $\Phi_1 \simeq \Phi_2$.

Theorem 21. There exists a one-to-one correspondence between the non-null ultimate set and the almost closed set A (modulo transient sets) under the following condition:

$$\Lambda \simeq \mathscr{L}(A) \tag{85}$$

A can be taken as $A = \{i: P_i(\Lambda) > \varepsilon\}$, $0 < \varepsilon < 1$.

Proof. See Chung (1967, I.17, Theorem 1) QED

Theorem 22. There exists a one-to-one correspondence between the non-negative bounded P-harmonic function u and the non-negative bounded ultimate random variable Φ under the following condition:

$$u_i = E_i\Phi \qquad \Phi = \lim_{n\to\infty} u(x_n) \tag{86}$$

Proof. See Chung (1967, I.17, Theorem 5). QED

Definition 9. The non-null ultimate set $\Lambda \in \mathscr{B}_\infty$ is said to be atomic if Λ cannot be decomposed into the union of two non-null ultimate sets; the non-null ultimate set $\Lambda \in B_\infty$ is said to be completely non-atomic if for any $\Lambda_1 \subset \Lambda$, $\Lambda_1 \in \mathscr{B}_\infty$, then Λ_1 is not atomic.

Theorem 23. Ω_∞ has the following decomposition:

$$\Omega_\infty = \Lambda_0 + \left(\bigcup_{a \in \mathscr{A}} \Lambda_a\right) \tag{87}$$

where Λ_0 is a completely non-atomic ultimate set and can be dropped. $\mathscr{A}$ is an empty set or a finite set or a denumerable infinite set, and moreover $0 \notin \mathscr{A}$. Also, Λ_a $(a \in \mathscr{A})$ is an atomic ultimate set. The decomposition is unique.

Proof. Making use of Theorem 2.3 and Theorem 5.6, which is to be proved latter, we obtain the proof of this theorem. QED

7.3.11 Martin entrance boundary

The Martin boundary discussed previously is actually the Martin exit boundary, that is, to depict how the process 'goes to infinity' from a finite state. As for the description of how the process comes to a finite state from infinity, we need the Martin entrance boundary.

Suppose the decomposition of the state space E for the Markov chain $P = (p_{ij}, i, j \in E)$ is (2.15) and E_0 is the set of non-recurrent states, we write

$$E_{00} = \{i: i \in E_0, \text{ there exists } j \in E - E_0 \text{ such that } i \Rightarrow j\}$$

$$E_1 = E - E_{00} \qquad P_1 = (p_{ij}, i, j \in E_1)$$

Zhen-ting Hou (1974, Chapter 8) points out: any finite excessive measure $(\alpha_j, j \in E)$ of P must be zero on E_{00}, that is, $\alpha_j = 0 (j \in E_{00})$, and moreover, there exist excessive measures that are positive on E_1. We fix such a P-excessive measure $(\alpha_j, j \in E)$, then the measure $(V_j, j \in E)$ is a finite excessive (finite harmonic) measure of P if and only if $V_j = 0$ $(j \in E_{00})$ and $(V_j/\alpha_j, j \in E_1)$ is the finite excessive (finite harmonic) function of $\tilde{P}$. Here

$$\tilde{P} = (\tilde{p}_{ij}, i, j \in E_1) \qquad \tilde{p}_{ij} = \alpha_j p_{ji}/\alpha_i$$

Select a standard measure $\tilde{\gamma}$ of $\tilde{P}$. According to $\tilde{\gamma}$ and $\tilde{P}$, we can well define the Martin boundary $\partial\tilde{E}_1$ of $\tilde{P}$, the essential Martin boundary $\tilde{B}$, the atomic boundary $\tilde{B}_1$ and the non-atomic boundary $\tilde{B}_2$. We call $\partial\tilde{E}_1$, $\tilde{B}, \tilde{B}_1$ and $\tilde{B}_2$ respectively the Martin entrance boundary of P, the essential Martin entrance boundary, the atomic entrance boundary and the non-atomic entrance boundary. Of course the boundaries mentioned above are relative to α and $\tilde{\gamma}$. Therefore the study of the entrance boundary and excessive measure can be obtained with the help of the study of the exit boundary and excessive function. We are not going to discuss it in detail.

7.4 PROBABILITY REPRESENTATION OF A STOPPING POTENTIAL

Definition 1. We call a non-negative (including $+\infty$) function u the potential of the chain P if there exists a non-negative function v such that

$$u = Gv = \sum_{n=0}^{\infty} P^n v$$

that is,

$$u_i = \sum_j G(i, j) v_j \qquad i \in E \tag{1}$$

The non-zero potential u is said to be minimal if, when $u = u^1 + u^2$, where u^1 and u^2 are potentials, then $u = c^a u^a$ $(a = 1, 2,)$, where c^a is a constant.

If $u = Gv$, then

$$Pu = PGv = (G - I)v = u - v$$

Hence if $u_i < \infty$, then $v_i = u_i - (Pu)_i$ is uniquely determined by u.

Definition 2. We call u the stopping potential of P if $u = Gv$, and furthermore $v_i = 0$ $(i \in E - H)$, where H is the stopping state set.

Theorem 1. u is a stopping potential of P if and only if there exists a non-negative function f_a $(a \in H)$ such that

$$u_i = E_i\{f_{x(\beta)}, \beta < \infty\} = \sum_{a\in H} f_a u_i^a \qquad i \in E \tag{2}$$

where

$$u_i^a = P_i\{x_\beta = a\} = G(i, a)d_a \qquad a \in H \tag{3}$$

is a minimal potential whereas d_a is the stopping quantity of the state a.

If u is finite then there exists a one-to-to correspondence between u and f, and moreover f_a $(a \in H)$ is also finite

Proof. Sufficiency: By (3.77),

$$\begin{aligned} u_i &= E_i\{f_{x(\beta)}, \beta < \infty\} \\ &= E_i\{f_{x(\beta)}, \beta < \infty, x(\beta) \in H\} \\ &= \sum_{n=0}^{\infty} E_i\{f_{x(n)}, \beta = n, x(n) \in H\} \\ &= \sum_{n=0}^{\infty} \sum_{a\in H} p_{ia}^n d_a f_a \\ &= \sum_{a\in H} G(i, a) d_a f_a \end{aligned} \tag{4}$$

Hence the u determined by (2) is a stopping potential. In particular, (3) is valid.

Now we are going to prove that u^a is a minimal potential. Suppose

$$u^a = u_1 + u_2 \qquad u_b = Gv_b \qquad (b = 1, 2)$$

are potentials. Then

$$d_a \delta_a(i) = (u^a - pu^a)_i = \sum_{b=1}^{2} (u_b - pu_b)_i = v_1(i) + v_2(i)$$

Thereby we obtain $v_b = c^b(d_a\delta_a)$ $(b = 1, 2, c^b \geq 0$ are constants); consequently $u_b = c^b u^a$ $(b = 1, 2)$.

Necessity: Suppose $u = Gv$ is a stopping potential. Take $f_a = v_a/d_a$ $(a \in H)$. By (4) we know that (2) holds.

The one-to-one correspondence follows from the paragraph preceding Definition 2. The proof is terminated. QED

7.5 SOJOURN SOLUTION, ULTIMATE SET, ALMOST CLOSED SET AND BOUNDARY

Theorem 1. The (non-negative) bounded harmonic function h corresponds one-to-one to the non-negative bounded Borel function f on B according to the following relation:

$$h_i = \int_B K(i, \xi) f(\xi) \mu(d\xi) \tag{1}$$

$$\lim_{n\to\infty} h(x_n) = f(x_\infty) \qquad E_i f(x_\infty) = h_i \tag{2}$$

where the measure μ is the distribution of the ultimate state, determined by (3.34).

Proof. According to Theorem 3.18, the function h defined by (1) is bounded and harmonic. Conversely, suppose h is a bounded harmonic function. We shall assume its bound to be 1. Then both h and $g = 1 - h$ are excessive. By Theorem 3.18, $\mu_h(E_0) = 0$, and moreover

$$1 = \int_{E_0 \cup B} K(i, \xi) u(d\xi) = \int_{E_0 \cup B} K(i, \xi)(\mu_h + \mu_g)(d\xi)$$

Therefore $\mu = \mu_h + \mu_g$, and μ_h has a density function f bounded by 1 relative to $\mu = \mu_h(d\xi) = f(\xi)\mu(d\xi)$. Hence (3.83) becomes (1).

Rewrite (1) as $h_i = E_i\{f(x_\beta), \Omega_\infty\}$. Consequently the formula (2) follows from Theorem 3.22. The proof is completed. QED

Theorem 2. Write $\mathscr{B}_{x(\infty)}$ for the Borel field on Ω_∞ produced by sets $\{x(\infty) \in \Gamma\}$ (Borel sets $\Gamma \subset B$), then $\mathscr{B}_\infty = \mathscr{B}_{x(\infty)}$.

Proof. Since the non-negative bounded $\mathscr{B}_{x(\infty)}$-measurable function Φ can be written in the form $f(x_\infty)$, where f is a non-negative bounded Borel function on B, it follows that by (2) we know $\Phi = f(x_\infty)$ is $\mathscr{B}_\infty$-measurable. That is, $\mathscr{B}_{x(\infty)} \subset \mathscr{B}_\infty$.

Suppose that Φ is a non-negative bounded $\mathscr{B}_\infty$-measurable function. According to Theorem 3.22, $h_i = E_i\Phi$ is bounded harmonic and $\Phi = \lim_{n\to\infty} h_{x(n)}$.

By Theorem 1, there exists a bounded Borel-measurable function f on B such that (2) holds. As a result, $\Phi = f(x_\infty)$. Consequently Φ is $\mathscr{B}_{x(\infty)}$-measurable, that is, $\mathscr{B}_\infty \subset \mathscr{B}_{x(\infty)}$, and the proof is completed. QED

Theorem 3. For any almost closed set A, there exists a Borel set $\Gamma \subset B$ such that

$$\mathscr{L}(A) \simeq \{x_\infty \in \Gamma\} \tag{3}$$

Apart from μ-null sets, Γ is uniquely defined.

Proof. As $\mathscr{L}(A) \in \mathscr{B}_\infty$, making use of Theorem 2 we can prove the theorem. QED

Definition 1. Call non-zero $u \in \mathscr{M}^+(1)$ a sojourn solution, if for any $v \in \mathscr{M}^+(1)$: $v \leqslant u, v \leqslant \bar{u} - u$ we have $v = 0$. Here

$$\bar{u}_i = P_i(\Omega_\infty) = P_i(\mathscr{L}(E)) \in \mathscr{M}^+(1) \tag{4}$$

Theorem 4. There exists a one-to-one correspondence between sojourn solutions u and the closed sets A (modulo transient sets) under the following condition:

$$u_i = P_i\{\mathscr{L}(A)\} \tag{5}$$

Proof. Suppose that u is a sojourn solution so that u is harmonic; according to Theorem 3.22, there exists a non-zero ultimate random variable $\Phi: 0 \leqslant \Phi \leqslant 1$ such that $u_i = E_i\Phi$. Evidently

$$\psi = \min\{\Phi, (1-\Phi)I_{(\Phi<1)}\}$$

is an ultimate random variable. Hence $v_i = E_i\psi \in \mathscr{M}^+(1)$, $v \leqslant u, v \leqslant \bar{u} - u$. So $v = 0$, and therefore $\psi = 0$. From this we obtain $\Phi = I_{(\Phi=1)}$. Because $(\Phi = 1)$ is a non-negative ultimate set, according to Theorem 3.21, there exists an almost closed set A such that $(\Phi = 1) \simeq \mathscr{L}(A)$. Thus

$$u_i = E_i\Phi = E_i I_{(\Phi=1)} = E_i I_{\mathscr{L}(A)} = P_i(\mathscr{L}(A))$$

Now suppose that u is defined by (5) and assume $v \in \mathscr{M}^+(1), v \leqslant u, v \leqslant \bar{u} - u$. According to Theorems 3.21 and 3.22,

$$I_{\mathscr{L}(E-A)} = \lim_{n\to\infty} u_{x(n)} \qquad I_{\mathscr{L}(E-A)} = \lim_{n\to\infty} (\bar{u}_{x(n)} - u_{x(n)})$$

and there exists an ultimate random variable $\Phi: 0 \leqslant \Phi \leqslant 1$, such that

$$v_i = E_i\Phi \qquad \Phi = \lim_{n\to\infty} v_{x(n)}$$

Since $v \leqslant \min(u, \bar{u} - u)$, it follows that $\Phi \leqslant \min(I_{\mathscr{L}(A)}, I_{\mathscr{L}(E-A)}) = 0$.
Accordingly $v_i = E_i\Phi = 0$, and the proof is concluded. QED

Theorem 5. The sojourn solution u, the non-null ultimate set Λ, the almost closed set A (modulo transient sets) and the μ-non-null Borel set Γ ($\subset B$ modulo μ-null sets) are one-to-one correspondent under the following condition:

$$u_i = P_i(\mathscr{L}(A)) \qquad \Lambda \simeq \mathscr{L}(A) \simeq \{x_\infty \in \Gamma\} \tag{6}$$

Proof. It suffices to summarize Theorems 3 and 4 and Theorem 3.21. QED

Definition 2. We say a sojourn solution u is minimal if u as a harmonic function is minimal; a sojourn solution u is said to be completely non-minimal if for any sojourn solution $v < u$, v is not minimal.

Evidently, if the completely non-atomic A_0 appears in the Blackwell decomposition, then $u_i^0 = P(\mathscr{L}(A_0))$ is the greatest completely non-minimal sojourn solution.

As a special case of Theorem 5, we have the following theorem.

Theorem 6. The completely non-minimal sojourn solution $u(\leqslant u^0)$ and the completely non-atomic ultimate set $\Lambda(\subset \Lambda_0$, see (3.87)), the completely non-atomic almost closed set A ($\subset A_0$, Modulo transient sets) and the μ-non-null Borel set $\Gamma(\subset B_2$, modulo μ-null sets) are one-to-one correspondent according to (6). And, the minimal sojourn solution u, the atomic ultimate set Λ, the atomic almost closed set A (modulo transient sets) and the atomic boundary point $\xi \in B_1$ are one-to-one correspondent according to the following relation:

$$\Lambda \simeq \mathscr{L}(A) \simeq \{x_\infty = \xi\} \qquad u_i = P_i\{\mathscr{L}(A)\} \tag{7}$$

Theorem 7. The bounded harmonic function u, the non-negative bounded ultimate random variable Φ and the non-negative bounded Borel function f (defined on B, apart from the difference of the function values on μ-null sets) are one-to-one correspondent according to the following relation:

$$u_i = E_i\Phi \qquad \Phi \simeq \lim_{n\to\infty} u_{x(n)} \qquad f(x_\infty) \simeq \Phi \tag{8}$$

Proof. Summing up Theorem 3.22 and Theorem 1, the conclusions of this theorem follow. QED

7.6 CANONICAL PROCESS

Suppose that E has a discrete topology. Compactifying the denumerable set E by one point '∞' ($\infty \notin E$), we get $\bar{E} = E \cup \{\infty\}$. Assume σ to be a non-negative (including ∞) random variable defined on the complete probability space

$(\Omega, \mathscr{F}, P)$, and call $X(\omega)=\{x_t(\omega), t<\sigma(\omega)\}$[1] $(\omega\in\Omega)$ a homogeneous Markov process, or simply process, if for any $\omega\in\Omega$ and $t<\sigma(\omega)$ we have $x(t,\omega)\in\bar{E}$, but

$$P\{x(t)=\infty\}=0 \qquad t\geqslant 0 \tag{1}$$

While for any $l\geqslant 2, 0\leqslant t_1<t_2<\cdots<t_{l+1}, i_1, i_2, \ldots, i_{l+1}\in E$, provided that $P\{x(t_a)=i_a, 1\leqslant a\leqslant l\}>0$, we have

$$P\{x(t_{\mu+1})=i_{l+1}|x(t_a)=i_a, 1\leqslant a\leqslant l\}=P\{x(t_{l+1})=i_{l+1}|x(t_l)=i_l\} \tag{2}$$

And moreover the value of the right-hand side of the formula above is independent of t_1 and only depends on $t_{l+1}-t_l$. We call

$$p_{ij}(t)=P\{x(s+t)=j|x(s)=i\} \tag{3}$$

the transition probability of the process, whose transition probability matrix $P(t)=\{p_{ij}(t)\}$ must satisfy conditions (2.2.A) and (2.2.B) and is supposed to satisfy (2.2.C) (see section 2.2). According to the notation of section 2.1, that is, $P(t)\in\mathscr{P}$. If two processes X and $\bar{X}$ have the same transition probability matrix $P(t)$, we say that X and $\bar{X}$ are the same process.

Conversely, for every $P(t)\in\mathscr{P}$, there exists a process $X=\{x(t), t<\sigma\}$ with $P(t)$ as its transition probability matrix. According to Zi-kun Wang and Xiang-qun Yang (1988) we can also suppose X is well separable, Borel-measurable, and right-lower semicontinuous. That is, for every $\omega\in\Omega$.

$$\varliminf_{s\downarrow t} x(s,\omega)=x(t,\omega)\in\bar{E} \qquad \text{for all } t<\sigma(\omega) \tag{4}$$

Such a process is said to be a canonical one. If for process X, (4) is valid for almost all $\omega\in\Omega$, and for exceptional $\omega, x(\omega)$ may even have no definition. In this case for the exceptional ω, we can revise or add to the definition as follows:

$$x(t,\omega)=i_0\in E \qquad t<\sigma(\omega) \tag{5}$$

Denote the class composed of all canonical processes X by $\mathscr{H}$. Evidently, $\mathscr{H}$ is one-to-one corresponding to $\mathscr{P}$. According to the notation in section 2.1, the class composed of processes $X\in\mathscr{H}$ corresponding to $P(t)\in\mathscr{P}_s$ is denoted by $\mathscr{H}_s$. The class composed of processes $X\in\mathscr{H}$ corresponding to $P(t)\in\mathscr{F}_s(Q)$ is denoted by $\mathscr{H}_s(Q)$ and X in $\mathscr{H}_s(Q)$ is known as a Q process.

When $P_i(\sigma=\infty)=1\,(i\in E)$, the process X is said to be honest. If and only if condition (2.2.D) is valid, X is honest.

For a stopping process $X=\{x(t), t<\sigma\}$, we can select $\Delta\notin\bar{E}$ arbitrarily, and then let

$$\tilde{x}(t)=\begin{cases} x(t) & \text{if } t<\sigma \\ \Delta & \text{if } t\geqslant\sigma \end{cases} \tag{6}$$

[1] Take $x_t(\omega)\equiv x(t,\omega)$ and we frequently omit ω.

Then $\tilde{X}=\{\tilde{x}(t), t<\infty\}$ is an honest process, whose transition probability matrix $\tilde{P}(t)=\{\tilde{p}_{ij}(t)\}\,(i,j\in E\cup\{\Delta\})$ is well defined by (2.2.3).

Let $X\in\mathscr{H}$. Write $\mathscr{F}_t^0$ for the Borel field generated by the sets $\{x(s)=i\}\,(s\leqslant t, i\in E)$. Write

$$\mathscr{F}_{t+0}^0=\bigcap_{s>t}\mathscr{F}_s^0$$

Theorem 1.

$$\mathscr{F}_{t+0}^0=\mathscr{F}_t^0$$

The proof is seen in Chung (1967, II.8, Theorem 1).

We call non-negative random variable β a Markov time for the process X, if $\beta\leqslant\sigma$, and for any $t\geqslant 0$,

$$(\beta\leqslant t<\sigma)\in\mathscr{F}_t^0 \tag{7}$$

or equivalently,

$$(\beta<t<\sigma)\in\mathscr{F}_t^0 \tag{8}$$

For a Markov time β, let $\mathscr{F}_\beta$ be the class of sets Λ that satisfy $\Lambda\subset\Omega_\beta\equiv(\beta<\sigma)$ and $\Lambda\cap(\beta\leqslant t<\sigma)\in\mathscr{F}_t^0, t\geqslant 0$; $\mathscr{F}_\beta$ is called the pre-β field. Let $\mathscr{F}'_\beta$ be the Borel field on Ω_β generated by the sets $\{x(\beta+t)=i\}\,(i\in E, t\geqslant 0)$, it is called the post-$\beta$ field.

For the Markov time β of X, we can define a set translation operator θ_β from $\mathscr{F}_\infty^0$ to $\mathscr{F}_\infty^0$ such that the operations union, intersection and complement are invariant under action of the operator θ_β; for instance, $\theta_\beta(A\cup B)=(\theta_\beta A)\cup(\theta_\beta B)$ and so on. Moreover

$$\theta_\beta\{x(t)=i\}=\{x(\beta+t)=i\} \tag{9}$$

Hence for a $\mathscr{F}_\infty^0$-measurable function ξ, we can define a function $\theta_\beta\xi$: $(\theta_\beta\xi)(\omega)=a$ if $\omega\in\theta_\beta(\xi=a)$, especially

$$\theta_\beta x(t)=x(\beta+t) \qquad (t\geqslant 0) \tag{10}$$

The details are found in Dynkin (1963, pp. 121–44).

Theorem 2. $X\in\mathscr{H}$ has the strong Markov property, that is, it possesses the following property. For any Markov time β, provided $P(\Omega_\beta)>0$,

(i) suppose $M\in\mathscr{F}_\infty^0, \Lambda\in\mathscr{F}_\beta$, then

$$P\{\theta_\beta M|\Lambda, x(\beta)=i\}=P_i\{M\} \qquad i\in E \tag{11}$$

$$P\{x(\beta+t)=\infty\}=0 \qquad t>0 \tag{12}$$

(ii) $\{X(\beta+t), 0<t<\sigma-\beta\}$ is an open Markov chain on $(\Omega_\beta, \Omega_\beta\mathscr{F}, P(\cdot|\Omega_\beta))$ whose transition probability is $(p_{ij}(t))$.

(iii) Let $\Delta' = \{x(\beta) \neq \infty\}, \sigma = \sigma - \beta, x'(t) = x(\beta + t)$. Then $X'(\omega) = \{x'(t, \omega), t < \sigma'(\omega)\}(\omega \in \Delta')$ is a canonical process defined on the probability space $(\Delta', \Delta'\mathscr{F}, P(\cdot|\Delta'))$; the state space E' of X' is included in E, and its transition probability matrix is the restriction to $E' \times E'$ of the transition probability matrix of X.

We call X' the post-β process.

The proof is seen in Zi-kun Wang and Xiang-qun Yang (1988, section 3.3) or Chung (1967, II.9).

Theorem 3. Suppose that $X \in \mathscr{H}$ and that β is a Markov time of X, and assume that α is a Markov time of the post-β process X', then $\beta + \alpha$ is a Markov time of X.

The proof is encountered in Chung (1967, II.15, Theorem 1).

7.7 PROBABILISTIC Q PROCESS

The probabilistic Q process $X \in \mathscr{H}_s(Q)$ and the analytical Q process $P(t) \in \mathscr{P}_s(Q)$ are identical. We say that a Q process X satisfies the Kolmogorov equations: certainly by saying so we mean that $P(t) \in \mathscr{P}_s(Q)$ corresponding to X satisfies the system of Kolomogorov equations, and so on.

For a Q process $X \in \mathscr{H}_s(Q)$, the right-lower semicontinuity property (6.4) becomes right-continuity in $\bar{E}$:

$$\lim_{s \downarrow t} x(s, \omega) = x(t, \omega) \in \bar{E} \qquad \text{for all } t < \sigma(\omega) \tag{1}$$

If Q process $X \in \mathscr{H}_s(Q)$ satisfies the stronger condition:

$$\lim_{s \downarrow t} x(s, \omega) = x(t, \omega) \in E \qquad \text{for all } t < \sigma(\omega) \tag{2}$$

then we call X the D-type process. The class composed of the D-type processes is written as $\mathscr{H}_D$. When Q is fixed, the class composed of the D-type Q processes X is written as $\mathscr{H}_D(Q)$.

Definition 1. Suppose $X \in \mathscr{H}$, $q_i < \infty$. We call $[a, b)$ an i-interval of $X(\omega)$, if $x(t, \omega) = i$ for all $t \in [a, b)$, but $x(t, \omega) \not\equiv i$ in any interval $[c, d)$ containing $[a, b)$ as a proper subset. When i is used in general an i-interval is called a constancy interval.

Theorem 1. Suppose $X \in \mathscr{H}, q_i < \infty$. Then for almost all $\omega \in \Omega, X(\omega)$ has only finitely many i-intervals in any finite interval.

The proof is seen in Zi-kun Wang and Xiang-qun Yang (1988, section 3.1, Theorem 2) or Chung (1967, II.5, Theorem 7).

Let $X \in \mathscr{H}_s(Q)$. We call

$$\tau_1 = \begin{cases} \inf\{t | 0 < t < \sigma, x(t) \neq x(0)\} \\ \sigma \qquad \text{if the set above is empty} \end{cases} \tag{3}$$

the first discontinuity of X.

Theorem 2. Let $X \in \mathscr{H}_s(Q)$. Then

$$P_i\{\tau_1 > t\} = e^{-q_i t} \tag{4}$$

when $q_i > 0$,

$$P_i\{x(\tau_1) = j\} = \Pi_{ij} \qquad j \in E \tag{5}$$

$$P_i\{x(\tau_1) = \infty\} = P_i\{\tau_1 < \sigma\} - \sum_j \Pi_{ij} \tag{6}$$

where $\Pi = (\Pi_{ij})$ is determined by (2.9.7).

For the proof see Zi-kun Wang and Xiang-qun Yang (1988, section 2.2, Theorems 5 and 6).

Theorem 3. Suppose that $X \in \mathscr{H}, 0 < q_i < \infty, \beta$ is a Markov time of X, $P\{x(\beta) \neq i\} = 0, \alpha$ is the first discontinuity after β, and let $\rho = \alpha - \beta$. Then for $\Lambda \in \mathscr{F}_\beta, M \in \mathscr{F}'_\alpha$,

$$P\{\Lambda, \rho > t, M\} = P\{\Lambda\} e^{-q_i t} P\{M | \Omega_\beta\}$$

In particular, $\mathscr{F}_\alpha$ and $\mathscr{F}'_\alpha$ are conditionally independent with regard to Ω_β. More particularly, $x(\tau_1)$ and τ_1 are conditionally independent with respect to the measure P_i.

The proof is to be seen in Zi-kun Wang and Xiang-qun Yang (1988, section 3.2, corollary 2 to Theorem 1 or Chung (1967, II.15, Theorem 2).

Suppose $X \in \mathscr{H}_s(Q)$. We call

$$\eta_i \simeq \begin{cases} \inf\{t | \tau_1 < t < \sigma, x(t) = i\} \\ \sigma \qquad \text{if the set above is empty} \end{cases} \tag{7}$$

the first time of returning to i after first discontinuity. If $P_i\{\eta_i^* < \sigma\} = 1$, we say that i is recurrent. If i is recurrent, and moreover

$$m_{ii} = E_i \eta_i^* < \infty \tag{8}$$

then we say i is ergodic. If all i are recurrent or ergodic, then we say process X is recurrent or ergodic.

Theorem 4.

(i) i is recurrent iff for almost all $\omega \in \{x(0)=i\}$, $X(\omega)$ has infinitely many i-intervals, or equivalently

$$\int_0^\infty p_{ii}(t)\,dt = \infty$$

(ii) Suppose i is recurrent, then $P_i\{\sigma = \infty\} = 1$, and i is ergodic iff

$$\lim_{t\to\infty} p_{ii}(t) = \pi_i > 0 \tag{9}$$

If i is ergodic, C is the irreducible recurrent class including i, then

$$\lim_{t\to\infty} p_{ji}(t) = \frac{1}{q_i m_{ii}} \qquad j \in C \tag{10}$$

The proof is found in Zi-kun Wang and Xiang-qun Yang (1988, section 4.2, Theorem 1) or Chung (1967, II. 10, Theorem 4 and its corollary in II. 12 formula (9)).

7.8 PROBABILISTIC MINIMAL PROCESS

Definition 1. Suppose $X \in \mathscr{H}_s$. We call $t \in (0, \sigma(\omega)]$ a jumping point of $X(\omega)$, if $t = \sigma(\omega) < \infty$ and there exist $i \in E$ and $\varepsilon > 0$ such that $x(u,\omega) = i$ for $u \in (t-\varepsilon, t)$ or $t < \sigma(\omega)$ and there exist different $i, j \in E$ and $\varepsilon > 0$ such that we have $x(u,\omega) = i$ for $u \in (t-\varepsilon, t)$, and $x(u,\omega) = j$ for $u \in (t, t+\varepsilon)$. We call $t \in (0, \sigma(\omega)]$ a leaping point of $X(\omega)$, if $t = \sigma(\omega) < \infty$, or $t < \sigma(\omega)$ and for any $\varepsilon > 0$, $X(\omega)$ has infinitely many jumping points in $(t-\varepsilon, t+\varepsilon)$. We agree that $t = 0$ is a jumping point and leaping point; it is called the zeroth jumping point and the zeroth leaping point.

Every Q process $X \in \mathscr{H}_s(Q)$ has first leaping point τ:

$$\tau = \begin{cases} \inf\{t \mid 0 < t < \sigma(\omega), \lim\limits_{s\to t} x(s,\omega) = \infty\} \\ \sigma(\omega) \qquad \text{if the set above is empty} \end{cases} \tag{1}$$

The first discontinuity τ_1 of X may not necessarily be a jumping point. But when Q is conservative, τ_1 is a jumping point. If $\tau_1\,(<\sigma)$ is not a jumping point, according to Theorem 7.2, then τ_1 is the first leaping point.

Theorem 1. Suppose $X \in \mathscr{H}_s(Q)$, $q_i > 0$. Then

$$P_i\{\tau_1 = \tau\} = d_i/q_i \tag{2}$$

where d is the non-conservative quantity of Q, determined by (2.2.6).

Proof. By (7.6)

$$\begin{aligned} P_i\{\tau_1 = \tau\} &= P_i\{x(\tau_1) = \infty\} + P_i\{\tau_1 = \sigma\} \\ &= 1 - \sum_j \Pi_{ij} = \frac{d_i}{q_i} \end{aligned}$$

QED

Theorem 2. Suppose $X \in \mathscr{H}_s(Q)$, the set Λ and the non-negative random variable ξ are all $\mathscr{F}^0_\infty$-measurable.

(i) If

$$P\{\Lambda = \theta_{\tau_1}\Lambda\} = P_i\{\xi = \tau_1 + \theta_{\tau_1}\xi\} = 1 \tag{3}$$

then $u_j = E_j\{\xi, \Lambda\}$ satisfies the equation

$$\sum_j q_{ij} u_j = -P_i\{\Lambda\} \tag{4}$$

and $u_j(\lambda) = E_j\{e^{-\lambda\xi}, \Lambda\}\,(\lambda > 0)$ satisfies the equation

$$\lambda u_i - \sum_j q_{ij} u_j = 0 \tag{5}$$

(ii) If

$$P_i\{\tau_1 = \tau, \Lambda\} = P_i\{\tau_1 = \tau\} \tag{6}$$

$$P_i\{\tau_1 < \tau, \theta_{\tau_1}\Lambda\} = P_i\{\tau_1 < \tau, \Lambda\} \tag{7}$$

then $u_j = P_j\{\Lambda\}$ satisfies the equation

$$\sum_j q_{ij} u_j = d_i \tag{8}$$

Proof. It is trivial when $q_i = 0$. Suppose $q_i > 0$.

By the strong Markov property, Theorem 7.2 and the independence between τ_1 and $x(\tau_1)$,

$$\begin{aligned} E_i\{\xi, \Lambda\} &= E_i\{\tau_1 + \theta_{\tau_1}\xi, \theta_{\tau_1}\Lambda\} \\ &= \sum_j \Pi_{ij} E_i\{\tau_1 + \theta_{\tau_1}\xi, \theta_{\tau_1}\Lambda \mid x(\tau_1) = j\} \\ &= \sum_j \Pi_{ij}[E_i\{\tau_1, \theta_{\tau_1}\Lambda \mid x(\tau_1) = j\} + E_i\{\theta_{\tau_1}\xi, \theta_{\tau_1}\Lambda \mid x(\tau_1) = j\}] \\ &= \sum_j \Pi_{ij}(E_i\{\tau_1\}P_j\{\Lambda\} + E_j\{\xi, \Lambda\}) \\ &= \frac{1}{q_i} P_i\{\Lambda\} + \sum_j \Pi_{ij} E_j\{\xi, \Lambda\} \end{aligned}$$

So we get (4). Secondly

$$\begin{aligned} E_i\{e^{-\lambda\xi},\Lambda\} &= \sum_j \Pi_{ij}E_i\{e^{-\lambda\tau_1}\theta_{\tau_1}e^{-\lambda\xi},\theta_{\tau_1}\Lambda|x(\tau_1)=j\} \\ &= \sum_j \Pi_{ij}E_i\{e^{-\lambda\tau_1}|x(\tau_1)=j\}E_i\{\theta_{\tau_1}e^{-\lambda\xi},\theta_{\tau_1}\Lambda|x(\tau_1)=j\} \\ &= \sum_j \Pi_{ij}E_i\{e^{-\lambda\tau_1}\}E_j\{e^{-\lambda\xi},\Lambda\} \\ &= \frac{q_i}{\lambda+q_i}\sum_j \Pi_{ij}E_j\{e^{-\lambda\xi},\Lambda\} \end{aligned}$$

(5) follows. Thirdly,

$$\begin{aligned} P_i\{\Lambda\} &= P_i\{\Lambda,\tau=\tau_1\} + P_i\{\theta_{\tau_1}\Lambda,\tau_1<\tau\} \\ &= P_i\{\tau=\tau_1\} + \sum_j \Pi_{ij}P_j\{\Lambda\} \end{aligned}$$

Noting (2), we obtain (8). The proof is completed. QED

Theorem 3. Suppose $X\in\mathscr{H}_s(Q)$. X satisfies the backward equations iff for almost all $\omega\in\Omega$, $X(\omega)$ the first discontinuity is a jumping point. X satisfies the forward equations iff for arbitrarily given $t>0$, for almost all $\omega\in(t<\sigma)$, $X(\omega)$ has a last discontinuity in $[0,t]$ and moreover, it is a jumping point.

For the proof see Zi-kun Wang and Xiang-qun Yang (1988, section 2.3, Theorems 1 and 2).

Theorem 4. Suppose $X\in\mathscr{H}_s(Q)$, τ is the first leaping point, then $\bar{X}=\{x(t),t<\tau\}\in\mathscr{H}_s(Q)$, whose transition probability matrix is the minimal solution $f(t)=\{f_{ij}(t)\}$ in section 2.9.

The proof is in Zi-kun Wang and Xiang-qun Yang (1988, section 2.3, Theorem 5).

Definition 2. We call $X=\{x(t),t<\sigma\}\in\mathscr{H}_s(Q)$ the minimal Q process, if $P\{\tau=\sigma\}=1$, where τ is the first leaping point.

Suppose $X\in\mathscr{H}_s(Q)$, τ_1 is the first discontinuity, $\tau_1\leqslant\tau$. If $\tau_1=\infty$, then $\tau=\infty$, and we put $\tau_n=\infty\,(n>1)$. If $\tau_1<\infty$, then either $\tau_1=\tau<\infty$ and in this case $\tau_n(n>2)$ is undefined, or $\tau_1<\tau$ and in this case τ_1 is a jumping point. Hence after τ_1 there exists a first discontinuity $\tau_2\leqslant\tau$. If $\tau_2=\infty$, then $\tau=\infty$, and we put $\tau_n=\infty\,(n>2)$. If $\tau_2<\infty$, then either $\tau_2=\tau<\infty$ and in this case $\tau_n(n>2)$ is undefined, or $\tau_2<\tau$, in which case τ_2 is a jumping point. Therefore after τ_2 there exists a first discontinuity $\tau_3\leqslant\tau$. And it continues like this.

Write $\tau_0=0,\Omega=\Omega_F\cup\Omega_\infty$

$$\begin{aligned} \Omega_F &= \bigcup_{n=1}^{\infty}(\tau_n=\tau<\infty) \\ \Omega_\infty &= \bigcap_{n=1}^{\infty}(\tau_n<\tau)+\bigcup_{n=1}^{\infty}(\tau_n=\tau=\infty) \end{aligned} \tag{9}$$

$$\beta=\begin{cases} +\infty & \text{if } \omega\in\Omega_\infty \\ \sup\{n: n\geqslant 0,\tau_n<\tau\} & \text{if } \omega\in\Omega_F \end{cases} \tag{10}$$

$$\tau_\beta=\lim_{n\to\beta}\tau_n \tag{11}$$

Then on Ω_F we have $\tau_\beta<\tau$, on Ω_∞ we have $\tau_\beta=\tau$.

When $\tau_n=\infty$, we set $x(\tau_n)=x(\tau_K)$, where $K=\max\{m:\tau_m<\infty\}$. Thus, when $n\leqslant\beta$, that is $\tau_n\leqslant\tau_\beta$, $x(\tau_n)$ is well defined.

By the strong Markov property and Theorem 7.2 we obtain the following.

Theorem 5. $X_T=\{X(\tau_n),n\leqslant\beta\}$ or write $X_T=\{X(\tau_n),\tau_n\leqslant\tau_\beta\}$ is a Markov chain. Its one-step transition probability matrix $\Pi=(\Pi_{ij})$ is defined by (2.9.7).

We call the Markov chain X_T the embedded chain of the process $\{X(t),t<\tau\}$, and call the matrix Π the embedding matrix of the matrix Q.

7.9 RESOLVENT PROCESS AND INDUCED PROCESS

Suppose $X=\{x(t),t<\sigma\}\in\mathscr{H}$, and ρ is a random variable independent of X, its distribution being the exponential distribution with parameter 1:

$$P\{\rho>t\}=e^{-t} \qquad t\geqslant 0 \tag{1}$$

For $\lambda>0$ let

$$\begin{aligned} \rho^\lambda &= \rho/\lambda & \sigma^\lambda &= \min(\sigma,\rho^\lambda) \\ x^\lambda(t) &= x(t) & t &< \sigma^\lambda \end{aligned} \tag{2}$$

Theorem 1. The process $X^\lambda=\{x^\lambda(t),t<\sigma^\lambda\}\in\mathscr{H}$. Has transition probabilities

$$p_{ij}^\lambda(t)=e^{-\lambda t}p_{ij}(t) \tag{3}$$

where $p_{ij}(t)$ are the transition probabilities of X.

Proof. Since the path of X^λ is the front section of the path of X, we need only prove the homogeneous Markov property of X^λ.

Suppose $0\leqslant t_1<t_2<\cdots<t_{n+1},i_1,i_2,\ldots,i_{n+1}\in E$. By the homogeneous

Markov property of X and the independence between ρ and X.

$$P\{x^\lambda(t_a)=i_a,\ 1\leqslant a\leqslant n+1\}=P\{x(t_a)=i_a,\ 1\leqslant a\leqslant n+1,\ t_{n+1}<\rho^\lambda\}$$
$$=P\{x(t_1)=i_1\}\prod_{a=1}^{n}P_{i_ai_{a+1}}(t_{a+1}-t_a)e^{-\lambda t_{n+1}}$$
$$=P\{x(t_1)=i_1\}P\{t_1<\rho^\lambda\}\prod_{a=1}^{n}p^\lambda_{i_ai_{a+1}}(t_{a+1}-t_a)$$
$$=P\{x^\lambda(t_1)=i_1\}\prod_{a=1}^{n}p^\lambda_{i_ai_{a+1}}(t_{a+1}-t_a) \qquad \text{QED}$$

We call X^λ the resolvent process of X.

Theorem 2. Suppose that $\bar{X}=\{x(t),t<\tau\}$ is the minimal Q process, Π is the embedding matrix. Then the resolvent process X^λ is the minimal Q^λ process, where

$$q^\lambda_{ij}=q_{ij}(i\neq j) \qquad q^\lambda_i=\lambda+q_i \tag{4}$$

The embedding matrix $\Pi(\lambda)$ is determined by (2.9.8).

Proof. It suffices to show that Q^λ has the same form as (4), the rest being obvious. Suppose that τ^λ_1 is the first discontinuity of X^λ, then $\tau^\lambda_1=\min(\tau_1,\rho^\lambda)$, so

$$\exp(-q^\lambda_it)=P_i\{\tau^\lambda_1>t\}=P_i\{\tau_1>t,\rho^\lambda>t\}$$
$$=P_i\{\tau_1>t\}P_i\{\rho^\lambda>t\}=e^{-q_it}e^{-\lambda t}=\exp[-(\lambda+q_i)t]$$

so that $q^\lambda_i=\lambda+q_i$. Secondly, for $j\neq i$,

$$\frac{q^\lambda_{ij}}{q^\lambda_i}=P_i\{x^\lambda(\tau^\lambda_1)=j\}=P_i\{x(\tau_1)=j,\ \tau_1<\rho^\lambda\}$$
$$=P_i\{x(\tau_1)=j\}P\{\tau_1<\rho^\lambda\}=\frac{q_i}{\lambda+q_i}\Pi_{ij}$$

hence $q^\lambda_{ij}=q_{ij}$ and the proof is completed. QED

Theorem 3. Suppose $\bar{X}=\{x(t),t<\tau\}$ is the minimal Q process, Λ is the ultimate set of the embedding chain X_T, $P(\Lambda)>0$. Then the process $X^\lambda(\omega)=\{x(t,\omega),t<\tau(\omega)\}$ $(\omega\in\Lambda)$ confined to Λ is the minimal Q^λ process on the probability space $(\Lambda,\Lambda\mathscr{F},P\{\cdot|\Lambda\})$, and its state space is

$$E^\Lambda=\{i|P_i(\Lambda)>0\} \tag{5}$$

Its transition probability is

$$P^\Lambda_{ij}(t)=\frac{f_{ij}(t)P_j(\Lambda)}{P_i(\Lambda)} \tag{6}$$

The Q matrix $Q^\Lambda=(q^\Lambda_{ij})$ $(i,j\in E^\Lambda)$ is conservative, and furthermore

$$q^\Lambda_{ij}=\frac{q_{ij}D_j(\Lambda)}{P_j(\Lambda)} \tag{7}$$

Proof. First notice: for any $t\geqslant 0$, we have $\Lambda=\theta_t\Lambda$ on $(t<\tau)$ where θ_t is the translation operator. Actually, because Λ is a ultimate set, there exists an infinite-dimensional invariant function g such that

$$1_\Lambda=g(x(\tau_n),x(\tau_{n+1}),\ldots) \qquad \text{for all } n \tag{8}$$

So by Dynkin (1963, p. 122),

$$1_{\theta_t\Lambda}=g(\theta_tx(\tau_n),\theta_tx(\tau_{n+1}),\ldots)$$
$$=g(x(\theta_t\tau_n),x(\theta_t\tau_{n+1}),\ldots) \tag{9}$$

On $(t<\tau)$ there must exist l such that $\tau_1\leqslant t<\tau_{l+1}$; therefore $\theta_t\tau_m=\tau_{m+l}$. Accordingly by (8) and (9),

$$1_{\theta_t\Lambda}=g(x(\tau_{n+l}),x(\tau_{n+l+1}),\ldots)=1_\Lambda$$

that is, we have $\Lambda=\theta_j\Lambda$ on $(t<\tau)$.

Suppose $0\leqslant t_1<t_2<\cdots<t_{n+1},i_1,i_2,\ldots,i_{n+1}\in E^\Lambda$. By the homogeneous Markov property of X

$$P\{x^\Lambda(t_{n+1})=i_{n+1}|\Lambda,x^\Lambda(t_a)=i_a,\ 1\leqslant a\leqslant n\}$$
$$=P\{x(t_{n+1})=i_{n+1}|\Lambda,x(t_a)=i_a,\ 1\leqslant a\leqslant n\}$$
$$=\frac{P\{x(t_a)=i_a,\ 1\leqslant a\leqslant n+1,\Lambda\}}{P\{x(t_a)=i_a,\ 1\leqslant a\leqslant n,\Lambda\}}$$
$$=\frac{P\{x(t_a)=i_a,\ 1\leqslant a\leqslant n+1,\theta_{t_{n+1}}\Lambda\}}{P\{x(t_a)=i_a,\ 1\leqslant a\leqslant n,\theta_{t_n}\Lambda\}}$$
$$=\frac{P\{x(t_a)=i_a,\ 1\leqslant a\leqslant n\}f_{i_ni_{n+1}}(t_{n+1}-t_n)P_{i_{n+1}}(\Lambda)}{P\{x(t_a)=i_a,\ 1\leqslant a\leqslant n\}P_{i_n}(\Lambda)}$$
$$=P^\Lambda_{i_ni_{n+1}}(t_{n+1}-t_n)$$

It remains to be proved that Q^λ is conservative, and this follows from

$$P_i(\Lambda)=P_i(\theta_{\tau_1}\Lambda)=\sum_j\Pi_{ij}P_j(\Lambda)$$

The proof is over. QED

Corollary

Suppose that X is the minimal Q process, $P(\Omega_\infty)>0$. Then X^{Ω_∞} is the minimal Q^{Ω_∞} process on $(\Omega_\infty, \Omega_\infty\mathscr{F}, P(\cdot|\Omega_\infty))$ and moreover is conservative.

Definition 1. Call process X^Λ the induced process of the minimal Q process on the ultimate set Λ.

7.10 PROBABILITY REPRESENTATION OF THE $\Pi(\lambda)$ POTENTIAL

Suppose that $X=\{x(t), t<\tau\}$ is the minimal Q process, H the set of non-conservative states of Q, d the non-conservative quantity of Q, and $\Pi(\lambda)$ the embedding chain of the resolvent process X^λ of X. For convenience, we write $f_i \equiv f(i)$.

Theorem 1. For the chain $\Pi(\lambda), u(\lambda)$ is the potential of a non-negative function v equal to zero on $E-H$ iff $u(\lambda)$ has the representation:

$$u_i(\lambda)=E_i\{e^{-\lambda\tau}f(x_{\tau-0}), \Omega_F\}=[\phi(\lambda)df]_i \tag{1}$$

where $f(a)\geqslant 0$ $(a\in H)$.

Proof. Sufficiency: By (1)

$$u_i(\lambda)=\sum_{n=1}^{\infty}E_i\{e^{-\lambda\tau_n}f(x_{\tau_n-0}), \tau_n=\tau\}=\sum_{n=1}^{\infty}T_i^n$$

where

$$T_i^1=E_i\{e^{-\lambda\tau_1}f(x_{\tau_1-0}), \tau_1=\tau\}=f_i\frac{d_i}{q_i}\frac{q_i}{\lambda+q_i}=\frac{1}{\lambda+q_i}d_if_i$$

$$T_i^n=E_i\{e^{-\lambda\tau_n}f(x_{\tau_n-0}), \tau_n=\tau\}$$

$$=\sum_j \Pi_{ij}\frac{q_i}{\lambda+q_i}T_j^{n-1}=\sum_j \Pi_{ij}(\lambda)T_j^{n-1}$$

$$T^n=\Pi(\lambda)T^{n-1}=\cdots=\Pi^{n-1}(\lambda)(\lambda+q)^{-1}df$$

$$u(\lambda)=\sum_{n=1}^{\infty}\Pi^{n-1}(\lambda)(\lambda+q)^{-1}df=\phi(\lambda)df$$

So $u(\lambda)$ is the potential of the function $v_j=d_jf_j/(\lambda+q_j)$ while v is zero on $E-H$.

Necessity: Take

$$f_a=\frac{(\lambda+q_a)v_a}{d_a} \qquad (a\in H)$$

By the proof of the sufficiency, we know $u(\lambda)$ has the representation (1), and the proof is terminated. QED

Theorem 2. Suppose

$$G(\lambda)=\sum_{n=0}^{\infty}\Pi^n(\lambda)=\phi(\lambda)(\lambda+q)$$

and column vector $v\geqslant 0$. Then

$$[\Pi^n(\lambda)v]_i=E_i\{e^{-\lambda\tau_n}v(x_{\tau_n})\} \tag{2}$$

$$[G(\lambda)v]_i=\sum_{n=0}^{\infty}E_i\{e^{-\lambda\tau_n}v(x_{\tau_n})\} \tag{3}$$

Proof. Following Theorem 1 we can prove (2), and hence (3) follows. QED

Corollary

$$\Pi_{ij}^n(\lambda)=E_i\{e^{-\lambda\tau_n}, x(\tau_n)=j\}.$$

7.11 λ-IMAGE AND STANDARD IMAGE

Recall that a harmonic function is non-negative. We write the class of finite Π- (or $\Pi(\lambda)$-) harmonic functions as $\hat{\mathscr{M}}^+$ (or $\hat{\mathscr{M}}_\lambda^+$), denote the class of bounded Π- (or $\Pi(\lambda)$-) harmonic functions by $\mathscr{M}^+$ (or $\mathscr{M}_\lambda^+$), and denote the class of Π- (or $\Pi(\lambda)$-) harmonic functions having the upper bound K by $\mathscr{M}^+(K)$ (or $\mathscr{M}_\lambda^+(K)$).

Suppose $u\in\hat{\mathscr{M}}^+$. Then $\Pi(\lambda)u\leqslant\Pi u=u$, $\Pi^{n+1}(\lambda)u\leqslant\Pi^n(\lambda)u\leqslant u$, so $\Pi^n(\lambda)u\downarrow u(\lambda)\leqslant u$, and moreover $u(\lambda)\in\hat{\mathscr{M}}_\lambda^+$. If $u\in\mathscr{M}^+(K)$, then $u(\lambda)\in\mathscr{M}_\lambda^+(K)$.

Definition 1. Suppose $u\in\hat{\mathscr{M}}^+$, λ is fixed. We call $u(\lambda)=\lim_{n\to\infty}\Pi^n(\lambda)u\in\hat{\mathscr{M}}_\lambda^+$ the λ-image of u.

Theorem 1. $u\in\hat{\mathscr{M}}^+$ and its λ-image $u(\lambda)\in\hat{\mathscr{M}}_\lambda^+$ have the following relation:

$$\lambda\phi(\lambda)u=u-u(\lambda) \tag{1}$$

Proof. Obviously

$$\sum_{a=0}\Pi^a(\lambda)=\sum_{a=0}\Pi^a(\lambda)(\lambda I+q)^{-1}q\Pi+I \tag{2}$$

Consequently if we let

$$\phi^n(\lambda)=\sum_{a=0}^{n-1}\Pi_a(\lambda)(\lambda I+q)^{-1}$$

then (2) becomes

$$\phi^{n+1}(\lambda)(\lambda+q)=\phi^n(\lambda)q\Pi+I$$

$$\lambda\phi^{n+1}(\lambda)+\phi^{n+1}(\lambda)q=\phi^n(\lambda)q\Pi+I \tag{3}$$

Multiplying from the right by u and noticing that $\Pi u=u$, we obtain

$$\lambda\phi^{n+1}(\lambda)u+\phi^{n+1}(\lambda)q\Pi u=\phi^n(\lambda)q\Pi u+u$$

$$\lambda\phi^{n+1}(\lambda)u+\Pi^n(\lambda)(\lambda I+q)^{-1}q\Pi u=u \tag{4}$$

$$\lambda\phi^{n+1}(\lambda)u+\Pi^{n+1}(\lambda)u=u$$

Setting $n\to\infty$ we obtain (1). The proof is concluded. QED

Suppose that $u(\lambda)$ is a $\Pi(\lambda)$-harminoc function but not necessarily finite. Then $\Pi u(\lambda)\geqslant\Pi(\lambda)u(\lambda)=u(\lambda)$, $\Pi^{n+1}u(\lambda)\geqslant\Pi^n u(\lambda)$. Therefore $\Pi^n u(\lambda)\uparrow u\geqslant u(\lambda)$, and moreover $\Pi u=u$; that is, u is a $\Pi(\lambda)$-harmonic function, but not necessarily finite.

Definition 2. Let $u(\lambda)$ be a $\Pi(\lambda)$-harmonic function. We call the Π-harmonic function $u=\lim_{n\to\infty}\Pi^n u(\lambda)$ the standard image of $u(\lambda)$.

Theorem 2. Assume that $u(\lambda)\in\hat{\mathscr{M}}_\lambda^+$. Then its standard image is[1]

$$u=u(\lambda)+\lambda\Gamma u(\lambda) \tag{5}$$

where

$$\Gamma=\lim_{\lambda\downarrow 0}\phi(\lambda)=\sum_{n=0}\Pi^n\frac{1}{q}$$

Proof. Rewrite $\Pi(\lambda)u(\lambda)=u(\lambda)$ as

$$u(\lambda)+\lambda q^{-1}u(\lambda)=\Pi u(\lambda)$$

Multiplying from the left by Π^n and summing over $0\leqslant n\leqslant a-1$, we obtain

$$u(\lambda)+\lambda\sum_{n=0}^{a-1}\Pi^n q^{-1}u(\lambda)=\Pi^a u(\lambda)$$

Letting $n\to\infty$, (5) follows, and the proof is completed. QED

Remark

From (5) it can be seen that the standard image of the harmonic exit family $u(\lambda)$ $(\lambda>0)$ agrees with Definition 2.10.2. But the λ in Definition 2 may be a fixed non-negative number.

[1] When $q_j=0$, set $1/q_j=\infty$. But from $\Pi(\lambda)u(\lambda)=u(\lambda)$ we know $u_j(\lambda)=0$, and q_j; hence $(1/q_j)u_j(\lambda)=\infty$. $0=0$.

Theorem 3. Suppose $u(\lambda)\in\hat{\mathscr{M}}_\lambda^+$. The standard image of $u(\lambda)$ is $u\in\hat{\mathscr{M}}^+$. Then (1) is true.

Proof. Equation (3) is already known to be true. Multiplying (3) from the right by $u=u(\lambda)+\lambda\Gamma u(\lambda)\in\hat{\mathscr{M}}^+$ we have

$$\lambda\phi^{n+1}(\lambda)u+\phi^{n+1}(\lambda)q\Pi u=\phi^n(\lambda)q\Pi u+u$$

$$\lambda\phi^{n+1}(\lambda)u+\sum_{a=0}^{n}\Pi^a(\lambda)(\lambda+q)^{-1}q\Pi[u(\lambda)+\lambda\Gamma u(\lambda)]$$

$$=\sum_{a=0}^{n-1}\Pi^a(\lambda)(\lambda+q)^{-1}q\Pi[u(\lambda)+\lambda\Gamma u(\lambda)]+u$$

$$\lambda\phi^{n+1}(\lambda)u+\Pi^{n+1}(\lambda)[u(\lambda)+\lambda\Gamma u(\lambda)]=u$$

Noticing that $u(\lambda)\in\hat{\mathscr{M}}_\lambda^+$, that is,

$$\lambda\phi^{n+1}(\lambda)u+u(\lambda)+\Pi^{n+1}(\lambda)\lambda\Gamma u(\lambda)=u \tag{6}$$

However as $u\in\hat{\mathscr{M}}_\lambda^+$,

$$\Pi^{n+1}(\lambda)\lambda\Gamma u(\lambda)\leqslant\Pi^{n+1}\lambda\Gamma u(\lambda)=\Pi^{n+1}[u-u(\lambda)]=u-\Pi^{n+1}u(\lambda)\downarrow u-u=0$$

Letting $n\to\infty$ in (6), (1) follows. The proof is terminated. QED

Corollary

Suppose $u(\lambda)\in\hat{\mathscr{M}}_\lambda^+$, and its standard image is $u\in\hat{\mathscr{M}}^+$. Then

$$\phi(\lambda)u=\Gamma u(\lambda) \tag{7}$$

Proof. The assertion follows from Theorems 2 and 3. QED

Let

$$\bar{\mathscr{M}}_\lambda^+=\{u(\lambda)\in\hat{\mathscr{M}}_\lambda^+\mid\text{the standard image of } u(\lambda) \text{ is } u\in\hat{\mathscr{M}}^+\} \tag{8}$$

$$\bar{\mathscr{M}}^+=\{u\in\bar{\mathscr{M}}^+\mid u \text{ is the standard image of some } u(\lambda)\in\bar{\mathscr{M}}_\lambda^+\}. \tag{9}$$

Theorem 4. Between $\bar{\mathscr{M}}_\lambda^+$ and $\bar{\mathscr{M}}^+$ we have established a one-to-one correspondence between the standard image and the λ-image. Furthermore, the standard map and the λ-map are mutually inverse maps.

Proof. Clearly, the standard map maps $\bar{\mathscr{M}}_\lambda^+$ onto $\bar{\mathscr{M}}^+$ by Theorem 3. The standard map is one-to-one. Now we suppose the standard image of $u(\lambda)\in\bar{\mathscr{M}}_\lambda^+$ is $u\in\bar{\mathscr{M}}^+$, and the λ-image of u is $v(\lambda)$. By Theorems 1 and 3, $v(\lambda)=u(\lambda)$, which means that $u(\lambda)$ is the λ-image of u. The proof is finished. QED

Theorem 5. Under the standard map or the λ-map, there exists a one-to-one correspondence between the $\Gamma(\lambda)$-minimal $u(\lambda)\in\bar{\mathscr{M}}_\lambda^+$ and the Π-minimal $u\in\bar{\mathscr{M}}$.

Proof. According to the definition of minimality, and by using Theorem 4, the assertion follows. QED

7.12 BOUNDARY OF MINIMAL Q PROCESS

Suppose that $X=\{x(t), t<\tau\}$ is the minimal Q process and that $X_T=\{x(\tau_n), n\leqslant\beta\}$ is its embedding chain with its embedding matrix denoted by Π. According to section 7.3, for the chain Π and X_T we can introduce the concepts of its Martin boundary ∂E, essential Martin boundary B, atomic boundary B_1, non-atomic boundary B_2, ultimate Borel field $\mathscr{B}_\infty$ and so on. We still use the notation employed in sections 7.3 to 7.5. But in the present case, Ω_F and Ω_∞ in (2.17) should be understood according to (8.9) and (8.10); H should be the set of non-conservative states of Q; (3.11) becomes

$$x(\tau-0)=\lim_{\tau_n\uparrow\tau} x(\tau_n)\in H\cup B \tag{1}$$

or more precisely, $x(\tau-0)\in H$ on Ω_F, $x(\tau-0)\in B$ on Ω_∞; whereas the measure μ in (3.34) becomes

$$\mu(\Gamma)=P_\gamma\{x(\tau-0)\in\Gamma\}\qquad \Gamma\subset H\cup B \tag{2}$$

Obviously, $K(\cdot,\xi)\in\hat{\mathscr{M}}^+$ when $\xi\in B$. Suppose the λ-image of $K(\cdot,\xi)$ is $K_\lambda(\cdot,\xi)$. By Theorem 11.1 we have

$$\lambda\phi(\lambda)K(\cdot,\xi)=K(\cdot,\xi)-K_\lambda(\cdot,\xi)\qquad \xi\in B \tag{3}$$

According to the resolvent equation of $\phi(\lambda)$, and from the formula above, we obtain

$$K_\lambda(\cdot,\xi)-K_\nu(\cdot,\xi)+(\lambda-\nu)\phi(\lambda)K_\nu(\cdot,\xi)=0\qquad (\xi\in B,\ \lambda,\nu>0) \tag{4}$$

That is, $K_\lambda(\cdot,\xi)$ is a harmonic exit family. Thus either for all $\lambda>0$, $K_\lambda(\cdot,\xi)=0$ or $\neq 0$. And so we can let

$$B_e=\{\xi\in B\,|\,K_\lambda(\cdot,\xi)\neq 0\} \tag{5}$$

$$B_p=\{\xi\in B\,|\,K_\lambda(\cdot,\xi)=0\} \tag{6}$$

Definition 1. We call B_e and B_p respectively the Martin exit boundary and the passive boundary of the minimal Q process X or of the matrix Q.

Theorem 1. B_e and B_p are Borel sets.

Proof. Since B is a Borel set and moreover $K(i,\xi)$ $(\xi\in B)$ is a Borel-measurable function for every $i\in E$, it follows that $K_\lambda(i,\xi)$ $(\xi\in B)$ is also a Borel-measurable function. Accordingly

$$B_e=\bigcup_{i\in E}\{\xi\,|\,K_\lambda(i,\xi)\neq 0\}$$

is a Borel set, and so is $B-B_e=B_p$. The proof is completed. QED

Theorem 2. Suppose $\xi\in B_e$, then $K(\cdot,\xi)\in\bar{\mathscr{M}}^+$, $K_\lambda(\cdot,\xi)\in\bar{\mathscr{M}}_\lambda^+$. Moreover the standard image of $K_\lambda(\cdot,\xi)$ is $K(\cdot,\xi)$, and the λ-image of $K(\cdot,\xi)$ is $K_\lambda(\cdot,\xi)$.

Proof. By (3), $K_\lambda(\cdot,\xi)\leqslant K(\cdot,\xi)$, $\Pi^n K_\lambda(\cdot,\xi)\leqslant\Pi^n K(\cdot,\xi)=K(\cdot,\xi)$. So the standard image of $K_\lambda(\cdot,\xi)$ is $u=\lim_{n\to\infty}\Pi^n K_\lambda(\cdot,\xi)\leqslant K(\cdot,\xi)$, that is $K_\lambda(\cdot,\xi)\in\bar{\mathscr{M}}_\lambda^+$, $u\in\bar{\mathscr{M}}^+$. According to Theorem 11.4, $u=K(\cdot,\xi)$. QED

Corollary

$\{u(\lambda)\in\bar{\mathscr{M}}_\lambda^+\,|\,u(\lambda)$ is $\Pi(\lambda)$-minimal, the standard image of $u(\lambda)$ is γ-integrable$\}=\{cK_\lambda(\cdot,\xi)\,|\,\xi\in B_e$, constant $c>0\}$.

Proof. If follows from Theorem 2 and Theorem 8.19. QED

For any Borel set $\Gamma\subset B$, let

$$X_i^\Gamma\equiv P_i\{x(\tau-0)\in\Gamma\}=\int_\Gamma K(i,\xi)\mu(\mathrm{d}\xi) \tag{7}$$

$$X_i^\Gamma(\lambda)\equiv E_i\{e^{-\lambda\tau}, x(\tau-0)\in\Gamma\} \tag{8}$$

Theorem 3. For any Borel set $\Gamma\subset B$,

$$\lambda\phi(\lambda)X^\Gamma=X^\Gamma-X^\Gamma(\lambda) \tag{9}$$

Proof. By the homogeneous Markov property of X and $\Lambda=\{x(\tau-0)\in\Gamma\}$ being the ultimate set,

$$\begin{aligned}X_i^\Gamma(\lambda)&=\int_0^\infty e^{-\lambda t}\,\mathrm{d}P_i\{\Lambda,\tau\leqslant t\}\\&=\int_0^\infty e^{-\lambda t}\frac{\mathrm{d}}{\mathrm{d}t}P_i\{\Lambda,\tau\leqslant t\}\,\mathrm{d}t\\&=\lambda\int_0^\infty e^{-\lambda t}P_i\{\Lambda,\tau\leqslant t\}\,\mathrm{d}t\\&=\lambda\int_0^\infty e^{-\lambda t}[P_i\{\Lambda\}-P_i\{\Lambda,\tau>t\}]\,\mathrm{d}t\end{aligned}$$

$$= X_i^{\Gamma} - \lambda \sum_j \int_0^{\infty} e^{-\lambda t} P_i\{t < \tau, x(t) = j, \theta_t \Lambda\} dt$$

$$= X_i^{\Gamma} - \lambda \sum_j \int_0^{\infty} e^{-\lambda t} f_{ij}(t) P_j\{\Lambda\} dt$$

$$= X_i^{\Gamma} - \lambda \sum_j \phi_{ij}(\lambda) X_j^{\Gamma}$$ QED

Theorem 4. For any Borel set $\Gamma \subset B$,

$$X_i^{\Gamma}(\lambda) = \int_{\Gamma} K_{\lambda}(i, \xi)\mu(d\xi) = \int_{\Gamma \cap B_e} K_{\lambda}(i, \xi)\mu(d\xi) = X_i^{\Gamma \cap B_e}(\lambda) \tag{10}$$

The standard image of $X^{\Gamma}(\lambda)$ is $X^{\Gamma \cap B_e} \in \bar{\mathcal{M}}^{+}$. And the λ-image of X^{Γ} is $X^{\Gamma}(\lambda)$.

Proof. By the definition of B_p, we have the second equality in (10). Since

$$\Pi^n(\lambda) X^{\Gamma} = \int_{\Gamma} \Pi^n(\lambda) K(\cdot, \xi)\mu(d\xi)$$

setting $n \to \infty$, it follows that the λ-image of X^{Γ} is $\int_{\Gamma} K_{\lambda}(\cdot, \xi)\mu(d\xi)$. By Theorem 11.1 and Theorem 3 we obtain the first equality in (10). Hence by (10),

$$\Pi^n X^{\Gamma}(\lambda) = \int_{\Gamma \cap B_e} \Pi^n K_{\lambda}(\cdot, \xi)\mu(d\xi)$$

Taking the limit, we know that the standard image of $X^{\Gamma}(\lambda)$ is

$$\int_{\Gamma \cap B_e} K(\cdot, \xi)\mu(d\xi) = X^{\Gamma \cap B_e}$$

and the proof is completed. QED

Theorem 5. For all $i \in E$,

$$P_i\{\tau < \infty \mid x(\tau - 0) \in B_e\} = 1 \tag{11}$$

$$P_i\{\tau = \infty \mid x(\tau - 0) \in B_p\} = 1 \tag{12}$$

Proof. By (10)

$$X_i^{B_p}(\lambda) = E_i\{e^{-\lambda\tau}, x(\tau - 0) \in B_p\} = E_i\{e^{-\lambda\tau}, \tau < \infty, x(\tau - 0) \in B_p\}$$

$$= \int_{B_p} K_{\lambda}(i, \xi)\mu(d\xi) = 0$$

From what precedes we get (12). Secondly

$$\Pi^n X_i^{\Gamma}(\lambda) = E_i\{E_{x(\tau_n)}[e^{-\lambda\tau}, \tau < \infty, x(\tau - 0) \in \Gamma]\}$$

$$= E_i\{\theta_{\tau_n}[e^{-\lambda\tau}, \tau < \infty, x(\tau - 0) \in \Gamma]\}$$

$$= E_i\{e^{-\lambda(\tau - \tau_n)}, \tau < \infty, x(\tau - 0) \in \Gamma\}$$

Taking the limit we know that the standard image of $X^{\Gamma}(\lambda)$ is $P_i\{\tau < \infty, x(\tau - 0) \in \Gamma\}$. Especially, the standard image of $X^{B_e}(\lambda)$ is $P_i\{\tau < \infty, x(\tau - 0) \in B_e\}$. But according to Theorem 4, the standard image of $X^{B_e}(\lambda)$ is X^{B_e}. And so $P_i\{\tau < \infty, x(\tau - 0) \in B_e\} = P_i\{x(\tau - 0) \in B_e\}$. From this we get (11), and the proof is terminated. QED

Theorem 6. Suppose f is a non-negative Borel function on B_e. Then the standard image of

$$u_i(\lambda) = E_i\{e^{-\lambda\tau} f[x(\tau - 0)], \Omega_{\infty}\} = \int_{B_e} K_{\lambda}(i, \xi) f(\xi)\mu(d\xi) \tag{13}$$

is

$$u_i = E_i\{f[x(\tau - 0)], x(\tau - 0) \in B_e\} = \int_{B_e} K(i, \xi) f(\xi)\mu(d\xi) \tag{14}$$

Conversely, if $u_i < \infty$ $(i \in E)$ determined by the formula above or f is a non-negative Borel function on B, and furthermore

$$\bar{u}_i = E_i\{f[x(\tau - 0)], \Omega_{\infty}\} = \int_B K(i, \xi) f(\xi)\mu(d\xi) < \infty. \tag{15}$$

then the λ-image of u or $\bar{u}$ is $u(\lambda)$.

Proof. Similar to the proof of Theorem 4. QED

Theorem 7. $(\partial E)_1 = \{\xi_a, a \in \mathscr{A}\} \subset B_p$.

Proof. Fix a $j \in E_a$. Suppose the time that the minimal Q process X stays at j after reaching j for the nth time is ρ_j^n. Then $\rho_j^n (n \geqslant 1)$ are mutually independent with respect to P_j, $\tau \geqslant \sum_{n=1}^{\infty} \rho_j^n = \infty$. That is $P_i\{\tau = \infty\} = 1$. Hence for any i, $P_i\{\tau = \infty \mid x(\tau - 0) = \xi_a\} = 1$. According to Theorem 5, $\xi_a \in B_p$. QED

Theorem 8. For any $i \in E$,

$$P_i\{\tau < \infty \mid x(\tau - 0) \in H\} = 1 \tag{16}$$

Proof. Note that $q_j > 0$ when $j \in H$, that is, j is non-absorbing. If $x(\tau - 0) = j$ and $\tau = \infty$, then there exists t_0 such that for all $t \geqslant t_0$ we have $x(t, \omega) = j$. Since j is non-absorbing, except for the ω sets whose probability is 0, this is impossible. The proof is completed. QED

Combining Theorems 5 and 8, we have P_γ almost surely QED

$$x(\tau-0)\in H\cup B_e \qquad \text{if } \tau<\infty \tag{17}$$

$$x(\tau-0)\in B_p \qquad \text{if } \tau=\infty \tag{18}$$

7.13 PROBABILITY REPRESENTATION OF $\mathscr{M}_\lambda^+$

Theorem 1. Suppose that f is a non-negative, bounded Borel function on $B, u(\lambda)$ is defined by (12.13). Then we have P_γ-almost surely

$$\lim_{n\to\infty} u_{x(\tau_n)}(\lambda)=\begin{cases} f[x(\tau-0)] & \text{when } x(\tau-0)\in B_e \\ 0 & \text{when } x(\tau-0)\in B_p \end{cases} \tag{1}$$

Proof. Because on Ω_∞,

$$\begin{aligned} u_{x(\tau_n)}(\lambda) &= E_i\{\theta_{\tau_n}(e^{-\lambda\tau}f[x(\tau-0)], x(\tau-0)\in B_e)|\mathscr{F}_{\tau_n}\} \\ &= E_i\{e^{-\lambda(\tau-\tau_n)}f[x(\tau-0)], x(\tau-0)\in B_e|\mathscr{F}_{\tau_n}\} \\ &= e^{-\lambda\tau_n}E_i\{e^{-\lambda\tau}f[x(\tau-0)], x(\tau-0)\in B_e|\mathscr{F}_{\tau_n}\} \end{aligned}$$

When $n\to\infty$, making use of the convergence theorem for martingales, we get

$$\lim_{n\to\infty} u_{x(\tau_n)}(\lambda)=e^{-\lambda\tau}e^{-\lambda\tau}f[x(\tau-0)]1_{(x(\tau-0)\in B)}$$

By this formula and using Theorem 12.5 we obtain (1), and the proof is concluded. QED

Theorem 2. Suppose that for a Borel set $\Gamma\subset B, X^\Gamma$ is defined by (12.7). Then we have P_γ-almost surely

$$\lim_{n\to\infty} X^\Gamma_{x(\tau_n)}=\begin{cases} 1 & \text{if } x(\tau-0)\in\Gamma \\ 0 & \text{if } x(\tau-0)\in B-\Gamma \end{cases} \tag{2}$$

Proof. This is a special case of Theorem 3.22. QED

Suppose that f is a bounded Borel function which is zero on B_p and is non-negative on B_e. The class composed of such functions f is written as $\mathscr{F}_e^+$. If $f_1, f_2\in\mathscr{F}_e^+$ and $\mu\{f_1\neq f_2\}=0$, then regard f_1 and f_2 as the same function.

Theorem 3. The element $u(\lambda)$ in $\mathscr{M}_\lambda^+$ has the following general form:

$$u_i(\lambda)=E_i\{e^{-\lambda\tau}f[x(\tau-0)]\}=\int_{B_e}K_\lambda(i,\xi)f(\xi)\mu(d\xi) \qquad f\in\mathscr{F}_e^+. \tag{3}$$

The element u of $\mathscr{M}^+\cap\bar{\mathscr{M}}^+$ has the following general form:

$$u_i=E_i\{f[x(\tau-0)]\}=\int_{B_e}K(i,\xi)f(\xi)\mu(d\xi) \qquad f\in\mathscr{F}_e^+ \tag{4}$$

$u(\lambda)\in\mathscr{M}_\lambda^+, u\in\mathscr{M}^+\cap\bar{\mathscr{M}}^+$, $f\in\mathscr{F}_e^+$ are one-to-one correspondent under the following conditions:

$$u_i(\lambda)=E_i\{e^{-\lambda\tau}f[x(\tau-0)]\}=E_i\left\{e^{-\lambda\tau}\lim_{n\to\infty}u_{x(\tau_n)}\right\} \tag{5}$$

$$u_i=E_if[x(\tau_\infty-0)]=E_i\left\{\lim_{n\to\infty}u_{x(\tau_n)}(\lambda)\right\} \tag{6}$$

$$f[x(\tau-0)]\simeq\lim_{n\to\infty}u_{x(\tau_n)}(\lambda)\simeq\lim_{n\to\infty}u_{x(\tau_n)} \tag{7}$$

Proof. By Theorem 12.6, if $f\in\mathscr{F}_e^+$, then for $u(\lambda)$ and u, determined by (5) and (6), $u(\lambda)\in\mathscr{M}_\lambda^+, u\in\mathscr{M}^+\cap\bar{\mathscr{M}}^+$ hold.

Suppose $u(\lambda)\in\mathscr{M}_\lambda^+$, whose standard image is $u\in\mathscr{M}^+\cap\bar{\mathscr{M}}^+$. According to Theorem 5.7, there exists a non-negative bounded Borel function g on B such that $u_i=E_ig[x(\tau-0)]$. By Theorem 12.6, the λ-image of u is $u_i(\lambda)=E_i\{e^{-\lambda\tau}g[x(\tau-0)]\}$. Taking $f=g1_{B_e}\in\mathscr{F}_e^+$ we know $u(\lambda)$ has the form (3).

By Theorem 1 we derive the first formula in (7). Again by (4) and Theorem 5.7 we obtain $f[x(\tau-0)]=\lim_{n\to\infty}u_{x(\tau_n)}$, hence we get the second formula in (7). The proof is terminated. QED

7.14 ATOMIC AND NON-ATOMIC EXIT BOUNDARY OF THE MINIMAL Q PROCESS

Definition 1. We call $B_{ea}=B_e\cap B_a\,(a=1,2)$ the atomic exit boundary and the non-atomic exit boundary of the minimal Q process (or matrix Q) respectively. Similarly we can define the atomic passive boundary and the non-atomic passive boundary.

Theorem 1. When $\xi\in B_{e1}, K_\lambda(\cdot,\xi)$ is bounded; when $\xi\in B_{e2}, K_\lambda(\cdot,\xi)$ is unbounded.

Proof. By the definition of standard image and λ-image and by Theorem 12.2, when $\xi\in B_e$, both $K(\cdot,\xi)$ and $K_\lambda(\cdot,\xi)$ are bounded, or unbounded.

Suppose $\xi\in B_{e1}$, then $\mu(\xi)>0$. By (3.40),

$$1\geqslant p_i(x(\tau-0)=\xi)=K(i,\xi)\mu(d\xi)$$
$$K(i,\xi)\leqslant 1/\mu(\xi)<\infty$$

That is, $K(\cdot,\xi)$ is bounded.

Suppose $\xi\in B_e, K(\cdot,\xi)$ is bounded. So $K(\cdot,\xi)$ is the standard image of $K_\lambda(\cdot,\xi)$

and moreover is a bounded Π-harmonic function, that is $K(\cdot,\xi)\in\mathcal{M}^+\cap\tilde{\mathcal{M}}^+$. According to Theorem 13.3, there exists a non-negative bounded Borel function f on B_e such that

$$K(\cdot,\xi)=\int_{B_e}K(\cdot,\zeta)f(\zeta)\mu(d\zeta)$$

By the uniqueness Theorem 3.18, surely $f(\zeta)\mu(d\zeta)=\delta_\xi d\zeta$, so that $f(\xi)\mu(\xi)=1$. But f is bounded, therefore $\mu(\xi)>0$, that is $\xi\in B_{e1}$. The proof is finished. QED

Definition 2. We call the elements of $\hat{\mathcal{M}}_\lambda^+$ the non-negative solutions of the equation

$$\lambda u-Qu=0 \qquad \lambda>0 \tag{1}$$

We call $u(\lambda)\in\hat{\mathcal{M}}_\lambda^+$ the minimal solution or the completely non-minimal solution of the equation (1), if $u(\lambda)$ as a $\Pi(\lambda)$-harmonic function is minimal or completely non-minimal.

Theorem 2. $B_{e1}=\phi$ iff there does not exist any bounded non-negative minimal solution to equation (1); $B_{e2}=\phi$ iff there does not exist any bounded non-negative completely non-minimal solution to equation (1), or equivalently, there does not exist a non-negative minimal solution $u(\lambda)$ to equation (1) such that $u(\lambda)$ is unbounded and the standard image of $u(\lambda)$ is γ-integrable.

Proof. It follows from Theorem 1, the corollary to Theorem 12.2 and Theorem 13.3. QED

7.15 EXITING ALMOST CLOSED SET AND THE BLACKWELL DECOMPOSITION OF THE MINIMAL Q PROCESS

Assume that X is the minimal Q process. According to Theorem 5.6, and in accordance with the Blackwell decomposition (2.2.19) of the embedding matrix Π, the potency of the index set $\mathcal{A}$ is equal to that of the atomic boundary B_1, so we can assume $\mathcal{A}=B_1$. Hence we have

$$\mathcal{L}(A_a)=\{x(\tau-0)=a\} \qquad a\in B_1 \tag{1}$$

$$\mathcal{L}(A_0)=\{x(\tau-0)\in B_2\} \tag{2}$$

Definition 1. Suppose that A is an almost closed set. We say that A is exit if

$$P\{\tau<\infty\,|\,\mathcal{L}(A)\}=1 \tag{3}$$

We say that A is passive if

$$P\{\tau=\infty\,|\,\mathcal{L}(A)\}=1 \tag{4}$$

Obviously, if A is an atomic almost closed set, and not exit, then it must be passive. But if A is a non-atomic almost closed set, the conclusion above is not true.

As for the completely non-atomic almost closed set A_0, according to Theorems 5.6 and 12.5, we can decompose it into $A_0=A_{0e}\cup A_{0p}$, where both A_{0e} and A_{0p} are almost closed sets, so that

$$\mathcal{L}(A_{0e})=\{x(\tau-0)\in B_{e2}\}=\{x(\tau-0)\in B_2,\tau<\infty\}$$
$$\mathcal{L}(A_{0p})=\{x(\tau-0)\in B_{p2}\}=\{x(\tau-0)\in B_2,\tau=\infty\}$$

Accordingly A_{0e} is exit while A_{0p} is passive. Thus we follow the Blackwell decomposition of the minimal Q process.

Theorem 1. Suppose that $X=\{x(t),t<\tau\}$ is the minimal Q process. Then its state space E has the following decomposition:

$$E=A_{0e}\cup\Big(\bigcup_{a\in B_{e_1}}A_a\Big)\cup A_{0p}\cup\Big(\bigcup_{a\in B_{p_1}}A_a\Big) \tag{5}$$

where A_{0e} and A_{0p} are completely non-atomic almost closed sets, exit and passive respectively, and therefore may be not present. $A_a(a\in B_{e1})$ are exiting atomic almost closed sets while $A_a(a\in B_{p1})$ are passive atomic almost closed sets. The decomposition is unique modulo transient sets.

7.16 THE CONDITION FOR FINITE EXIT

Theorem 1. Suppose that finitely or denumerable infinitely many exit almost closed sets A_a are mutually disjoint. Write

$$X_i^a(\lambda)=E_i\{e^{-\lambda\tau},\mathcal{L}(A_a)\} \tag{1}$$

If $\sum_a|c_a|<\infty,\sum_a c_aX^a(\lambda)=0$, then $c_a=0$.

Proof. By Theorem 5.6, there exist mutually disjoint Borel sets Γ_a such that $\mathcal{L}(A_a)=\{x(\tau-0)\in\Gamma_a\}$ and $\mu(\Gamma_a)>0$. According to Theorem 12.4, the standard image of $X^a(\lambda)$ is $X_i^a=P_i\{x(\tau-0)\in\Gamma_a\}$. Consequently by $\sum_a c_aX^a(\lambda)=0$ we get $\sum_a c_aX^a=0$. According to Theorem 3.22 $\sum_a c_a 1_{(x(\tau-0)\in\Gamma_a)}=0$, that is $\sum_a c_a 1_{\Gamma_a}=0$ (μ, a.s.). Thus $c_a=0$. The proof is concluded. QED

Theorem 2. Suppose $a,b\in B_{e1}$. Then

$$X_i^a(\lambda)=E_i\{e^{-\lambda\tau},x(\tau-0)=a\}\to\begin{cases}1 & \text{when } i\to a\\ 0 & \text{when } i\to b\neq a\end{cases} \tag{2}$$

Proof. According to Zi-kun Wang and Xiang-qun Yang (1988, section 0.2,

Theorem 4), for any $b \in B_{e1}$, the limit $\lim_{i \to b} X_i^a(\lambda)$ exists, and so we only need to calculate the limit, which follows from Theorem 13.2. QED

Theorem 3. Suppose $0 \leqslant n < \infty$. The following conditions are equivalent

(i) In the decomposition (15.5), A_{0e} vanishes, the potency of the set B_{e1} is n.

(ii) The dimension of $\mathscr{M}_\lambda^+$ is n.

(iii) B_e is composed of only n atomic boundary points.

Proof. It is obvious that (i) and (ii) are equivalent.

(i)$\Rightarrow$(ii). Suppose that $A_1, \ldots, A_n$ are exit atomic. By Theorem 1, $X^a(\lambda)(1 \leqslant a \leqslant n)$ are mutually independent. Secondly, by Theorem 13.3, for $u(\lambda) \in \mathscr{M}_\lambda^+$, there exists f such that

$$u_i(\lambda) = E_i\{e^{-\lambda\tau} f[x(\tau - 0)]\} = \sum_{a=1}^{n} E_i\{e^{-\lambda\tau} f[x(\tau - 0)], \mathscr{L}(A_a)\}$$

$$= \sum_{a=1}^{n} f(a) X_i^a(\lambda)$$

That is, the dimension of $\mathscr{M}_\lambda^+$ is n.

(ii)$\Rightarrow$(i). First we show $P_i\{\mathscr{L}(A_{0e})\} = 0 (i \in E)$. Otherwise, there exists i such that $P_i(\mathscr{L}(A_{0e})\} > 0$. Since A_{0e} is completely non-atomic, it follows that there exist infinitely many disjoint almost closed sets. Thereby by Theorem 1, to $\mathscr{M}_\lambda^+$ there are more than n linear independent solutions. This is in contradiction with (ii). So A_{0e} vanishes. By the process of proof of (i)$\Rightarrow$(ii) we know the number of exit atoms is identical to the dimension of $\mathscr{M}_\lambda^+$, and the proof is completed. QED

7.17 A CONDITIONAL INDEPENDENCE THEOREM

Suppose that $X = \{x(t), t < \sigma\} \in \mathscr{H}_s(Q)$ is the non-minimal Q process, and that α is a Markov time of $X, P\{x(\alpha) = \infty\} = 0, P\{\alpha < \sigma\} < 0$. We write the first leaping point after α as τ_α; on$(\alpha < \sigma), \theta_\alpha \tau = \tau_\alpha$; let $\tau_\alpha^n = \theta_\alpha \tau_n$. We write $\mathscr{F}_{\tau_\alpha^n}$ for the pre-τ_α^n field, and $\mathscr{F}_{\tau_\alpha^n}$ for the minimal Borel field on Ω_{τ_α} containing all $\mathscr{F}_{\tau_\alpha^n}(n \geqslant 1)$.

Theorem 1. Suppose $A \in \mathscr{F}_\infty^0, \Lambda \in \mathscr{F}_{\tau_\alpha^-}$. There exists a Borel function f which is defined on $H \cup B_e$ and is independent of the conditional distribution of $X(\alpha)$, relative to $(\alpha < \sigma)$, such that on $\{x(\tau_\alpha - 0) \in H \cup B_e\}, P_\gamma$-almost surely holds

$$P\{\theta_{\tau_\alpha} A \mid \Lambda, x(\tau - 0)\} = f\{x(\tau_\alpha - 0)\} \tag{1}$$

The condition on the left side of the above formula should be understood as $|_A, x(\tau_\alpha - 0)$.

In order to prove Theorem 1, we need to quote a theorem in Dynkin (1963, p. 782), which is stated as follows.

Theorem 2. Suppose that $(\Omega, \mathscr{F}, P)$ is a probability space, $E|\xi| < \infty, \Omega_n \in \mathscr{F}$, $\Omega_n \downarrow \Omega'(n \uparrow \infty), \mathscr{A}_n \subseteq \mathscr{F}, \mathscr{A}_n$ is a Borel field on Ω_n and moreover $\mathscr{A}_m \cap \Omega_n \subseteq \mathscr{A}_n$ $(m < n)$. Then on Ω' we have almost surely

$$\lim_{n \to \infty} E(\xi \mid \mathscr{A}_n) = E(\xi \mid \mathscr{A})$$

where $\mathscr{A}$ is the minimal Borel field on Ω including all $\mathscr{A}_n$.

Proof of Theorem 1. First we prove (1) is almost surely true on $M \equiv \{X(\tau_\alpha - 0) \in H\}$. For this, let

$$\beta = \begin{cases} \tau_\alpha^{n-1} & \text{if } \omega \in M, \tau_\alpha = \tau_\alpha^n, \tau_\alpha^{n-1} < \tau_\alpha^n \\ \infty & \text{if } \omega \notin M \end{cases}$$

Note that when $\Lambda \in \mathscr{F}_{\tau_\alpha^-}$, $\Lambda(X(\beta) = i) \in \mathscr{F}_\beta$. What is more, $\{X(\tau_\alpha - 0) = i\} = \{X(\beta) = i\}$. According to Theorem 6.2(i), when $X(\tau_\alpha - 0) = i \in H$, (1) is true, and furthermore the restriction of f on H, $f(a), a \in H$, is independent of the conditional distribution of $X(\alpha)$, relative to $(\alpha < \sigma)$. So (1) is almost everywhere true on M.

Secondly, we verify that (1) is almost surely true on $N \equiv \{X(\tau_\alpha - 0) \in B_e\}$. That is, we must prove that

$$1_N P\{\theta_{\tau_\alpha} A \mid \Lambda, X(\tau_\alpha - 0\} = 1_N f\{X(\tau_\alpha - 0)\} \tag{2}$$

According to the definition of conditional probability, on N we have almost surely

$$P\{\theta_{\tau_\alpha} A, M \mid \Lambda, X(\tau_\alpha - 0)\} = 0$$

that is

$$1_N P\{\theta_{\tau_\alpha} A, M \mid \Lambda, X(\tau_\alpha - 0)\} = 0$$

so that the left side of (2) is equal to

$$1_N P\{\theta_{\tau_\alpha} A, M \mid \Lambda, X(\tau_\alpha - 0)\} \tag{3}$$

Now let $\Omega_n = (\tau_\alpha^n < \tau_\alpha), \mathscr{A}_n = \mathscr{F}_{\tau_\alpha^n}, \xi = 1_{N \cap \theta_{\tau_\alpha} A}$. Then the conditions in Theorem 2 are satisfied. Obviously, $N \subset \Omega' = \bigcap_n \Omega_n$; on Ω_n we have $\xi = \theta_{\tau_\alpha^n} \eta$, where $\eta = 1_{(\theta_\tau A) \cap (X(\tau - 0) \in B_e)}$. And it is easy to see

$$u(i) \equiv E_i \eta \in \mathscr{M}^+$$

Thus by the strong Markov property and Theorem 2, on Ω' we have almost

surely

$$E(\xi|\mathscr{A}) = \lim_{n\to\infty} E(\xi|\mathscr{A}_n) = \lim_{n\to\infty} E_{X(\tau_\alpha^n)}\eta$$

$$= \lim_{n\to\infty} u\{X(\tau_\alpha^n)\}$$

Thereby we have

$$1_N E(\xi|\mathscr{A}) = \lim_{n\to\infty} 1_N u\{X(\tau_\alpha^n)\}$$

We can easily see that $\phi = \lim_{n\to\infty} 1_N u\{X(\tau_\alpha^n)\}$ is a non-negative ultimate random variable; so there exists a non-negative Borel-measurable function f on B such that $\phi = f\{x(\tau_\alpha - 0)\} = 1_N f\{x(\tau_\alpha - 0)\}$.

Consequently

$$1_N E(\xi|\mathscr{A}) = 1_N f\{x(\tau_\alpha - 0)\}$$

Because $\Lambda \in \mathscr{F}_{\tau_\alpha^-} \subset \mathscr{A}$, $x(\tau_\alpha - 0)$ is $\mathscr{A}$-measurable, and hence the left side of (2), that is, expression (3), equals

$$\begin{aligned}
1_N E\{\xi|\Lambda, x(\tau_\alpha - 0)\} &= 1_N E\{E(\xi|\mathscr{A})|\Lambda, x(\tau_\alpha - 0)\} \\
&= E\{1_N E(\xi|\mathscr{A})|\Lambda, x(\tau_\alpha - 0)\} \\
&= E\{1_N f[x(\tau_\alpha - 0)]|\Lambda, x(\tau_\alpha - 0)\} \\
&= 1_N f\{x(\tau_\alpha - 0)\}
\end{aligned}$$

Thus we have proved (2).

We are going to prove that the restriction of f to B_e, $f(a), a \in B_e$, is independent of the conditional distribution of $x(\alpha)$ with respect to $(\alpha < \sigma)$, but we should think of the function and a function which is equal to f μ-almost surely as the same function, μ being the ultimate state distribution. For this, it suffices to apply the following fact: Suppose (x_n) is a Markov chain, and $u(i) (i \in E)$ is a real-valued function. If for any initial distribution, with probability 1 limit, the $\lim_{n\to\infty} u(x_n)$, exists, then this limit is independent of the initial distribution. In the foregoing fact, taking the Markov chain $x_n = x(\tau_\alpha^n)$ defined on the probability space $((\alpha < \sigma), (\alpha < \sigma)\mathscr{F}, P(\cdot|\alpha < \sigma))$ we get what we intend to prove.

The proof of the above fact is as follows. Suppose the distribution $v = v(i)$ satisfies $v_i > 0$, for all i. By $P_\gamma(u(x_n) \to A(v)) = 1$ we derive $P_i(u(x_n) \to A(v)) = 1$ for all i. On the other hand under the supposition that $P_i(u(x_n) \to A(\delta_i)) = 1, \delta_i$ represents the unit distribution concentrated at i. So $P_i(A(v) = A(\delta_i)) = 1$. Now suppose v' is any distribution. When $v'_i > 0$, repeating the argument above we obtain

$$P_i(A(v) = A(\delta_i) = A(v')) = 1 \qquad P_i(A(v) = A(v')) = 1$$

Thus we have $P_{v'}(A(v) = A(v')) = 1$. By symmetry, $P_v(A(v) = A(v')) = 1$. If $\tilde{v}$ is another distribution, from the above $P_v(A(v) = A(\tilde{v})) = 1$. It follows that

$$P_v(A(v') = A(\tilde{v})) = 1 \qquad P_i(A(v') = A(\tilde{v})) = 1 \qquad \text{(for all } i)$$
$$P_\gamma(A(v') = A(\tilde{v})) = 1$$

According to Theorem 5.7, there exist non-negative Borel functions $f_{v'}$ and $f_{\tilde{v}}$, such that $A(v') = f_{v'}(x_\infty)$, $A(\tilde{v}) = f_{\tilde{v}}(x_\infty)$. Therefore $P_\gamma(f_{v'}(x_\infty) = f_{\tilde{v}}(x_\infty)) = 1$. That is, $f_{v'} = f_{\tilde{v}}$ is true μ-almost surely. The proof is over. QED

Corollary

Suppose $\Lambda \in \mathscr{F}_{\tau^-}$, $A \in \mathscr{F}_\infty^0$, then under $\{\tau < \infty, x(\tau - 0)\}$, Λ and $\theta_\tau A$ are conditionally independent. That is, on $\{x(\tau - 0) \in H \cup B_e\}$, almost surely holds

$$P\{\Lambda\theta_\tau A|x(\tau - 0)\} = P\{\Lambda|x(\tau - 0)\}P\{\theta_\tau A|x(\tau - 0)\} \tag{4}$$

7.18 FURTHER DESCRIPTION OF GENERAL Q PROCESSES

Suppose that $X = \{x(t), t < \sigma\} \in \mathscr{H}_s(Q)$ is a non-minimal Q process and its resolvent operator is $\psi(\lambda)$.

Theorem 1. $\psi(\lambda)$ has the following representation:

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \sum_{a \in H \cup B_{e1}} X_i^a(\lambda) F_j^a(\lambda) + \int_{B_{e2}} X_i(\lambda, d_a) F_j(\lambda, a) \tag{1}$$

where

$$X_i^a(\lambda) = E_i\{e^{-\lambda\tau}, x(\tau - 0) = a\} = \phi_{ia}(\lambda) d_a \qquad a \in H \tag{2}$$

$$X_i^a(\lambda) = E_i\{e^{-\lambda\tau}, x(\tau - 0) = a\} \qquad a \in B_{e1} \tag{3}$$

$$X_i(\lambda, \Gamma) = E_i\{e^{-\lambda\tau}, x(\tau - 0) \in \Gamma\} \qquad \Gamma \subset B_{e2} \tag{4}$$

and

$$F^a(\lambda) \geqslant 0 \qquad \lambda[F^a(\lambda), 1] \leqslant 1 \qquad a \in H \cup B_{e1} \tag{5}$$

$$F(\lambda, a) \geqslant 0 \qquad \lambda[F(\lambda, a), 1] \leqslant 1 \qquad \text{for } \mu\text{-almost all } a \in B_{e2} \tag{6}$$

Proof.

$$\begin{aligned}
\psi_{ij}(\lambda) &= E_i \int_0^\sigma e^{-\lambda t} 1_j[x(t)]\, dt \\
&= E_i \int_0^\tau e^{-\lambda t} 1_j[x(t)]\, dt + E_i \int_\tau^{\sigma - \tau} e^{-\lambda t} 1_j[x(t)]\, dt \\
&= \phi_{ij}(\lambda) + E_i e^{-\lambda\tau} \int_\tau^{\sigma - \tau} e^{-\lambda(t - \tau)} 1_j[x(t)]\, dt
\end{aligned}$$

By the corollary to Theorem 17.1, the second term of the above formula is equal to

$$E_i\left\{e^{-\lambda\tau}1_{(\tau<\infty)}\theta_\tau\int_0^\sigma e^{-\lambda\tau}1_j[x(t)]\,dt\right\}$$

$$=\int_{H\cup B_e}E_i\{e^{-\lambda\tau},x(\tau-0)\in d_a\}E_i\left\{\theta_\tau\int_0^\sigma e^{-\lambda t}1_j[x(t)]\,dt\,|\,x(\tau-0)=a\right\}$$

$$=\int_{H\cup B_e}X_i(\lambda,d_a)F_j(\lambda,a)$$

Parts (1), (5) and (6) follow from this. As for the second equality in (2), it is derived from Theorem 10.1, and the proof is concluded. QED

7.19 INSTANTANEOUS RETURN PROCESS AND ITS BOUNDARY

Recall the class $\mathscr{H}_D$ of D-type processes, defined in section 7.7, and the definition 8.1 of the jumping and leaping points. Declare the minimal process to be a zero-order instantaneous return process.

Definition 1. Suppose $X=\{x(t),t<\sigma\}\in\mathscr{H}_D$ is a non-minimal process. If for almost all ω and any $t<\sigma(\omega)$, $X(\omega)$ has at most only finitely many leaping points in $[0,t)$, then we call X a first-order instantaneous return process, or simply a first-order process. The class composed of first-order processes (or first-order Q processes) is written as $\mathscr{H}_1$ (or $\mathscr{H}_1(Q)$).

Let $X\in\mathscr{H}_1$. Then its discontinuities can be arranged in an order:

$$0=\tau_0,\tau_1,\tau_2,\ldots;\tau_\omega,\tau_{\omega+1},\ldots;\tau_{\omega^2},\tau_{\omega^2+1},\ldots;\tau_{\omega^n},\tau_{\omega^n+1},\ldots \tag{1}$$

where either for some ordinal number $\alpha<\omega^2$ we have $\tau_\alpha=\sigma$, or for all ordinal numbers $\alpha<\omega^2$ we have $\tau_\alpha<\alpha$, and $\lim_{n\to\infty}\tau_{\omega^n}=\sigma$.

If $X=\{x(t),t<\sigma\}\in\mathscr{H}_D$, then its discontinuouities can also be arranged in this order:

$$0=\tau_0,\tau_1,\tau_2,\ldots;\tau_\omega,\tau_{\omega+1},\ldots;\tau_{\omega^2},\ldots,\tau_{\omega^n},\ldots,\tau_{\sum_{i=0}^{l}\omega^{l-i}k_i},\ldots \tag{2}$$

where the k_i are non-negative integers and $l>n$. We shall define the kth-order instantaneous return process by induction.

Definition 2. We say $X=\{x(t),t<\sigma\}\in\mathscr{H}_D$ is a kth-order instantaneous return process, if X is not an nth-order $(0\leqslant n\leqslant k-1)$ instantaneous return process, and either there exists some ordinal number $\alpha<\omega^{k+1}$ such that $\tau_\alpha=\sigma$, or for all ordinal numbers $\alpha<\omega^{k+1}$ it is true that $\tau_\alpha<\sigma$, and furthermore $\tau_{\omega^{k+1}}=\sigma$. The class composed of kth-order instantaneous return processes is written as $\mathscr{H}_k$. The class composed of kth-order instantaneous return Q processes is denoted by $\mathscr{H}_k(Q)$.

Suppose $X\in\mathscr{H}_k(Q)$. We call the embedding chain $X_T^0=\{x(\tau_n),\tau_n\leqslant\tau_\beta\}$ the zero-order embedding chain of X, where τ_β is defined by (8.9)–(8.11), the one-step transition probability of X_T^0 is

$$_0\Pi_{ij}=P_i\{x(\tau_1)=j\} \tag{3}$$

We call $X_T^1=\{x(\tau_{\omega n}),\tau_{\omega n}\leqslant\tau_{\omega\beta}\}$ the first-order embedding chain, where $\tau_{\omega\beta}$ can be defined by following (8.9)–(8.11). The one-step transition probability of X_T^1 is

$$_1\Pi_{ij}=P_i\{x(\tau_\omega)=j\} \tag{4}$$

$x^l=\{x(\tau_{\omega^l n}),\tau_{\omega^l n}\leqslant\tau_{\omega^l\beta}\}$ is the lth-order embedding chain, where $\tau_{\omega^l\beta}$ can be defined by following (8.9)–(8.11). The one-step transition probability of X_T^l is

$$_l\Pi_{ij}=P_i\{x(\tau_{\omega^l})=j\} \tag{5}$$

Similarly, we can define

$$_0\Pi_{ij}(\lambda)=E_i\{e^{-\lambda\tau_1},x(\tau_1)=j) \qquad \lambda>0 \tag{6}$$

$$_l\Pi_{ij}(\lambda)=E_i\{e^{-\lambda\tau_{\omega^l}},x(\tau_{\omega^l})=j\} \qquad \lambda>0 \tag{7}$$

Just as in sections 7.12 and 7.14 according to $_0\Pi, {}_0\Pi(\lambda)$ we could determine Martin boundary ∂E, essential Martin boundary B, Martin exit boundary B_e, Martin passive boundary B_p, atomic exit boundary B_{e1}, non-atomic exit boundary B_{e2}, atomic passive boundary B_{p1}, non-atomic passive boundary B_{p2}, and so on, so according to $_l\Pi$ and $_l\Pi(\lambda)$ we can determine lth-order Martin boundary $_l(\partial E)$, l-order essential Martin boundary $_lB$, lth-order exit boundary $_lB_e$, lth-order passive boundary $_lB_p$, lth-order atomic exit boundary $_lB_{e1}$, lth-order non-atomic exit boundary $_lB_{e2}$, lth-order atomic passive boundary $_lB_{p1}$, lth-order non-atomic passive boundary $_lB_{p2}$, and so on. Much of the boundary theory of the minimal process, i.e. zero-order process, can be transplanted into kth-order instantaneous return processes. For instance, for a kth-order instantaneous return process $X=\{x(t),t<\sigma\}\in\mathscr{H}_k$, if we let

$$_k\Omega_F=\{\text{there exists an ordinal number }\alpha<\omega^{k+1}\text{ such that }\tau_\alpha=\sigma<\infty\}$$

and

$$_k\Omega_\infty=\{\text{for all ordinal numbers }\alpha<\omega^{k+1}\text{ it is true that }\tau_\alpha<\sigma\}$$

$$+\{\text{there exists an ordinal number }\alpha<\omega^{k+1}\text{ such that }\tau_a=\sigma=\infty\}$$

$$H=({}_0H)\cup({}_0B_e)\cup({}_1B_e)\cup\cdots\cup({}_{k-1}B_e) \qquad {}_0H=H \qquad {}_0B_e=B_e$$

Then on $_k\Omega_F$ we have $x(\sigma-0)\in{}_kH$, on $_k\Omega_\infty$ we have $x(\sigma-0)\in{}_kB$. Similarly to Theorem 12.5 and Theorem 12.8 we have the following.

Theorem 1. Suppose $X = \{x(t), t < \sigma\} \in \mathscr{H}_k$. Then for all $i \in E$ we have

$$P_i\{\sigma < \infty \mid x(\sigma - 0) \in_k B_e\} = 1 \tag{8}$$

$$P_i\{\sigma = \infty \mid x(\sigma - 0) \in_k B_p\} = 1 \tag{9}$$

$$P_i\{\sigma < \infty \mid x(\sigma - 0) \in_k H\} = 1 \tag{10}$$

CHAPTER 8

Construction of Q Processes with Finite Non-conservative States and Finite Exit Boundary

8.1 INTRODUCTION

For construction of general Q processes we often suppose that Q is conservative. Under this supposition, Feller (1957a) for the case of finite exit boundary and finite entrance boundary has constructed all Q processes satisfying the system of forward equations. Under the same supposition, Xiang-qun Yang (1966a) has constructed all Q processes. Williams (1964, 1966) and Chung (1963, 1966) in the case that Q is conservative and has a finite exit boundary found all the Q processes. Xiang-qun Yang (1982, 1983a), for the case of Q having a finite set of non-conservative states and a finite exit boundary (simply called bifinite), has constructed all the Q processes, but the results obtained are not very obviously related with those in Williams (1964, 1966). For this reason, by means of methods similar to those used in Williams (1964, 1966), Da-guo Xiong (1980, 1981) has constructed all the bifinite Q processes, but his results are still not very obviously associated with those in Feller (1957a) and Xiang-qun Yang (1966a). In this chapter, we shell, under the bifinite condition and according to the methods in Feller (1957a) and Xiang-qun Yang (1966a), construct all the Q processes, and point out the relation with the results obtained in Williams (1964, 1966) and Da-guo Xiong (1980, 1981). The content of this chapter is taken from Xiang-qun Yang (1983b).

8.2 BASIC HYPOTHESIS AND THE CONDITION SATISFIED BY $F^{\alpha}(\lambda)$

Given a matrix $Q = (q_{ij})$ satisfying (2.2.6) and with $d = Q1$ as the non-conservative quantity, let $H_e = \{i : d_i > 0\}$ be the set of non-conservative states and B_e be the Martin exit boundary induced by Q.

Basic Hypothesis

Suppose that $A_e = H_e \cup B_e$ is a finite set, H_e or B_e may be empty. But when B_e is empty, either $\mathscr{L}_\lambda^+$ is not empty, or

$$\inf_\lambda \lambda \sum_j \phi_{ij}(\lambda) = 0 \qquad \lambda > 0 \tag{1}$$

Under the basic hypothesis the Q process is not unique. When $a \in H_e$, $X_i^a(\lambda) = \phi_{ia}(\lambda) d_a$ is an exit family, and

$$X_i^a(\lambda) \uparrow X_i^a = \Gamma_{ia} d_a \qquad \lambda \downarrow 0 \tag{2}$$

$$\lambda X_i^a(\lambda) \to \delta_{ia} d_a \qquad \lambda \to \infty \tag{3}$$

Write $\bar{X}(\lambda)$ for the maximal solution of $\mathscr{M}_\lambda^+(1)$, $\bar{X}(\lambda) \uparrow \bar{X}(\lambda \downarrow 0)$. If B_e is non-empty and finite, according to the discussion of sections 7.13 to 7.16, we can select an exit family $X^a(\lambda)$ $(a \in B_e)$ such that $\bar{X}(\lambda) = \sum_{a \in B_e} X^a(\lambda)$. Moreover,

$$X_i^a(\lambda) \to \delta_{ab} \qquad a, b \in B_e, i \to b \tag{4}$$

$$X^a(\lambda) \uparrow X^a \qquad \lambda \downarrow 0, a \in B_e \tag{5}$$

$$\lambda X^a(\lambda) \to 0 \qquad a \in B_e, \lambda \to \infty \tag{6}$$

$$X_i^a(\lambda) \to 0 \qquad a \in H_e, i \to b \in B_e \tag{7}$$

Thus under the basic hypothesis, we can select a harmonic exit family $X^a(\lambda)$ $(a \in A_e)$ whose standard image is X^a $(a \in A_e)$, such that

$$\lambda \phi(\lambda) 1 = 1 - Z(\lambda) \qquad Z(\lambda) = \sum_{a \in A_e} X^a(\lambda) \tag{8}$$

$$Z(\lambda) \uparrow Z = \sum_{a \in A_e} X^a \qquad \lambda \downarrow 0 \tag{9}$$

$\bar{X} = \sum_{a \in B_e} X^a$ and $X^0 = 1 - Z$ are the maximal exit solution and the maximal passive solution of Q, respectively. The $X^a(\lambda)(a \in A_e)$ are linearly independent. In fact, suppose $\sum_{a \in A_e} c^a X^e(\lambda) = 0$. By (7) we get $\sum_{a \in B_e} c^a X^a(\lambda) = 0$ and so by (4) we obtain $c^a = 0$ $(a \in B_e)$. Hence $\sum_{a \in H_e} c^a X^a(\lambda) = 0$. By Lemma 2.11.6 $c^a = 0$ $(a \in H_e)$.

According to Theorem 7.18.1, and Q process $\psi(\lambda)$ has the following form:

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \sum_{a \in A_e} X_i^a(\lambda) F_j^a(\lambda) \tag{10}$$

where

$$F^a(\lambda) \geqslant 0 \qquad \lambda [F^a(\lambda), 1] \leqslant 1 \tag{11}$$

Lemma 1. For $\psi_{(\lambda)}$ in (10), the norm condition is equivalent to that (11) holds for every $a \in A_e$. The resolvent equation is equivalent to that for every $a \in A_e$ the following holds:

$$F^a(\lambda) A(\lambda, \mu) = F^a(\mu) + (\mu - \lambda) \sum_{b \in A_e} [F^a(\lambda), X^b(\mu)] F^b(\mu) \tag{12}$$

where $A(\lambda, \mu)$ is as in (2.10.7); (12) is also equivalent to

$$F^a(\lambda) - F^a(\mu) = (\mu - \lambda) F^a(\lambda) \psi(\mu) \tag{13}$$

The Q condition is equivalent to that, for every $a \in H_e$, the following holds:

$$\lim_{\lambda \to \infty} \lambda F^a(\lambda) = 0 \tag{14}$$

Proof. The necessity of condition (11) for the norm condition has already been pointed out previously. The sufficiency follows from

$$\lambda \psi(\lambda) 1 = 1 - \sum_{a \in A_e} X^a(\lambda) + \sum_{a \in A_e} X^a(\lambda) \lambda [F^a(\lambda), 1]$$

and (11).

Substituting $\psi_{(\lambda)}$ in (10) into the resolvent equation, because $\phi(\lambda)$ satisfies the resolvent equation and $X^a(\lambda)(a \in A_e)$ is a harmonic exit family, and moreover, is linearly independent, we know that the resolvent equation for $\psi(\lambda)$ is equivalent to (12). Substituting $\psi(\lambda)$ (10) into (13), we find (13) is equivalent to (12).

Noting that $\phi(\lambda)$ satisfies the Q condition and observing the finiteness of A_e, we find that the Q condition for $\psi(\lambda)$ is equivalent to

$$\lim_{\lambda \to \infty} \lambda X_i^a(\lambda) \lambda F_j^a(\lambda) = 0 \qquad a \in A_e$$

By (3) and (6), the formula above is equivalent to that for every $a \in H_e$ (14) holds.

QED

8.3 SIMPLIFICATION OF THE PROBLEM

Definition 1. Suppose a and $b \in A_e$. We call a and b indistinguishable if for all $\lambda > 0$, $F^a(\lambda) = F^b(\lambda)$ holds.

Obviously, the indistinguishable relation is an equivalence relation. According to the indistinguishable relation, A_e can be decomposed into disjoint equivalence classes $a_1, a_2, \ldots,$. We can think of the equivalence class a_n as a new boundary point, and write

$$A = \{a_1, a_2, \ldots\} \qquad A = H \cup B \tag{1}$$

$$H = \{a : a \in A, a \cap H_e \neq \varnothing\} \qquad B = \{a : a \in A, a \cap H_e = \phi\} \tag{2}$$

Write

$$Y^a(\lambda)=\sum_{b\in a} X^b(\lambda)\uparrow Y^a=\sum_{b\in a} X^b \qquad a\in A, \lambda\downarrow 0 \tag{3}$$

Then (2.10) becomes

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+\sum_{a\in A} Y^a(\lambda)F^a(\lambda) \tag{4}$$

All the boundary points in A are distinguishable.

Lemma 1. There exists a subset $J\subset A$ and an $A\times J$ matrix $G=(G^{ab})$, whose columns are non-zero and whose rows satisfy

$$G^{ab}\geqslant 0 \qquad \sum_{b\in J} G^{ab}\leqslant 1\,(a\in A) \qquad G^{aa}=1\,(a\in J) \tag{5}$$

such that

$$F^g(\lambda)=\sum_{b\in J} G^{ab}F^b(\lambda) \qquad a\in A, \lambda>0 \tag{6}$$

And for each $a\in J$, $F^a(\lambda)$ cannot be represented by the following form:

$$F^a(\lambda)=\sum_{b\in J} h^{ab}F^b(\lambda) \qquad \lambda>0 \tag{7}$$

where

$$h^{aa}=0 \qquad h^{ab}\geqslant 0 \qquad \sum_{b\in J} h^{ab}\geqslant 1 \tag{8}$$

Especially, we have $F^a(\lambda)\neq 0\,(\alpha\in J)$.

Proof. It is obvious that there exist a subset $J\subset A$ and an $A\times J$ matrix G such that (5) and (6) hold. For instance, it suffices to take A to be J, and to take G to be the unit matrix. But then (7) and (8) may hold for some $a\in J$.

Suppose there exist $J\subset A$ and an $A\times J$ matrix G such that (5) and (6) hold, and there exist $a_0\in J$ such that (7) and (8) hold for $a=a_0$. Then by (6) and (7) we obtain

$$F^a(\lambda)=\sum_{b\in J_0} G_0^{ab}F^b(\lambda) \qquad a\in A, \lambda>0$$

where $J_0=J-\{a_0\}$, and the element of the $A\times J_0$ matrix $G_0=(G_0^{ab})$ are $G_0^{ab}=G^{ab}+G_0^{aa}h^{a_0b}, a\in A,\ b\in J_0$.

By (5) and (8), G satisfies

$$G_0^{ab}\geqslant 0 \qquad \sum_{b\in J_0} G_0^{ab}\leqslant 1\,(a\in A) \qquad G_0^{aa}=1\,(a\in J_0)$$

Thus, after substituting J_0 and $A\times J_0$ matrix G_0 for J and G, (5) and (6) still hold. Going on like this, we can finally get a subset J and an $A\times J$ matrix G such that (5) and (6) hold, while for every $a\in J$, (7) and (8) cannot hold simultaneously. Finally if some columns in G are zero columns, then substituting $J-\{b\in J$: the bth column of G is a zero column$\}$ for J we can meet the needs of the lemma. The proof is completed. QED

Lemma 2. Every Q process $\psi(\lambda)$ has the form

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+\sum_{a\in J} Z_i^a(\lambda)F_j^a(\lambda) \tag{9}$$

where the subset $J\subset A$ and all the boundary points in J are distinguishable, and

$$Z^b(\lambda)=\sum_{a\in A} Y^a(\lambda)G^{ab} \qquad b\in J \tag{10}$$

is a linerly independent harmonic exit family. Each column of the $A\times J$ matrix $G=(G^{ab})$ is non-zero and satisfies (5), and is such that (6) holds; for every $a\in J$, (7) and (8) cannot hold simultaneously. Especially, $F^a(\lambda)\neq 0(a\in J)$.

Proof. Substituting into (4) we get (9). Since $X^a(\lambda)$, $a\in A_e$, is a linearly independent harmonic exit family, it follows that $Z^b(\lambda)(b\in J)$ in (10) is a linearly independent harmonic exit family. The remainder of the conclusion follows from Lemma 1. The proof is completed. QED

Lemma 3. Suppose the Q process $\psi(\lambda)$ has the form (9) in Lemma 2. Then the norm condition of $\psi(\lambda)$ is equivalent to

$$F^a(\lambda)\geqslant 0 \qquad \lambda[F^a(\lambda),1]\leqslant 1 \qquad a\in J \tag{11}$$

The resolvent equation is equivalent to

$$F^a(\lambda)A(\lambda,\mu)=F^a(\mu)+(\mu-\lambda)\sum_{b\in J}[F^a(\lambda),Z^b(\mu)]F^b(\mu) \qquad a\in J \tag{12}$$

The Q condition is equivalent to

$$\lim_{\lambda\to\infty}\lambda F^a(\lambda)=0 \qquad a\in J_H \tag{13}$$

where

$$J_H=J\cap H=\{a\in J: a\cap H_e\neq\varnothing\} \tag{14}$$

$$J_B=J\cap B=\{a\in J: a\cap H_e=\varnothing\} \tag{15}$$

$$J=J_H\cup J_B \tag{16}$$

Proof. It follows by using Lemma 2.1 and Lemma 3.2. QED

Write

$$Z^b = \sum_{a\in A} Y^a G^{ab} \qquad b\in J \tag{17}$$

$$Z^*(\lambda) = Z(\lambda) - \sum_{b\in J} Z^b(\lambda) = \sum_{a\in A} Y^a(\lambda)\left(1 - \sum_{b\in J} G^{ab}\right) \tag{18}$$

$$Z^* = Z - \sum_{b\in J} Z^b = \sum_{a\in A} Y^a\left(1 - \sum_{b\in J} G^{ab}\right) \tag{19}$$

Then obviously

$$Z^b(\lambda)\uparrow Z^b(b\in J) \qquad Z^*(\lambda)\uparrow Z^* \qquad \lambda\downarrow 0 \tag{20}$$

$$X^0 + Z^* + \sum_{a\in J} Z^a = 1 \tag{21}$$

8.4 GENERAL FORM OF $F^a(\lambda)$

Lemma 1. Suppos $g\in l$, then

$$C_{\nu\lambda}^{-1}\|g\| \leqslant \|gA(\lambda,\nu)\| \leqslant C_{\lambda\nu}\|g\| \tag{1}$$

Here $\|g\|$ represents the norm in the Banach space l, and

$$C_{\lambda\nu} = 1 + \frac{|\lambda-\nu|}{\nu} \tag{2}$$

Proof. From the definition of $A(\lambda,\nu)$,

$$\|gA(\lambda,\nu)\| \leqslant \|g\| + \frac{|\lambda-\nu|}{\nu}[|g|, \nu\phi(\nu)1]$$

$$\leqslant \|g\| + \frac{|\lambda-\nu|}{\nu}[|g|,1] = C_{\lambda\nu}\|g\|$$

The right-hand inequality in (1) follows. The left-hand inequality follows from

$$\|g\| = \|gA(\lambda,\nu)A(\nu,\lambda)\| \leqslant C_{\nu\lambda}\|gA(\lambda,\nu)\| \qquad \text{QED}$$

Lemma 2. Suppose the non-zero $F^a(\lambda)(a\in J)$ satisfies (3.11) and (3.12). Then there exist non-negative numbers $M^{ab}(\lambda)(a,b\in J)$ and a harmonic entrance family $\eta^a(\lambda)(a\in J)$ such that

$$F^a(\lambda) = \sum_{b\in J} M^{ab}(\lambda)\eta^b(\lambda) \qquad a\in J \tag{3}$$

Furthermore we can also select numbers $M^{ab}(\lambda)$ and $\eta^b(\lambda)$ having the following

properties:

(i) $\eta^b(\lambda) = 0$ if and only if $M^{ab}(\lambda) = 0\,(a\in J)$,

(ii) there exists a non-negative $J\times J$ matrix $H = (H^{ab})$ such that

$$\eta^a(\lambda) = \sum_{b\in J} H^{ab}\eta^b(\lambda) \qquad a\in J \tag{4}$$

(iii) there exist positive numbers $\nu_a(a\in J)$ such that when μ moves along some subsequence $\mu_n\to\infty$ then

$$H^{ab}(\mu) = \frac{M^{ab}(\mu)}{\|F^a(\mu)A(\mu,\nu_a)\|} \to H^{ab} \qquad a,b\in J \tag{5}$$

If $\eta^a(\lambda)\neq 0$ then in the sense of strong convergence, when $\mu=\mu_n\to\infty$,

$$\eta^a(\lambda,\mu) = \frac{F^a(\mu)A(\mu,\lambda)}{\|F^a(\mu)A(\mu,\nu_a)\|} \to \eta^a(\lambda) \tag{6}$$

Proof. First note that for all $\lambda,\mu>0$, $F^a(\lambda)A(\lambda,\mu)$ is non-negative. In fact, by the definition of $A(\lambda,\mu)$ when $\lambda\geqslant\mu$ the conclusion is obvious. When $\lambda<\mu$, since the right-hand side of (3.12) is non-negative, so is the left-hand side. Because $F^a(\lambda)$ is non-zero we know $\|F^a(\lambda)A(\lambda,\mu)\|>0$ by (1). Multiplying (3.12) by $A(\mu,\lambda)$ from the right we obtain

$$F^a(\lambda) = \sum_{b\in J} M^{ab}(\nu_b,\lambda,\mu)\eta^b(\nu_b,\lambda,\mu) \tag{7}$$

where $\nu_b>0$ is arbitrary, and

$$M^{ab}(\nu,\lambda,\mu) = \sigma^{ab}(\lambda,\mu)\|F^b(\mu)A(\mu,\nu)\| \tag{8}$$

$$\sigma^{ab}(\lambda,\mu) = \delta_{ab} + (\mu-\lambda)[F^a(\lambda), Z^b(\mu)] \tag{9}$$

$$\eta^b(\nu,\lambda,\mu) = \frac{F^b(\mu)A(\mu,\lambda)}{\|F^b(\mu)A(\mu,\nu)\|} \tag{10}$$

Obviously

$$\|\eta^b(\nu,\nu,\mu)\| = 1 \tag{11}$$

On account of (2.10.9) and Lemma 1

$$C_{\lambda\nu}^{-1} \leqslant \|\eta^b(\nu,\lambda,\mu)\| \leqslant C_{\nu\lambda} \tag{12}$$

$$C_{\nu\lambda}^{-1}M^{ab}(\lambda,\lambda,\mu) \leqslant M^{ab}(\nu,\lambda,\mu) \leqslant C_{\lambda\nu}M^{ab}(\lambda,\lambda,\mu) \tag{13}$$

Taking $\nu_b=\nu>0$ in (7) we have

$$F^a(\lambda) = \sum_{b\in J} M^{ab}(\nu,\lambda,\mu)\eta^b(\nu,\lambda,\mu) \tag{14}$$

From (12) we derive

$$1/\lambda \geqslant \|F^a(\lambda)\| \geqslant \sum_{b\in J} M^{ab}(\nu,\lambda,\mu)C_{\lambda\nu}^{-1} \tag{15}$$

$$C_{\lambda\nu}/\lambda \geqslant \sum_{a\in J} M^{ab}(\nu,\lambda,\mu) \tag{16}$$

Especially, taking $\lambda=\nu$ in (15) we get

$$\|F^a(\nu)\| \geqslant \sum_{b\in J} M^{ab}(\nu,\nu,\mu) \tag{17}$$

Consequently for every $b\in J$, either

$$\overline{\lim_{\mu\to\infty}}\, M^{ab}(\nu,\lambda,\mu)=0 \qquad \text{for all } \nu>0, \lambda>0, a\in J \tag{18}$$

or

$$\overline{\lim_{\mu\to\infty}}\, M^{ab}(\nu,\lambda,\mu)>0 \qquad \text{for some } \lambda>0, \text{ some } \nu>0 \text{ and some } a\in J \tag{19}$$

By (13) the formula above is equivalent to

$$\overline{\lim_{\mu\to\infty}}\, M^{ab}(\nu,\nu,\mu)>0 \qquad \text{for some } \nu>0 \text{ and some } a\in J \tag{20}$$

Assume the whole of $b\in J$ that makes (18) hold to be G_0; then $J-G_0\neq\varnothing$. Because, otherwise, by (14), we derive $F^a(\lambda)=0$ which contradicts the hypothesis that $F^a(\lambda)$ is non-zero. Therefore there exists $b_1\in J-G_0$, $\nu(b_1)>0$, $a(b_1)\in J$ and a subsequence $\mu_n(1)\to\infty$ such that

$$M^{a(b_i)b_i}(\nu(b_i),\nu(b_i),\mu_n(i))\to M^{a(b_c)b_c}(\nu(b_i))>0 \tag{21}$$

hold for $i=1$. Assume the whole of $b\in J$ that makes (18) hold to be G_1 when $\mu=\mu_n(1)\to\infty$. Then $G_0\subset G_1$, $b_1\in J-G_1$. If $G_1\cup\{b_1\}=J$, then we take a subsequence $\mu_n=\mu_n(1)$. Otherwise, there exist $b_2\in J-(G_1\cup\{b_1\})$, $\nu(b_2)>0$, $a(b_2)\in J$ and a subsequence $\mu_n(2)$ of $\mu_n(1)$ such that (21) is true for $i=2$. Because J is finite, there exist $\Delta=\{b_1,b_2,\ldots,b_k\}\subset J$ and $a(b_1)$, $a(b_2),\ldots,a(b_k)\in J$, $\nu(b_1)$, $\nu(b_2),\ldots,\nu(b_k)>0$, and a subsequence $\mu_n\to\infty$, such that

$$\lim_{n\to\infty} M^{ab}(\nu,\lambda,\mu_n)=0 \tag{22}$$

For $b\in J-\Delta$ and all $a\in J$, $\nu>0$, $\lambda>0$,

$$\lim_{n\to\infty} M^{a(c)c}(\nu(c),\nu(c),\mu_n)=M^{a(c)c}(\nu(c))>0 \qquad c\in\Delta \tag{23}$$

On account of (11) and (17), and by applying the diagonal method, we can

select a subsequence of μ_n, again to be denoted by μ_n, such that for every $c\in\Delta$ we have

$$\lim_{n\to\infty} M^{ab}(\nu(c),\nu(c),\mu_n)=M^{ab}(\nu(c)) \qquad a,b\in J \tag{24}$$

$$\lim_{n\to\infty} \eta_j^b(\nu(c),\nu(c),\mu_n=\eta_j^b(\nu(c)) \qquad b\in J, j\in E \tag{25}$$

and moreover

$$M^{a(c)c}(\nu(c))>0\,(c\in\Delta) \qquad M^{ab}(\nu(c))=0\,(c\in\Delta, a\in J, b\in J-\Delta) \tag{26}$$

Letting $\lambda=\nu=\nu(c)$ in (14) we obtain

$$F^a(\nu(c))=\sum_{b\in J} M^{ab}(\nu(c),\nu(c),\mu)\eta^b(\nu(c),\nu(c),\mu) \qquad c\in\Delta \tag{27}$$

Letting $\mu=\mu_n\to\infty$, we get

$$F^a(\nu(c))=\sum_{b\in\Delta} M^{ab}(\nu(c))\eta^b(\nu(c)) \tag{28}$$

Thus

$$\|F^a(\nu(c))\|=\sum_{b\in\Delta} M^{ab}(\nu(c))\|\eta^b(\nu(c))\| \tag{29}$$

Letting $\nu=\nu(c)$ in (17) and taking the limit we derive

$$\|F^a(\nu(c))\|\geqslant\sum_{b\in\Delta} M^{ab}(\nu(c)) \tag{30}$$

By (11) and Fatou's lemma we have $\|\eta^b(\nu(c))\|\leqslant 1$ $(c\in\Delta, b\in J)$. Comparing (29) and (30) we obtain

$$\|\eta^b(\nu(b))\|=1 \qquad b\in\Delta \tag{31}$$

Thus when $b\in\Delta$, $\eta^b(\nu(b),\nu(b),\mu_n)$ converges to $\eta^b(\nu(b))$ in coordinates, and the sequence of norms $\|\eta^b(\nu(b),\nu(b),\mu_n)\|=1$ obviously converges to the norm $\|\eta^b(\nu(b))\|=1$. Thus the convergence is a strong one in the Banach space l, that is

$$\|\eta^b(\nu(b),\nu(b),\mu_n)-\eta^b(\nu(b))\|\to 0 \qquad b\in\Delta \tag{32}$$

Write

$$\eta^b(\lambda)=\eta^b(\nu(b))A(\nu(b),\lambda)$$

Then $\eta^b(\lambda)\,(b\in\Delta)$ is obviously a non-zero harmonic entrance family. By Lemma 1

$$\|\eta^b\nu(b),\lambda,\mu_n)-\eta^b(\lambda)\|\leqslant C_{\nu(b)\lambda}\|\eta^b(\nu(b),\nu(b),\mu_n)-\eta^b(\nu(b))\|\to 0 \tag{33}$$

Now for every $a\in J$ and $\lambda>0$, we can select a subsequence of μ_n, and still write it as μ_n, such that

$$M^{ab}(v(b),\lambda,\mu_n)\to M^{ab}(\lambda) \qquad b\in\Delta \tag{34}$$

Letting $\mu=\mu_n\to\infty$ in (7) and noting (34) and (32) we find

$$F^a(\lambda)=\sum_{b\in\Delta} M^{ab}(\lambda)\eta^b(\lambda)$$

For $b\in J-\Delta$, we add the definition that $\eta^b(\lambda)=0$, $M^{ab}(\lambda)=0\,(a\in J)$. Thus we have proved (3), and (i) in Lemma 2, and (6).

By (3) we have

$$\eta^a(\lambda,\mu)=\frac{F^a(\mu)A(\mu,\lambda)}{\|F^a(\mu)A(\mu,v_a)\|}=\sum_{b\in J} H^{ab}(\mu)\eta^b(\lambda) \tag{35}$$

where $v_a=v(a)$, and

$$H^{ab}(\mu)=\frac{M^{ab}(\mu)}{\|F^a(\mu)A(\mu,v_a)\|}=\frac{M^{ab}(\mu)}{\sum_{b\in J}M^{ab}(\mu)\|\eta^b(v_a)\|} \tag{36}$$

Because

$$\sum_{b\in J} H^{ab}(\mu)\|\eta^b(v_a)\|=1 \tag{37}$$

and $\|\eta^b(v_a)\|>0$ when $b\notin\Delta$, then $H^{ab}(\mu)=0$ when $b\in J-\Delta$. So that when μ tends to infinity along some subsequence of μ_n the limit in (5) exists and is finite.

When $\eta^a(\lambda)\neq 0$, that is, $a\in\Delta$, by (33), $\eta^a(\lambda,\mu_n)=\eta^a(v_a,\lambda,\mu_n)$ converges strongly to $\eta^a(\lambda)$. Hence when $\eta^a(\lambda)\neq 0$, (4) follows from (35). When $\eta^a(\lambda)=0$, taking $H^{ab}=0\,(b\in J)$ (4) obviously holds. The lemma is proved. QED

8.5 NON-STICKY CASE

Definition 1. A harmonic exit family $(\eta(\lambda),\ \lambda>0)$ is said to be sticky if $\lim_{\lambda\to\infty}\lambda\|\eta(\lambda)\|=\infty$, otherwise it is said to be non-sticky. If in the representation (3.9) and (4.3) of a Q process $\psi(\lambda)$ every $\eta^a(\lambda)$ is non-sticky, then we say that the process $\psi(\lambda)$ is non-sticky.

Theorem 1. (i) Decompose $A_e=H_e\cup B_e$ into $A_e=a_1\cup a_2\cup\ldots$, where every a_m is non-empty and they are mutually disjoint. Write $A=\{a_1,a_2,\ldots\}$. Take a non-empty subset J of A, and $J=J_H\cup J_B$, where

$$J_H=\{a\in J: a\cap H_e\neq\varnothing\} \qquad J_B=\{a\in J: a\cap H_e=\varnothing\}$$

(ii) Take an $A\times J$ matrix $G=(G^{ab})$ satisfying (3.5), where each column of G

is non-zero. Write

$$Z^b(\lambda)=\sum_{a\in A}\sum_{c\in a}X^c(\lambda)G^{ab} \qquad Z^b=\sum_{a\in A}\sum_{c\in a}X^cG^{ab} \qquad b\in J \tag{1}$$

$$Z^*(\lambda)=Z(\lambda)-\sum_{a\in J}Z^a(\lambda)=\sum_{a\in A}\Big(\sum_{c\in a}X^c(\lambda)\Big)\Big(1-\sum_{b\in J}G^{ab}\Big) \tag{2}$$

$$Z^*=Z-\sum_{a\in J}Z^a=\sum_{a\in A}\Big(\sum_{c\in a}X^c\Big)\Big(1-\sum_{b\in J}G^{ab}\Big)$$

(iii) Take a non-zero and non-sticky harmonic exit family $(\eta^a(\lambda),\ \lambda>0)(a\in J)$. Write

$$\bar W^{ab}(\lambda)=\lambda[\bar\eta^a(\lambda),Z^b]\uparrow\bar W^{ab} \qquad \lambda\uparrow\infty \tag{3}$$

$$\bar W^{a*}(\lambda)=\lambda[\bar\eta^a(\lambda),Z^*]\uparrow\bar W^{a*} \qquad \lambda\uparrow\infty \tag{4}$$

$$\bar\sigma^a=\lambda[\bar\eta^a(\lambda),X^0] \qquad (\bar\sigma^a \text{ is independent of } \lambda) \tag{5}$$

$$\bar\alpha^a=\lim_{\lambda\to\infty}\lambda\bar\eta^a(\lambda) \tag{6}$$

such that

$$\bar\sigma^a+\bar W^a+\sum_{b\in J}\bar W^{ab}\leqslant 1 \tag{7}$$

$$\bar\alpha^a=0 \qquad \text{if } a\in J_H \tag{8}$$

(iv) Let

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+\sum_{a\in J}\sum_{b\in J}Z_i^a(\lambda)K^{ab}(\lambda)\bar\eta^b(\lambda) \tag{9}$$

where the $J\times J$ matrix $K(\lambda)$ is given by

$$\begin{aligned} K(\lambda)&=(K^{ab}(\lambda))=[I-\bar W+\bar W(\lambda)]^{-1}\\ \bar W(\lambda)&=(\bar W^{ab}(\lambda))\uparrow\bar W=(\bar W^{ab}) \qquad \lambda\uparrow\infty\end{aligned} \tag{10}$$

and $I=(\delta_{ab})$ is the $J\times J$ unit matrix.

Then $\psi(\lambda)$ is a non-sticky Q process, which is honest if and only if

$$\bar W^{a*}=0 \qquad \bar\sigma^a+\sum_{b\in J}\bar W^{ab}=1 \tag{11}$$

The $\psi(\lambda)$ is of B-type if and only if $J_H=\varnothing$. The $\psi(\lambda)$ is of F-type if and only if

$$\bar\alpha^a=0 \qquad a\in J_B \tag{12}$$

Every non-minimal and non-sticky Q process $\psi(\lambda)$ can be derived in the way above.

Proof. (a) Suppose that the Q process $\psi(\lambda)$ is non-minimal and non-sticky. Substituting (4.3) into (3.11) and (3.12) we obtain

$$\sum_{b\in J} M^{ab}(\lambda)\sigma^b + S^{a*}(\lambda) + \sum_{b\in J} S^{ab}(\lambda) \leqslant 1 \tag{13}$$

$$\sum_{b\in J} M^{ab}(\lambda)\eta^b(\mu) = \sum_{b\in J} [I - S(\lambda) + M(\lambda)W(\mu)]^{ab}F^b(\mu) \tag{14}$$

where $W(\lambda) = (W^{ab}(\lambda), S(\lambda) = (S^{ab}(\lambda))$ are $J \times J$ matrices,

$$W^{ab}(\lambda) = \lambda[\eta^a(\lambda), Z^b] \uparrow W^{ab} \qquad \lambda\uparrow\infty \tag{15}$$

$$W^{a*}(\lambda) = \lambda[\eta^a(\lambda), Z^*] \uparrow W^{a*} \qquad \lambda\uparrow\infty \tag{16}$$

$$\bar{\sigma}^a = \lambda[\bar{\eta}^a(\lambda), X^0] \qquad (\sigma^a \text{ being independent of } \lambda) \tag{17}$$

$$S^{ab}(\lambda) = \sum_{c\in J} M^{ac}(\lambda)W^{cb}(\lambda) \tag{18}$$

$$S^{a*}(\lambda) = \sum_{c\in J} M^{ac}(\lambda)W^{c*}(\lambda) \tag{19}$$

As $\psi(\lambda)$ is non-sticky, $W = (W^{ab})$ and W^{a*} are all finite.

Select a subsequence $\lambda \to \infty$ such that

$$M^{ab}(\lambda) \to M^{ab} \qquad S^{ab}(\lambda) \to S^{ab} \qquad S^{a*}(\lambda) \to S^{a*} \tag{20}$$

It is plain that S^{ab}, S^{a*} are all finite. We are going to prove M^{ab} finite. When $\eta^b(\lambda) = 0$ obviously $M^{ab}(\lambda) = 0 \to M^{ab} = 0$. Suppose $\Delta = \{b\in J : \eta^b(\lambda) \neq 0\}$. By (13), (18), (19) and Fatou's lemma we have

$$\sum_{c\in J} M^{ac}\sigma^c + S^{a*} + \sum_{b\in J} S^{ab} \leqslant 1 \tag{21}$$

$$\sum_{c\in J} M^{ac}W^{c*} \leqslant S^{a*} \tag{22}$$

$$\sum_{c\in J} M^{ac}W^{cb} \leqslant S^{ab} \tag{23}$$

(setting $\infty\cdot 0 = 0$). So by (21)–(23),

$$\begin{aligned}\sum_{c\in\Delta} M^{ac}\lambda[\eta^c(\lambda), 1] &= \sum_{c\in J} M^{ac}\lambda[\eta^c(\lambda), 1] \\ &= \sum_{c\in J} M^{ac}\left(\sigma^c + W^{c*}(\lambda) + \sum_{b\in J} W^{cb}(\lambda)\right) \\ &\leqslant \sum_{c\in J} M^{ac}\sigma^c + \sum_{c\in J} M^{ac}W^{c*} + \sum_{b\in J}\left(\sum_{c\in J} M^{ac}W^{cb}\right) \\ &\leqslant 1\end{aligned}$$

and therefore when $b\in\Delta$, $M^{ab} < \infty$. Thus the $J \times J$ matrix $M = (M^{ab})$ is finite. Hence the equalities hold in (22) and (23).

Taking the limit in (14) we get

$$\sum_{b\in J} M^{ab}\eta^b(\mu) = \sum_{b\in J} [I - MW + MW(\mu)]^{ab}F^b(\mu) \tag{24}$$

Write

$$\bar{\eta}^a(\lambda) = \sum_{c\in J} M^{ac}\eta^c(\lambda) \tag{25}$$

Then (24) becomes

$$\bar{\eta}^a(\mu) = \sum_{b\in J} [I - \bar{W} + \bar{W}(\mu)]^{ab}F^b(\mu) \tag{26}$$

Note that $\bar{\eta}^a(\mu)$ is a non-zero and harmonic entrance family. Actually, if $\bar{\eta}^a(\mu) = 0$, then $\bar{W}^{ab}(\mu) = 0$, $\bar{W}^{ab} = 0$, and hence by (26) we have $F^a(\mu) = 0$, which contradicts the non-zero property of $F^a(\mu)$. Thus $\mu[\bar{\eta}^a(\mu), 1] > 0$.

The matrix $\bar{W} - \bar{W}(\mu)$ is non-negative, and the row sum satisfies

$$\begin{aligned}\sum_{c\in J} [\bar{W}^{ac} - \bar{W}^{ac}(\mu)] &= \sum_{c\in J} \bar{W}^{ac} - \mu[\bar{\eta}^a(\mu), \mathbf{1} - X^0 - Z^*] \\ &= \bar{\sigma}^a + \bar{W}^{a*} + \sum_{c\in J} \bar{W}^{ac} - \mu[\bar{\eta}^a(\mu), \mathbf{1}] \\ &\leqslant 1 - \mu[\bar{\eta}^a(\mu), \mathbf{1}] < 1\end{aligned}$$

Accordingly the inverse matrix in (10) exists and is non-negative, and furthermore

$$K(\lambda) = \sum_{n=0}^{\infty} [\bar{W} - \bar{W}(\lambda)]^n \tag{27}$$

By (26) we have

$$F^a(\mu) = \sum_{b\in J} K^{ab}(\mu)\bar{\eta}^b(\mu) \tag{28}$$

An so we obtain (9).

Noting that

$$\lim_{\lambda\to\infty} K(\lambda) = \sum_{n=0}^{\infty} (\bar{W} - \bar{W})^n = I \tag{29}$$

by (28) we know that the Q condition (3.13) is just (8).

(b) Suppose that $\psi(\lambda)$ is determined according to (i)–(iv) in Theorem 1. Because

$$\bar{W}(\lambda) = K(\lambda)^{-1} - I + \bar{W}$$

by (28) and after simple computations we derive

$$\lambda[F^a(\lambda), 1] = 1 - \sum_{b\in J} K^{ab}(\lambda)\left(1 - \bar{\sigma}^b - \bar{W}^{b*} - \sum_{c\in J} \bar{W}^{bc}\right)$$

$$- \sum_{b\in J} K^{ab}(\lambda)[\bar{W}^{b*} - \bar{W}^{b*}(\lambda)] \leqslant 1 \tag{30}$$

The equality in the formula above holds if and only if the equality in (7) holds and, moreover, $\bar{W}^{b*}(\lambda) = \bar{W}^{b*}\,(b\in J)$. Considering (2.11.40) we have

$$(\mu - \lambda)[\bar{\eta}^b(\mu), Z^*(\lambda)] = \bar{W}^{b*}(\mu) - \bar{W}^{b*}(\lambda)$$

Therefore when $\mu \neq \lambda$, $[\bar{\eta}^b(\mu), Z^*(\lambda)] = 0$. Whence $[\bar{\eta}^b(\mu), Z^*] = 0$, $\bar{W}^{b*}(\mu) = 0$. Thus $\bar{W}^{b*}(\lambda) = \bar{W}^{b*}$ is equivalent to $\bar{W}^{b*} = 0$.

A simple computation shows that

$$K(\lambda)[\bar{W}(\lambda) - \bar{W}(\mu)]K(\mu) = K(\lambda)[K(\lambda)^{-1} - K(\mu)^{-1}]K(\mu) = K(\mu) - K(\lambda) \tag{31}$$

Applying the above formula we can easily verify that (3.12) holds for the resolvent equation of $\psi(\lambda)$. We have pointed out that (8) is just the norm condition (3.13); therefore $\psi(\lambda)$ is a non-sticky Q process.

(c) Suppose that $\psi(\lambda)$ is of B-type, that is, the B condition is satisfied. We proceed to prove $J_H = \varnothing$. If Q is conservative, the conclusion is trivially true. If Q is non-conservative, by Theorem 2.12.1, $F^a(\lambda) = 0\ (a\in H_e)$ holds in (2.10), and so $F^a(\lambda) = 0\,(a\in J_H)$ in (3.10). But by (28), every $F^a(\mu) \neq 0\,(a\in J)$. Hence surely $J_H = \varnothing$. Or else, if $J_H = \varnothing$, then for every $a\in J = J_B$, $(\lambda I - Q)Z^b(\lambda) = 0$ holds. Thus the B condition of $\psi(\lambda)$ is satisfied, that is, $\psi(\lambda)$ is of B-type.

The F condition of $\psi(\lambda)$ is equivalent to

$$0 = \sum_{a\in J}\sum_{b\in J} Z^a(\lambda)K^{ab}(\lambda)\bar{\eta}^b(\lambda)(\lambda I - Q) = \sum_{a\in J} Z^a(\lambda)\sum_{b\in J} K^{ab}(\lambda)\bar{\alpha}^b$$

Since $Z^a(\lambda)\,(a\in J)$ are linearly independent, $K(\lambda)^{-1}$ exists. The above formula is equivalent to $\bar{\alpha}^b = 0$, $b\in J$. By (8) it is also equivalent to (12). The proof is completed. QED

8.6 GENERAL CONSTRUCTION

Lemma 1. Suppose $a\in J$, and (3.11) and (3.12) hold. If for some $\lambda > 0$, we have

$$S^{aa}(\lambda) \equiv \lambda[F^a(\lambda), Z^a] = 1 \tag{1}$$

then (1) holds for all $\lambda > 0$ and there exists a non-zero and harmonic entrance family $\eta^a(\lambda)$, $\lambda > 0$, such that

$$F^a(\lambda) = \frac{\eta^a(\lambda)}{W^{aa}(\lambda)} \qquad \frac{F^a(\mu)A(\mu,\lambda)}{\| F^a(\mu)A(\mu, v_a)\|} = \eta^a(\lambda) \tag{2}$$

Here v_a is a positive number chosen such that $\|\eta^a(v_a)\| = 1$, and $W^{aa}(\lambda)$ is determined by (5,15).

Proof. By (1) and (3.11) we have $\lambda[F^a(\lambda), Z^b] = 0$ for $b\in J$, $b \neq a$, so that $\lambda[F^a(\lambda), Z^b(\mu)] = 0$. Consequently (3.12) becomes

$$F^a(\lambda)A(\lambda,\mu) = F^a(\mu) + (\mu - \lambda)[F^a(\lambda), Z^a(\mu)]F^a(\mu)$$

Since $F^a(\mu) \neq 0$, it follows that $1 + (\mu - \lambda)[F^a(\lambda), Z^b(\mu)] > 0$. By imitation of the proof of Theorem 3.2.1, we know there exist a non-zero harmonic entrance family $\eta^a(\lambda), \lambda > 0$, and a constant c^a such that

$$F^a(\lambda) = \frac{\eta^a(\lambda)}{c^a + W^{aa}(\lambda)} \qquad \lambda > 0$$

By (1) we know $c^a = 0$ so (1) and the first expression in (2) hold for all $\lambda > 0$. Normalizing $\eta^a(\lambda)$ appropriately so that $\|\eta^a(v_a)\| = 1$, the second expression in (2) holds. The proof is terminated. QED

Theorem 2. (i) Select the set J and $Z^*(\lambda)$, $Z^a(\lambda)$, $a\in J$ and $Z^*, Z^a, a\in J$ according to (i) and (ii) in Theorem 5.1.

(ii) Take a subset L (L may be empty) of J, and set

$$L^a = \begin{cases} 1 & \text{if } a\in L \\ 0 & \text{if } a\in J - L \end{cases} \tag{3}$$

(iii) Take a non-zero and harmonic entrance family $(\bar{\eta}^a(\lambda), \lambda > 0), a\in J$, such that

$$\lim_{\lambda\to\infty} \lambda[\bar{\eta}^a(\lambda), 1 - Z^a] < \infty \qquad a\in J \tag{4}$$

or equivalently, according to the notation in (5.3)–(5.6),

$$\bar{W}^{a*} < \infty \qquad \bar{W}^{ab} < \infty \qquad a, b\in J,\ a \neq b \tag{5}$$

(iv) Take a non-negative $J\times J$ matrix $\bar{S} = (\bar{S}^{ab})$ such that

$$\bar{S}^{aa} = 0 \qquad \bar{S}^{a*} \geqslant \bar{W}^{a*} \qquad \bar{S}^{ab} \geqslant \bar{W}^{ab} \qquad a, b\in J,\ a \neq b \tag{6}$$

$$\bar{\sigma}^a + \bar{W}^{a*} + \sum_{b\in J} \bar{S}^{ab} \leqslant L^a \qquad a\in J \tag{7}$$

(v) Furthermore, the following two equations should hold:

$$(L^b + \bar{W}^{bb})^{-1}\bar{\alpha}^b = 0 \qquad \text{if } b\in J_H \tag{8}$$

$$N^{ab}(L^b + \bar{W}^{bb})^{-1}\bar{\alpha}^b = 0 \qquad \text{if } a\in J_H, b\in J_B \tag{9}$$

Here $J\in J$ matrix $N(\lambda)\downarrow N = (N^{ab})$, $\lambda\uparrow\infty$,

$$N(\lambda) = [I - R(\lambda)]^{-1} = \sum_{n=0}^{\infty} R^n(\lambda) \tag{10}$$

$$R^{aa}(\lambda)=0 \qquad R^{ab}(\lambda)=\frac{\bar{S}^{ab}-\bar{W}^{ab}(\lambda)}{L^a+\bar{W}^{aa}(\lambda)} \qquad (a\neq b) \tag{11}$$

$$N=\sum_{n=0}^{\infty}R^n \qquad R=(R^{ab}) \tag{12}$$

$$R^{aa}=0 \qquad R^{ab}=\frac{\bar{S}^{ab}-\bar{W}^{ab}}{L^a+\bar{W}^{aa}} \qquad (a\neq b) \tag{13}$$

(vi) Let

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+\sum_{a,b\in J}Z_i^a(\lambda)\bar{K}^{ab}(\lambda)\bar{\eta}_j^b(\lambda) \tag{14}$$

where the $J\times J$ matrix $\bar{K}(\lambda)$ is given by

$$\bar{K}(\lambda)=(\bar{K}^{ab}(\lambda))=[\bar{I}-\bar{S}+\bar{W}(\lambda)]^{-1}=N(\lambda)D(\lambda) \tag{15}$$

and $\bar{I}$ is a diagonal matrix with L^a, $a\in J$, as its diagonal elements. $D(\lambda)$ is a diagonal matrix whose diagonal elements are

$$D^{aa}(\lambda)=\frac{1}{L^a+\bar{W}^{aa}(\lambda)} \qquad a\in J \tag{16}$$

Then $\psi(\lambda)$ is a Q process. Every Q process $\psi(\lambda)$ can be obtained in the way stated above. The process is honest if and only if

$$\bar{S}^{a*}=0 \qquad \bar{\sigma}^a+\sum_{b\in J}\bar{S}^{ab}=L^a \qquad a\in J \tag{17}$$

Remark 1

For the process $\psi(\lambda)$ in Theorem 2, if $a\in J-L$, then

$$\lambda[\bar{\eta}^a(\lambda),1]=\bar{W}^{aa}(\lambda)>0 \tag{18}$$

$$F^a(\lambda)=\frac{\bar{\eta}^a(\lambda)}{\bar{W}^{aa}(\lambda)} \tag{19}$$

Actually, from (6) and (7) follows (18). Furthermore, from

$$F^a(\lambda)=\sum_{b\in J}K^{ab}(\lambda)\bar{\eta}^b(\lambda)$$

we obtain

$$\bar{\eta}^a(\lambda)=\sum_{b\in J}[\bar{I}-\bar{S}+\bar{W}(\lambda)]^{ab}F^b(\lambda) \tag{20}$$

By (6) and (7), the above expression becomes $\bar{\eta}^a(\lambda)=\bar{W}^{aa}(\lambda)F^a(\lambda)$; therefore we get (19).

Proof of Theorem 2. (a) Assume $\psi(\lambda)$ to be a Q process. Then (5.13) and (5.14)

still hold. By Lemma 1 we can write

$$L=\{a: a\in J, S^{aa}(\lambda)<1\} \tag{21}$$

Let

$$\delta^a(\lambda)=\begin{cases}1-S^{aa}(\lambda) & \text{if } a\in L\\ \|F^a(\lambda)A(\lambda,v_a)\| & \text{if } a\in J-L\end{cases} \tag{22}$$

Then $\delta^a(\lambda)>0$. Dividing (5.13) and (5.14) by $\delta^a(\lambda)$ we have

$$\sum_{b\in J}\bar{M}^{ab}(\lambda)\sigma^b+\bar{S}^{a*}(\lambda)+\sum_{b\in J}\bar{S}^{ab}(\lambda)\leqslant L^a \tag{23}$$

$$\sum_{b\in J}\bar{M}^{ab}(\lambda)\eta^b(\mu)=\sum_{b\in J}[\bar{I}-\bar{S}(\lambda)+\bar{M}(\lambda)W(\mu)]^{ab}F^b(\mu) \tag{24}$$

where

$$\bar{M}^{ab}(\lambda)=\frac{M^{ab}(\lambda)}{\delta^a(\lambda)} \qquad \bar{S}^{aa}(\lambda)=0 \qquad \bar{S}^{a*}(\lambda)=\frac{S^{a*}(\lambda)}{\delta^a(\lambda)}$$
$$\bar{S}^{ab}(\lambda)=\frac{S^{ab}(\lambda)}{\delta^a(\lambda)} \qquad (b\neq a) \tag{25}$$

satisfies

$$\bar{S}^{ab}(\lambda)=\sum_{c\in J}\bar{M}^{ac}(\lambda)W^{cb}(\lambda) \qquad b\neq a$$
$$\bar{S}^{a*}(\lambda)=\sum_{c\in J}\bar{M}^{ac}(\lambda)W^{c*}(\lambda) \tag{26}$$

Select a subsequence $\lambda\to\infty$ from the sequence μ_n of Lemma 4.2, such that for all $a,b\in J$ the following hold:

$$\bar{M}^{ab}(\lambda)\to M^{ab} \qquad \bar{S}^{ab}(\lambda)\to\bar{S}^{ab} \qquad \bar{S}^{a*}(\lambda)\to\bar{S}^{a*} \tag{27}$$

By (23), (26) and Fatou's lemma we obtain

$$\sum_{b\in J}\bar{M}^{ab}\sigma^b+\bar{S}^{a*}+\sum_{b\in J}\bar{S}^{ab}\leqslant L^a \tag{28}$$

$$\bar{S}^{aa}=0 \qquad \bar{S}^{ab}\geqslant\sum_{c\in J}\bar{M}^{ac}W^{cb} \qquad (b\neq a)$$
$$\bar{S}^{a*}\geqslant\sum_{c\in J}\bar{M}^{ac}W^{c*} \tag{29}$$

(setting $\infty\cdot 0=0\cdot\infty=0$).

Obviously $\bar{S}^{a*}$ $(a\in J)$ and the $J\times J$ matrix $\bar{S}=(\bar{S}^{ab})$ are finite. We proceed now to prove that the $J\times J$ matrix $\bar{M}=(\bar{M}^{ab})$ is finite.

When $a\in J-L$, by (4.6), $\bar{M}^{ab}=H^{ab}<\infty$ $(b\in J)$. Suppose $a\in L$, below. Let

$$\Lambda=\left\{c: c\in J, \eta^c(\lambda)\neq 0, \sigma^c+W^{c*}+\sum_{b\neq c}W^{cb}=0\right\}$$

If $\eta^c(\mu)=0$, we have $\bar{M}^{ac}=0\,(a\in J)$ from the fact that $M^{ac}(\lambda)=0\,(a\in J)$. If $\eta^c(\mu)\neq 0$, and moreover $c\in J-\Lambda$, then by (5.25) and (5.26) we have

$$\sum_{c\in J}\bar{M}^{ac}\left(\sigma^c+W^{c*}+\sum_{b\neq c}W^{cb}\right)\leqslant\sum_{c\in J}\bar{M}^{ac}\sigma^c+\bar{S}^{a*}+\sum_{b\in J}\bar{S}^{ab}\leqslant L^a.$$

so that $\bar{M}^{ac}<\infty$ for $c\in J-\Lambda$, and $a\in J$.

Note that when $c\in\Lambda$,

$$\mu[\eta^c(\mu),1-Z^a]=\sigma^c+W^{c*}(\mu)+\sum_{b\neq a}W^{cb}(\mu)=0$$

so that

$$\mu[\eta^c(\mu),1]=W^{ca}(\mu)>0$$

Hence

$$\mu[\eta^c(\mu)-W^{ca}(\mu)F^a(\mu),Z^a]=W^{ca}(\mu)\delta^a(\mu)>0 \tag{30}$$

Since

$$\sum_{b\in J}\sum_{c\in J}\bar{M}^{ac}(\lambda)W^{cb}(\mu)F^b(\mu)=\sum_{c\in J-\Lambda}\bar{M}^{ac}(\lambda)\sum_{b\in J}W^{cb}(\mu)F^b(\mu)+\sum_{c\in\Lambda}\bar{M}^{ac}(\lambda)W^{ca}(\mu)F^a(\mu)$$

(24) can be rewriten as

$$\sum_{c\in\Lambda}\bar{M}^{ac}(\lambda)[\eta^c(\mu)-W^{ca}(\mu)F^a(\mu)]$$

$$=F^a(\mu)-\sum_{b\neq a}\bar{S}^{ab}(\lambda)F^b(\mu)+\sum_{c\in J-\Lambda}\bar{M}^{ac}(\lambda)\left(\sum_{b\in J}W^{cb}(\mu)F^b(\mu)-\eta^c(\mu)\right)$$

Multiplying the above expression on both sides by μZ^a, we have

$$\sum_{c\in\Lambda}\bar{M}^{ac}(\lambda)W^{ca}(\mu)\delta^a(\mu)=S^{aa}(\mu)-\sum_{b\neq a}\bar{S}^{ab}(\lambda)S^{ba}(\mu)+\sum_{c\in J-\Lambda}\bar{M}^{ac}(\lambda)\sum_{b\in J}[W^{cb}(\mu)S^{ba}(\mu)-W^{ca}(\mu)]$$

When $\lambda\to\infty$, the limit on the right-hand side is finite, and so is the limit on the left. Noting (30) we have $\bar{M}^{ac}<\infty\,(c\in\Lambda)$. Thus we have completed the proof of $(\bar{M}^{ab})$ being finite.

Letting $\lambda\to\infty$ in (24) we have

$$\sum_{b\in J}\bar{M}^{ab}\eta^b(\mu)=\sum_{b\in J}[\bar{I}-\bar{S}+\bar{M}W(\mu)]^{ab}F^b(\mu) \tag{31}$$

If we write

$$\bar{\eta}^a(\mu)=\sum_{b\in J}\bar{M}^{ab}\eta^b(\mu) \tag{32}$$

then (28), (29) and (31) become (6), (7) and

$$\bar{\eta}^a(\mu)=\sum_{b\in J}[\bar{I}-\bar{S}+\bar{W}(\mu)]^{ab}F^b(\mu) \tag{33}$$

We proceed now to prove that the inverse matrix

$$\bar{K}(\mu)=[\bar{I}-\bar{S}+\bar{W}(\mu)]^{-1} \tag{34}$$

exists and is non-negative.

Actually, the diagonal elements of $\bar{I}-\bar{S}+\bar{W}(\mu)$ are non-negative, while the non-diagonal elements are non-positive. In order to prove that the inverse matrix exists, we only need to prove that the row sums are bigger than zero. When $a\in J-L$, by (6) and (7) we know the row sum is equal to $\bar{W}^{aa}(\mu)=\mu[\bar{\eta}^a(\mu),1]>0$. When $a\in L$ surely $\bar{\eta}^a(\mu)\neq 0$. Because if $\bar{\eta}^a(\mu)=0$, then $F^a(\mu)=\sum_{b\in J}\bar{S}^{ab}F^b(\mu)$, that is, (3.7) and (3.8) hold for $h^{ab}=\bar{S}^{ab}$. But by Lemma 3.2, this is impossible. Therefore when $a\in L$, we have $\bar{\eta}^a(\mu)\neq 0$. Thus by (6) and (7), the row sum

$$1-\sum_{b\in J}\bar{S}^{ab}+\sum_{b\in J}\bar{W}^{ab}$$

$$\geqslant 1-\left(\bar{\eta}^a+\bar{S}^{a*}+\sum_{b\in J}\bar{S}^{ab}\right)+\bar{\sigma}^a+W^*+\sum_{b\in J}\bar{W}^{ab}(\mu)\geqslant\mu[\bar{\eta}^a(\mu),\mathbf{1}]>0$$

Hence (15) holds, and from (33) we find that

$$F^a(\lambda)=\sum_{b\in J}\bar{K}^{ab}(\lambda)\eta^b(\lambda) \tag{35}$$

So (14) is proved.

(b) By (15) and (10)–(13), the Q condition is equivalent to

$$\sum_{b\in J}N^{ab}(L^b+\bar{W}^{bb})^{-1}\bar{\alpha}^b=0 \qquad a\in J_H \tag{36}$$

Since $N^{aa}\geqslant 1\,(a\in J_H)$, it follows that the above expression is equivalent to (8) and (9).

(c) Suppose $\psi(\lambda)$ is determined by (i)–(vi) in the theorem.

Noting $\bar{W}(\lambda)=K(\lambda)^{-1}-\bar{I}-\bar{S}$, and imitating the proof of the norm condition in Theorem 5.1, we know $\lambda(F^a(\lambda),\mathbf{1}]\leqslant 1$, and furthermore equality holds if and only if (17) holds.

But (5.31) still holds for $\bar{K}(\lambda)$ in (34), and hence we come to the conclusion that the resolvent equation of $\psi(\lambda)$ holds. We have already pointed out that (8) and (9) are equivalent to the Q condition. Therefore $\psi(\lambda)$ is a Q process. The theorem is proved. QED

Theorem 3. Suppose that $\psi(\lambda)$ is the Q process in Theorem 2. Then $\psi(\lambda)$ is of B-type if and only if $J_H=\varnothing$ and of F-type if and only if

$$\bar{\alpha}^a=0 \qquad (a\in J)$$

Proof. The same as in Theorem 5.1. QED

Theorem 4. If $\psi(\lambda)$ is the Q process of Theorem 2, then $\psi(\lambda)$ is non-sticky if and only if

$$\lim_{\lambda\to\infty} \lambda\|\bar{\eta}^a(\lambda)\| < \infty \qquad a\in J \tag{37}$$

Proof. Suppose (37) holds. According to Definition 5.1, $\psi(\lambda)$ is non-sticky. Conversely, suppose $\psi(\lambda)$ is non-sticky, that is $\psi(\lambda)$ has the representations (3.9) and (4.3). Moreover

$$\lim_{\lambda\to\infty} \lambda\|\eta^a(\lambda)\| < \infty \qquad a\in J \tag{38}$$

$\psi(\lambda)$ has the representations (5.9) and (6.14). As $Z^a(\lambda)$, $a\in J$, are linearly independent, so

$$\sum_{b\in J} K^{ab}(\lambda)\eta^b(\lambda) = \sum_{b\in J} \bar{K}^{ab}(\lambda)\bar{\eta}^b(\lambda) \qquad a\in J$$

Hence

$$\bar{\eta}^a(\lambda) = \sum_{b\in J} [\bar{K}(\lambda)^{-1}K(\lambda)]^{ab}\eta^b(\lambda)$$

Multiplying the above formula from the right by $A(\lambda,\mu)$, we find that

$$\begin{aligned}\bar{\eta}^a(\mu) &= \sum_{b\in J} [\bar{K}(\lambda)^{-1}K(\lambda)]^{ab}\eta^b(\mu)\\ \mu\|\bar{\eta}^a(\mu)\| &= \sum_{b\in J} [\bar{K}(\lambda)^{-1}K(\lambda)]^{ab}\mu\|\eta^b(\mu)\|\end{aligned} \tag{39}$$

By the above expression and (38) we obtain (37). The proof is completed.

QED

Remark 2

In the representation (14) of the Q process $\psi(\lambda)$ if $\bar{\eta}^a(\lambda)$, $a\in J$, and $\tilde{\eta}^a(\lambda)$, $a\in J$, correspond to the same $\psi(\lambda)$, from the proof (39) of Theorem 4 we see that there exists a $J\times J$ matrix $R=(r_{ab})$ such that the inverse matrix of R exists and, moreover,

$$\bar{\eta}^a(\lambda) = \sum_{b\in J} r_{ab}\tilde{\eta}^b(\lambda) \qquad \lambda>0$$

8.7 EQUIVALENT CONSTRUCTION

Theorem 1. (i) The same as (i) in Theorem 6.2.

(ii) Take a non-zero entrance family $(\eta^a(\lambda), \lambda>0)$, $a\in J$, such that

$$\lim_{\lambda\to\infty} \lambda[\eta^a(\lambda), 1-Z^a] < \infty \qquad a\in J \tag{1}$$

or equivalently

$$W^{a*}<\infty \qquad W^{ab}<\infty\ (b\neq a) \qquad a\in J \tag{2}$$

(iii) Take a $J\times J$ matrix $T=(T^{ab})$ such that

$$\sigma^a \leqslant -W^{a*} + \sum_{b\in J} T^{ab} \tag{3}$$

$$W^{ab} \leqslant -T^{ab} \qquad (b\neq a) \tag{4}$$

$$(T^{bb}+W^{bb})^{-1}\bar{\alpha}^b = 0 \qquad b\in J_H \tag{5}$$

$$F^{ab}(T^{bb}+W^{bb})^{-1}\bar{\sigma}^b = 0 \qquad a\in J_H,\quad b\in J_B \tag{6}$$

Here the $J\times J$ matrix

$$F(\lambda) = (F^{ab}(\lambda))\downarrow F = (F^{ab}) \qquad \lambda\uparrow\infty$$

and the $J\times J$ matrix

$$F(\lambda) = [I-V(\lambda)]^{-1} = \sum_{n=0}^{\infty} V(\lambda)^n \downarrow F = \sum_{n=0}^{\infty} V^n \qquad \lambda\uparrow\infty$$

while

$$V(\lambda) = (V^{ab}(\lambda))\downarrow V = (V^{ab}) \qquad \lambda\uparrow\infty$$

$$V^{aa}(\lambda)=0 \qquad V^{ab}(\lambda) = \frac{(-T^{ab})-W^{ab}(\lambda)}{T^{aa}+W^{aa}(\lambda)} \qquad (b\neq a)$$

$$V^{aa}=0 \qquad V^{ab} = \frac{(-T^{ab})-W^{ab}}{T^{aa}+W^{aa}} \qquad (b\neq a) \tag{7}$$

(iv) Set

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \sum_{a\in J}\sum_{b\in J} Z_i^a(\lambda)K^{ab}(\lambda)\eta_j^b(\lambda) \tag{8}$$

where

$$K(\lambda) = [T+W(\lambda)]^{-1} = [I-V(\lambda)]^{-1}E(\lambda) \tag{9}$$

and

$$E(\lambda) = \mathrm{diag}\{[T^{aa}+W^{aa}(\lambda)]^{-1}\} \tag{10}$$

Then $\psi(\lambda)$ is a non-minimal Q process. Any non-minimal Q process can be obtained according to the above method.

Remark

When Q is conservative the $\psi(\lambda)$ in Theorem 1 is precisely the construction in Williams (1964, 1966).

Theorem 2. Assume $\psi(\lambda)$ to be the Q process having the form of Theorem 6.2.

Let

$$T = \bar{I} - \bar{S} \qquad \eta^a(\lambda) = \bar{\eta}^a(\lambda) \qquad a \in J \tag{11}$$

Then $\psi(\lambda)$ is a Q process having the form of Theorem 1. Conversely suppose that $\psi(\lambda)$ is a Q process having the form of Theorem 1. Let

$$L = \{a : a \in J,\ T^{aa} > 0\} \tag{12}$$

$$\begin{aligned} &\bar{S}^{a*} = 0 \qquad \bar{S}^{ab} = 0\ (b \in J) \qquad \text{if } a \in J - L \\ &\bar{S}^{a*} = \frac{W^{a*}}{T^{aa}} \qquad \bar{S}^{aa} = 0 \qquad \bar{S}^{ab} = \frac{-T^{ab}}{T^{aa}} (b \neq a) \qquad \text{if } a \in L \end{aligned} \tag{13}$$

$$\bar{\eta}^a(\lambda) = \begin{cases} \eta^a(\lambda) & \text{if } a \in J - L \\ \eta^a(\lambda)/T^{aa} & \text{if } a \in L \end{cases} \tag{14}$$

Then $\psi(\lambda)$ is a Q process having the form of Theorem 6.2.

Proof of Theorems 1 and 2. Suppose that $\psi(\lambda)$ has the form of Theorem 1. Determining L, $\bar{S}^{a*}$, $\bar{S} = (\bar{S}^{ab})$, $\bar{\eta}^a(\lambda)$ $(a \in J)$ according to (12)–(14), we find that $\psi(\lambda)$ has the form of Theorem 6.2. Thus $\psi(\lambda)$ is a Q process.

Provided $\psi(\lambda)$ is a Q process, according to Theorem 6.2, it surely has the form of Theorem 6.2. And by determining $T = (T^{ab})$ and $\eta^a(\lambda), a \in J$, according to (11), we see that $\psi(\lambda)$ has the form of Theorem 1. The proof is completed.

QED

8.8 REMARKS ON THE CONSTRUCTION OF THE NON-BIFINITE Q PROCESS

Remark 1

Suppose that $A_e = H_e \cup B_e$ is an infinite set. Take a decomposition $A_e = a_1 \cup a_2 \cup \cdots$. The set $A = \{a_1, a_2, \ldots\}$ may be an infinite set. Take a non-empty subset J of A. So long as the subset J is finite, or J is finite, but if only we define $K(\lambda)$ in Theorem 5.1 according to (5.27) and define $\bar{K}(\lambda)$ in Theorem 6.2 as follows:

$$\bar{K}(\lambda) = \left\{ \sum_{n=0}^{\infty} R^n(\lambda) \right\} D(\lambda)$$

$R(\lambda) = (R^{ab}(\lambda))$ as in (6.11), the diagonal form matrix $D(\lambda)$ as in (6.16), $K(\lambda)$ in Theorem 7.1 is defined by the formula below:

$$K(\lambda) = \left\{ \sum_{n=0}^{\infty} V(\lambda)^n \right\} E(\lambda)$$

$V(\lambda) = (V^{ab}(\lambda))$ as in (7.7), diagonal form $E(\lambda)$ as in (7.10), then the $\psi(\lambda)$ obtained according to Theorems 5.1, 6.2 and 7.1 are all non-minimal Q processes.

Remark 2

Suppose that H_e is infinite and B_e is finite, and that after taking a decomposition of A_e, the set $A = \{a_1, a_2, \ldots\}$ may still be infinite. If only we select a non-empty subset J of A such that $J_H = \varnothing$ (at this moment, J must be a finite set), the $\psi(\lambda)$ constructed according to Theorems 5.1, 6.2 and 7.1 are non-minimal Q processes satisfying the system of backward equations, and moreover at this moment, the Q processes constructed by Theorem 6.2 or Theorem 7.1 have already exhausted all the non-minimal Q processes satisfying the sysem of backward equations.

PART IV PATH STRUCTURE OF DENUMERABLE MARKOV PROCESSES

CHAPTER 9

The W Transformation and Strong Limit

9.1 INTRODUCTION

Professor Zi-kun Wang introduced the transforms g_n and f_n, and the concept of strong limit for the processes, during his study of construction theory of birth–death processes, and successfully solved the construction problem of birth–death processes by using probability methods. This will be seen in Zi-kun Wang (1958, 1962) and Zi-kun Wang and Xiang-qun Yang (1978, 1979). Their basic idea is to transform general processes with more complex paths into processes with simpler paths; then they proved that a general process is a strong limit of simpler processes. Professor Zhen-ting Hou applied transformations to general denumerable Markov processes. The W transformation is the further extension of the transformations g_n and f_n. Therefore, the results about the transformations g_n and f_n in Zi-kun Wang (1962), Zi-kun Wang and Xiang-qun Yang (1978, 1979) and Zhen-ting Hou (1975) may all be obtained as special cases of our present fundamental results. The results in this chapter are taken from Xiang-qun Yang (1978, 1980a).

9.2 DEFINITION OF W TRANSFORMATION

We first discuss the W transformation for a function. Let $X = \{x(t), t < \sigma\}$ be a function that has a domain of values $\bar{E} = E \cup \{\infty\}$ and is defined in $[0, \sigma)$ $(\sigma \leqslant \infty)$. Then $\{\tau, \beta\}$ is called a pair for X, if

$$0 = \tau_0 \leqslant \beta_0 \leqslant \tau_1 \leqslant \beta_1 \leqslant \cdots \leqslant \sigma \tag{1}$$

For a pair $\{\tau, \beta\}$, we set

$$\sigma_n = \begin{cases} 0 & \text{if } n = 0 \\ \sum_{k=1}^{n} (\tau_k - \beta_{k-1}) & \text{if } n > 0 \end{cases} \tag{2}$$

$$\delta_n = \sum_{k=0}^{n} (\beta_k - \tau_k) \qquad n \geqslant 0 \tag{3}$$

$$\bar{\sigma} = \lim_{n\to\infty} \sigma_n \qquad \bar{\delta} = \lim_{n\to\infty} \delta_n \tag{4}$$

whereas for $t \in [0, \bar{\sigma})$, we set

$$a_t = \beta_n + (t - \sigma_n) \qquad \text{if } t \in [\sigma_n, \sigma_{n+1}) \tag{5}$$

$$\bar{x}(t) = x(a_t) \qquad t \in [0, \bar{\sigma}) \tag{6}$$

Definition 1. The transformation that transforms the function $X = \{x(t), t < \sigma\}$ into the function $\bar{X} = \{\bar{x}(t), t < \bar{\sigma}\}$ is called a W transformation for X, to be denoted by $W_{\tau,\beta}$. Hence $\bar{X} = W_{\tau,\beta}(X)$.

Intuitively speaking, we throw out those sections of the function X corresponding to the intervals $[\tau_i, \beta_i)$, reserve those sections corresponding to $[\beta_i, \tau_{i+1})$ and move them towards the left so that the section on $[\beta_0, \tau_1)$ gets to $[0, \tau_1 - \beta_0)$, and link up all other sections on $[\beta_i, \tau_{i+1})$ $(i = 1, 2, \ldots)$ according to their original order without intersecting them, and the function thus obtained is called X.

We may define the W transformation for a process X, in the light of the W transformation for a function.

Definition 2. Let $X = \{x(t), t < \sigma\} \in \mathscr{H}$. Assume that for each $\omega \in \Omega$, there exists a pair $\{\tau(\omega), \beta(\omega)\}$, and we may determine $W_{\tau(\omega),\beta(\omega)}(X(\omega)) = \bar{X}(\omega) = \{\bar{x}(t, \omega), t < \bar{\sigma}(\omega)\}$. We call $\bar{X}$ the W transformation for the process X and write $\bar{X} = W_{\tau,\beta}(X)$.

9.3 STRONG LIMIT THEOREM

9.3.1 In the case that X is a function

Definition 1. A set A is assumed to be a Lebesgue-measurable set in $[0, \infty)$. Then $[\lambda, \eta)$ is called a composition interval of A if the following conditions are satisfied:

(i) $(\lambda, \eta) \subset A$

(ii) maximality: if $[\bar{\lambda}, \bar{\eta}) \supset [\lambda, \eta)$, and $(\bar{\lambda}, \bar{\eta}) \subset A$, then

$$[\bar{\lambda}, \bar{\eta}) = [\lambda, \eta)$$

We denote the collection of all the composition intervals of A by $\mathscr{U}(A)$. Put

$$C_1(A) = \bigcup_{[\lambda,\eta) \in \mathscr{U}(A)} (\lambda, \eta) \tag{1}$$

$$C_2(A) = \bigcup_{[\lambda,\nu) \in \mathscr{U}(A)} [\lambda, \eta) \tag{2}$$

$$\begin{aligned} \bar{C}_2^+(A) &= \{t \mid \text{there exist strictly decreasing } t_n \in C_2(A) \text{ such that } t_n \downarrow t\} \\ &= \{t \mid \text{there exist non-increasing } t_n \in C_2(A) \text{ such that } t_n \downarrow t\} \end{aligned} \tag{3}$$

$\bar{C}_2^+(A)$ is called the right closure of set A.

Definition 2. A transformation $\gamma: u \to \gamma(u)$ is said to be the one determined by the pair $\{\tau, \beta\}$ or by the sequence of pairs $\{\tau^n, \beta^n\}$ $(n \geq 1)$, if

$$\gamma(u) = L\{A \cap [0, u)\} \tag{4}$$

where L is the Lebesgue measure, and

$$A = \begin{cases} \bigcup_{k=0}^{\infty} [\beta_k, \tau_{k+1}) & \text{for the pair } \{\tau, \beta\} \\ \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{\infty} [\beta_k^n, \tau_{k+1}^n) & \text{for the sequence of pairs } \{\tau^n, \beta^n\} \end{cases} \tag{5}$$

Definition 3. A function $X = \{x(t), t < \sigma\}$ is said to be right-continuous, if

$$\lim_{s \downarrow t} x(s) = x(t) \in \bar{E} \qquad \text{for all } t < \sigma \tag{6}$$

Theorem 1. Let $X = \{x(t), t < \sigma\}$ be a right-continuous function, and that there exists a sequence of pairs $\{\tau^n, \beta^n\}$ $(n \geq 1)$ such that

$$A^n = \bigcup_{k=0}^{\infty} [\beta_k^n, \tau_{k+1}^n) \uparrow A \tag{7}$$

Let v^n and v denote transformations determined by the pair $\{\tau^n, \beta^n\}$ for the given n and the sequence of pairs $\{\tau^n, \beta^n\}$ $(n \geq 1)$ respectively. Put

$$\tau_k^{n,m} = \gamma^n(\tau_k^m) \qquad \beta_k^{n,m} = \gamma^n(\beta_k^m) \qquad m < n \tag{8}$$

$$W_{\beta^n,\tau^n}(X) = X^n = \{x^n(t), t < \sigma^n\} \tag{9}$$

Then:

(i) $\{\tau^{n,m}, \beta^{n,m}\}$ $(m < n)$ is a pair for X^n, and

$$X^m = W_{\tau^{n,m},\beta^{n,m}}(X^n) \qquad m < n \tag{10}$$

(ii) As $n \to \infty$ there exist limits

$$\sigma^n \uparrow \bar{\sigma} = L(A) \tag{11}$$

$$\lim_{n\to\infty} x^n(t) = x(\alpha_t) \qquad \text{for all } t \in [0, \bar{\sigma}) \tag{12}$$

where α_t is the unique solution of the equation

$$L\{C_2(A)\cap[0,u)\}=t \quad u\in\bar{C}_2^+(A) \tag{13}$$

(iii) $\bar{X}=\{x(\alpha_t),t<\bar{\sigma}\}$ is right-continuous.

The intuitive meaning of this theorem is as follows: Suppose X becomes X^m when transformation W_{τ^m,β^m} is performed and, moreover, for $m<n$, the sections reserved when W_{τ^m,β^m} transformation is performed are still reserved as W_{τ^n,β^n} transformation is carried out. Then (10) indicates that transformation of X into X^m can be completed in two steps. First, we throw out the sections of X corresponding to $\bigcup_{i=0}^{\infty}[\tau_i^n,\beta_i^n)$, reserve the sections corresponding to $\bigcup_{i=0}^{\infty}[\beta_i^n,\tau_{i+1}^n)$, and, moreover, translate them towards the left according to their original order, then connect them up without intersecting them so that we have X^n. In this process those sections corresponding to $[\beta_k^m,\tau_k^m)$ are preserved and after they move to the left β_k^m becomes $\beta_k^{n,m}$ while τ_k^m changes into $\tau_k^{n,m}$. Secondly, we abandon the sections of X^n corresponding to $\bigcup_{i=0}^{\infty}[\tau_i^{n,m},\beta_i^{n,m})$, and reserve the sections corresponding to $\bigcup_{i=0}^{\infty}[\beta_i^{n,m},\tau_{i+1}^{n,m})$, and, furthermore, translate them towards the left according to their original order and link them up without rendering them intersected; thus we obtain X^m.

Conclusion (iii) shows: reserve all the sections of X corresponding to $A=\bigcup_{n=1}^{\infty}\bigcup_{k=0}^{\infty}[\beta_k^n,\tau_{k+1}^n)$, abandon the remaining sections and, moreover, translate the reserved sections towards the left in their original order and connect them up without intersecting them, and the function obtained thus is precisely $\bar{X}$. In this process of translation α_t turns into t. $\bar{X}$ is also a right-continuous function.

Conclusion (ii) indicates that the limit function of X^n is $\bar{X}$.

The proof of the theorem is left to section 9.4.

9.3.2 In the case that X is a process

Definition 4. Let $X=\{x(t),t<\sigma\}$, $X^n=\{x^n(t),t<\sigma^n\}$ $(n\geq 1)$ belong to $\mathscr{H}$. X is called the strong limit of a sequence of X^n and we write $X=\lim_{n\to\infty}X^n$, if for almost all ω

$$\sigma^n(\omega)\uparrow\sigma(\omega)$$

$$\lim_{n\to\infty}x^n(t,\omega)=x(t,\omega) \qquad \text{for all } t<\sigma(\omega) \tag{14}$$

By definition, obviously, we have the following.

Theorem 2. If $X,X^n\in\mathscr{H}$, $\lim_{n\to\infty}X^n=X$, then for all $i,j\in E$, $t\geq 0$,

$$\lim_{n\to\infty}P_{ij}^n(t)=P_{ij}(t) \tag{15}$$

where $\{P_{ij}^n(t)\}$, $\{P_{ij}(t)\}$ are the transition probabilities of X^n and X, respectively.

Definition 5. Suppose that $X\in\mathscr{H}$, that $\beta_0\leq\tau_1\leq\beta_1$ are Markov times relative to X, satisfying $P(\beta_0<\sigma)>0$, and that the following hold:

$$\beta_0+\theta_{\beta_0}\tau_1=\tau_1, \qquad \beta_0+\theta_{\beta_0}\beta_1=\beta_1 \tag{16}$$

on $(\beta_0<\tau_1)$, where θ is a transition operator. Assume that τ_n,β_n have been determined already. If $\beta_n=\sigma$, we define $\tau_{n+1}=\beta_{n+1}=\sigma$, otherwise we define

$$\tau_{n+1}=\beta_n+\theta_{\beta_n}\tau_1 \qquad \beta_{n+1}=\beta_n+\theta_{\beta_n}\beta_1 \tag{17}$$

We call such a pair $\{\tau,\beta\}$ a canonical one.

Recall the class $\mathscr{H}_s$ of processes in which the Q matrices are finite and the class $\mathscr{H}_D$ of D-type processes, and we have the following.

Theorem 3. Let $X\in\mathscr{H}_s$, and $\{\tau,\beta\}$ be a canonical pair satisfying the following conditions:

(a) $P\{t+\theta_t\tau_1=\tau_1 \text{ for all } t\in[\beta_0,\tau_1)\}=1$

$$P\{t+\theta_t\beta_1=\beta_1 \text{ for all } t\in[\beta_0,\tau_1)\}=1 \tag{18}$$

(b) For α_t determined by (2.5), we have

$$P\{x(\alpha_t)=\infty\}=0 \qquad t\geq 0 \tag{19}$$

Then the following conclusion holds true:

(i) A sufficient condition for (b) is that the following condition (c) holds, that is:

(c) $P\{x(t)\in E \text{ for all } t\in[\beta_0,\tau_1)\}=1$

$$P\{x(\beta_1)=\infty\}=0 \tag{20}$$

(ii) If conditions (a) and (b) are satisfied, then $\bar{X}=W_{\tau,\beta}(X)\in\mathscr{H}_s$.

(iii) If conditions (a) and (c) are satisfied, then $\bar{X}=W_{\tau,\beta}(X)\in\mathscr{H}_D$.

The proof of the theorem is left to section 9.6.

Theorem 4. Suppose that $X\in\mathscr{H}_s$, that for every $n\geq 1$ there exists a canonical pair $\{\tau^n,\beta^n\}$ satisfying the conditions (a) and (c), that

$$P\{x(\inf\beta_0^n)=\infty\}=0 \tag{21}$$

and that for almost all $\omega\in\Omega$ the following holds:

$$A^n(\omega)=\bigcup_{k=0}^{\infty}[\beta_k^n(\omega),\tau_{k+1}^n(\omega))\uparrow A(\omega) \tag{22}$$

Then

(i) $X^n = W_{\tau^n,\beta^n}(X) \in \mathscr{H}_D$ and (10) is satisfied, where $\tau_k^{n,m}, \beta_k^{n,m}$ $(m < n, k \geqslant 0)$ are

$$\tau_k^{n,m} = L\{A^n \cap [0, \tau_k^m)\} \qquad \beta_k^{n,m} = L\{A^n \cap [0, \beta_k^m)\} \tag{23}$$

(ii) There exists a strong limit of X^n $(n \geqslant 1)$

$$\lim_{n\to\infty} X^n = \bar{X} \tag{24}$$

where $\bar{X} = \{x(\alpha_t), t < \bar{\sigma}\} \in \mathscr{H}_s$, $\bar{\sigma}(\omega) = L\{A(\omega)\}$, $\alpha_t(\omega)$ is the unique solution of the equation

$$\begin{gathered} L\{C_2(A(\omega)) \cap [0, u)\} = t \\ u \in \bar{C}_2^+(A(\omega)) \end{gathered} \tag{25}$$

The proof of the theorem is left to section 9.6.

9.4 THE PROOF OF THEOREM 3.1

Assume that $\{\tau, \beta\}$ is a pair of the function $X = \{x(t), t < \sigma\}$. Set

$$A = \bigcup_{k=0}^{\infty} [\beta_k, \tau_{k+1}) \qquad B = \bigcup_{k=0}^{\infty} [\tau_k, \beta_k) \tag{1}$$

$$\gamma(u) = L\{A \cap [0, u)\} \qquad \rho(u) = L\{B \cap [0, u)\} \qquad u < \sigma \tag{2}$$

Obviously, we have

Lemma 1. (i) $\gamma(u) + \rho(u) = u$

(ii) $\gamma(u)$ and $\rho(u)$ are non-decreasing continuous functions.

(iii) If $u \in A, u < t < \sigma$, then $\gamma(u) < \gamma(t)$. If $u \in B, u < t < \sigma$, then $\rho(u) < \rho(t)$.

(iv) If $u \in [\beta_k, \tau_{k+1})$, then $\rho(u) = \rho(\beta_k) = \delta_k$. If $u \in [\tau_k, \beta_k)$, then $\gamma(u) = \gamma(\tau_k) = \sigma_k$.

(v) For $t \in [0, \bar{\sigma})$, $\bar{\sigma} = L(A) = \gamma(\sigma)$, the quantity α_t in (2.5) is the unique solution of the equation

$$\gamma(u) = t \qquad u \in A \tag{3}$$

Lemma 2. When $u < \sigma$, the following conditions are equivalent to each other:

(i) $\alpha_t < u$

(ii) $\gamma(u) > t$

(iii) $\rho(u) < u - t$

Proof. By (i) in Lemma 1, we know that (ii) and (iii) in this lemma are equivalent.
Let $\alpha_t < u$; (v) in Lemma 1 leads to $\alpha_t \in A$, $\gamma(\alpha_t) = t$, whereas (iii) in Lemma 1 leads to $\gamma(u) > \gamma(\alpha_t) = t$. Conversely, let $u \leqslant \alpha_t$. From (ii) in Lemma 1 we have $\gamma(u) \leqslant \gamma(\alpha_t) = t$. Therefore, (i) and (ii) in this lemma are equivalent, and the proof is completed. QED

Lemma 3. Let $\{\tau, \beta\}$ and $\{\tau', \beta'\}$ be two pairs for the function X. What follows contains notations $'$ or $\tilde{}$ whose meanings are quite clear. Assume that

$$A \subset A' \qquad (\text{or } B \supset B') \tag{4}$$

Put $\bar{X} = W_{\tau,\beta}(X)$, $\bar{X}' = W_{\tau',\beta'}(X)$. Then

(i) $\bar{\sigma} \leqslant \bar{\sigma}'$ and for $t \in [0, \bar{\sigma})$, we have $\alpha_t' \leqslant \alpha_t$.

(ii) Set

$$\tilde{\tau}_k = L\{A' \cap [0, \tau_k)\} \qquad \tilde{\beta}_k = L\{A' \cap [0, \beta_k)\} \tag{5}$$

Then $\{\tilde{\tau}, \tilde{\beta}\}$ is a pair for $\bar{X}'$ and, moreover,

$$\bar{X} = W_{\tilde{\tau},\tilde{\beta}}(\bar{X}') \tag{6}$$

Remark

The intuitive meaning of this lemma goes as follows. If what is abandoned in the process of carrying out transformation $W_{\tau,\beta}$ for X is more than what is thrown out in the process of performing transformation $W_{\tau'\beta'}$, then $\bar{X} = W_{\tau,\beta}(X)$ can be derived by abandoning certain sections in two steps. To begin with, conduct transformation $W_{\tau',\beta'}$ for X, that is, abandon the sections of X corresponding to B', reserve those sections corresponding to A' and, furthermore, move them to the left, and link them up so that $\bar{X}'$ is obtained. In this process of shift, τ^k, β_k become $\tilde{\tau}_k, \tilde{\beta}_k$. Then apply transformation $W_{\tilde{\tau},\tilde{\beta}}$ to $\bar{X}'$, that is, abandon the sections of $\bar{X}'$ corresponding to $\tilde{B} = \bigcup_{k=0}^{\infty} [\tilde{\tau}_k, \tilde{\beta}_k)$, reserve those sections corresponding to $\tilde{A} = \bigcup_{k=0}^{\infty} [\tilde{\beta}_k, \tilde{\tau}_{k+1})$ and, moreover, translate them to the left, and connect them up so that we have $\bar{X}$.

Proof of Lemma 3. From (4) we have $\bar{\sigma} = L(A) \leqslant L(A') = \bar{\sigma}'$. If $\alpha_t < \alpha_t'$ by using (iii) and (v) in Lemma 1 we have $\alpha_t \in A \subset A', \alpha_t \in A'$, whence $t = L\{A \cap [0, \alpha_t)\} \leqslant L\{A' \cap [0, \alpha_t)\} < L\{A' \cap [0, \alpha_t')\} = t$.
There exists an inconsistency between them. Therefore $\alpha_t' \leqslant \alpha_t$.
Obviously, $\{\tilde{\tau}, \tilde{\beta}\}$ is a pair for $\bar{X}'$. Suppose $W_{\tilde{\tau},\tilde{\beta}}(\bar{X}') = \tilde{X} = \{\tilde{x}(t), t < \tilde{\sigma}\}$. Then

$$\tilde{\sigma}_n = \sum_{k=1}^{n} (\tilde{\tau}_k - \tilde{\beta}_{k-1}) = \sum_{k=1}^{n} L\{A' \cap [\beta_{k-1}, \tau_k)\} = L\left\{A' \cap \left(\bigcup_{k=1}^{n} [\beta_{k-1}, \tau_k)\right)\right\}$$

By (4), $\tilde{\sigma}_n = L\{\bigcup_{k=1}^{n} [\beta_{k-1}, \tau_k)\} = \sigma_n$, whence $\tilde{\sigma} = \bar{\sigma}$. In order to prove $\tilde{X} = \bar{X}$, it only remains to prove that $\tilde{x}(t) = \bar{x}(t)$, while $t \in [\tilde{\sigma}_n, \tilde{\sigma}_{n+1}) = [\sigma_n, \sigma_{n+1})$, i.e.

$$\bar{x}'(\tilde{\beta}_n + t - \tilde{\sigma}_n) = x(\beta_n + t - \sigma_n) \tag{7}$$

Put $u=\tilde{\beta}_n+t-\tilde{\sigma}_n$ ($<\bar{\sigma}'$). In order to prove (7), we need only to prove $\beta_n+t-\sigma_n=\alpha'_u$. By (v) in Lemma 1, we need only prove

$$L\{A'\cap[0,\beta_n+t-\sigma_n)\}=\tilde{\beta}_n+t-\tilde{\sigma}_n \tag{8}$$
$$\beta_n+t-\sigma_n\in A'$$

Since $0\leqslant t-\sigma_n<\sigma_{n+1}-\sigma_n=\tau_{n+1}-\beta_n$, then $\beta_n+t-\sigma_n\in[\beta_n,\tau_{n+1})\subset A\subset A'$. Moreover

$$\begin{aligned}L\{A'\cap[0,\beta_n+t-\sigma_n)\}&=L\{A'\cap[0,\beta_n)\}+L\{A'\cap[\beta_n,\beta_n+t-\sigma_n)\}\\&=\tilde{\beta}_n+L\{[\beta_n,\beta_n+t-\sigma_n)\}=\tilde{\beta}_n+t-\sigma_n=\tilde{\beta}_n+t-\tilde{\sigma}_n\end{aligned}$$

which proves (8), and the proof is completed. QED

Lemma 4. Suppose that the function $X=\{x(t),t<\sigma\}$ satisfies the condition in Theorem 3.1. Moreover,

$$A=\bigcup_{n=1}^{\infty}\bigcup_{k=0}^{\infty}[\beta_k^n,\tau_{k+1}^n) \tag{9}$$

$\gamma(u)=L\{A\cap[0,u)\}$ and the expressions $C_1(A)$, $C_2(A)$, $\bar{C}_2^+(A)$ are determined by (3.1)–(3.3). Then

(i) The conclusion (i) in Theorem 3.1 holds.

(ii)
$$C_1(A)\subset A\subset C_2(A)\subset\bar{C}_2^+(A)\subset[0,\sigma) \tag{10}$$
$$L\{C_1(A)\}=L\{A\}=L\{C_2(A)\}=L\{\bar{C}_2^+(A)\}=\bar{\sigma} \tag{11}$$

(iii) (a) $\gamma(u)$ is non-decreasing on $[0,\sigma)$.
(b) If $[a,b)\cap C_2(A)=\varnothing$, then for $u\in[a,b)$, we have $\gamma(u)=\gamma(a)$.
(c) If $u\in\bar{C}_2^+(A)$ and $u<s<\sigma$, then $\gamma(u)<\gamma(s)$. Especially $\gamma(u)$ is strictly increasing on $\bar{C}_2^+(A)$.
(d) For transformation γ we have $\gamma[0,\sigma)=[0,\bar{\sigma})$, $\bar{\sigma}=L(A)$.
(e) γ as a transformation from $\bar{C}_2^+(A)$ onto $[0,\bar{\sigma})=\gamma[\bar{C}_2^+(A)]$ is one-to-one. Therefore, it has the inverse transformation γ^{-1}. Furthermore, γ and γ^{-1} are measure-preserving transformations. Particularly, if $[\lambda,\eta)\in\mathscr{U}(A)$, $t\in[\gamma(\lambda),\gamma(\eta))$, then
$$\gamma^{-1}(t)=\lambda+t-\gamma(\lambda) \tag{12}$$

(iv) As $n\to\infty$ there exist limits
$$\sigma^n=L(A^n)\uparrow\bar{\sigma}=L(A) \tag{13}$$
$$\alpha_t^n\downarrow t\equiv\gamma^{-1}(t)\qquad t\in[0,\bar{\sigma}) \tag{14}$$

Proof. By conclusion (ii) in Lemma 3, follows (i) in this lemma. Inclusions (10) are obvious. It is easily shown that $\mathscr{U}(A)$ is a denumrable set, therefore, sets $C_1(A)$, A, $C_2(A)$ are different from each other by one countable set at most. Thus, the first two equalities in (11) are proved.

(iii) (a), (b), (d) in this lemma are obvious. In order to prove (iii) (c), first notice that if $t\in C_2(A)$, $t<s<\sigma$, then $\gamma(t)<\gamma(s)$. Therefore, for $u\in\bar{C}_2^+(A)$, $u<s<\sigma$, there exists $t_n\in C_2(A)$, $t_n\downarrow u$. Hence we have $u\leqslant t_n<s<\sigma$, for sufficiently large n. From (iii) (a) and what was mentioned just now, we have $\gamma(u)\leqslant\gamma(t_n)<\gamma(s)$. Thus (iii) (c) in this lemma is proved.

We now proceed to prove (iii) (d). By (iii) (c), γ is a one-to-one transformation from $\bar{C}_2^+(A)$ onto $\gamma(\bar{C}_2^+(A))$ so that γ^{-1} exists. In order to prove measure-preservation, we assume $\mathscr{L}$ to be the collection of Lebesgue sets F satisfying $F\subset\bar{C}_2^+(A)$ and $L\{\gamma(F)\}=L\{F\}$. Put $\mathscr{C}=\{[a,b)|[a,b)\subset C_2^+(A)\}$. It is easily found that $\mathscr{L}$ is a λ-system, $\mathscr{C}$ is a π-system, and $\mathscr{C}\subset\mathscr{L}$, whence $\mathscr{L}$ contains all Borel sets in $\bar{C}_2^+(A)$; see Dynkin (1959, Lemma 1.1). Consequently, $\mathscr{L}$ contains all Lebesgue-measurable sets in $\bar{C}_2^+(A)$.

From conclusion (i) in Lemma 3, it follows that the limits in (13) and (14) exist. For $t\in[0,\bar{\sigma})$, when n is sufficiently large, we have $t\in[0,\sigma^n)$. By conclusion (v) in Lemma 1, we have

$$L\{A^n\cap[0,\alpha_t^n)\}=t\qquad\alpha_t^n\in A^n \tag{15}$$

Since $A^n\subset A\subset C_2(A)$, letting $n\to\infty$, we find that the limit $\alpha_t=\lim_{n\to\infty}\alpha_t^n$ satisfies

$$\gamma(\alpha_t)=L\{A\cap[0,\alpha_t)\}=t\qquad\alpha_t\in\bar{C}_2^+(A) \tag{16}$$

Noticing (10) and (11), we know that α_t is a solution of (3.13). By (iii) (c) in this lemma, α_t is a unique solution, and $\gamma^{-1}(t)$ is a solution, too. Whence $\alpha_t=\gamma^{-1}(t)$, $\gamma(\bar{C}_2^+(A))=[0,\bar{\sigma})$ and $L\{\bar{C}_2^+(A)\}=L\{\gamma(\bar{C}_2^+(A))\}=\bar{\sigma}$. The proof is terminated. QED

Proof of Theorem 3.1. From the right-continuity of X, and (13) and (14), (3.11) and (3.12) can be proved, and hence by Lemma 4, conclusions (i) and (ii) in Theorem 3.1 can be verified. We are now going to prove conclusion (iii) in Theorem 3.1. Let $t\in[0,\bar{\sigma})$ and $s\downarrow t$. From (iii) (c) and (e) in Lemma 4, we have $\gamma^{-1}(s)\downarrow$. Let the limit be u. By (3.13), we know

$$L\{C_2(A)\cap[0,\gamma^{-1}(s))\}=s\qquad\gamma^{-1}(s)\in\bar{C}_2^+(A)$$

Putting $s\downarrow t$, we find that u satisfies (3.13); therefore $u=\gamma^{-1}(t)$, i.e. $\gamma^{-1}(s)\downarrow\gamma^{-1}(t)$. Consequently,

$$\lim_{s\downarrow t}\bar{x}(s)=\lim_{s\downarrow t}x(\alpha_s)=\lim_{s\downarrow t}x(\gamma^{-1}(s))=x(\gamma^{-1}(t))=x(\alpha_t)=\bar{x}(t)$$

and the proof is concluded. QED

9.5 SEVERAL LEMMAS

Let $X = \{x(t), t < \sigma\} \in \mathscr{H}_s$, β be a Markov time relative to X, $\mathscr{F}_\beta$ denote the pre-β field and $\mathscr{F}'_\beta$ denote the post-β field. Set

$$x'_\beta(t) = x(\beta + t) \qquad t < \sigma'_\beta = \sigma - \beta \tag{1}$$

We call $X'_\beta = \{x'_\beta(t), t < \sigma'_\beta\}$, or more simply X' the post-β process. Let symbols $\mathscr{L}^0_t(\beta)$, or simply $\mathscr{L}^0_t$, denote the complete Borel field which is on $\Omega_\beta = (\beta < \sigma)$ and is generated by $\{x'_\beta(u), u \leqslant t\}$, $\mathscr{L}^0_\infty(\beta) = \mathscr{F}'_\beta$.

Obviously, if $\tau \leqslant \beta$ are two Markov times relative to X, then

$$\mathscr{F}_\tau \cap \Omega_\beta \subset \mathscr{F}_\beta \qquad \mathscr{F}'_\tau \supset \mathscr{F}'_\beta. \tag{2}$$

Lemma 1. Let β be a Markov time relative to X. Then

$$\theta_\beta \mathscr{F}^0_t = \mathscr{L}^0_t(\beta) \subset \mathscr{F}'_\beta \cap \mathscr{F}_{\beta+t} \tag{3}$$

Proof. Let $\Lambda \in \mathscr{F}^0_t$. By Dynkin (1959, Lemma 1.5), there exist $t_k \in [0, t]$, and a Borel set Γ in $E^\infty = E \times E \times E \times \cdots$ such that

$$\Lambda = \{[x(t_1), x(t_2), \ldots] \in \Gamma\} \tag{4}$$

Therefore by Dynkin (1963, p. 122).

$$\theta_\beta \Lambda = \{[x'_\beta(t_1), x'_\beta(t_2), \ldots] \in \Gamma\} \in \mathscr{L}^0_t(\beta) \tag{5}$$

Hence we obtain $\theta_\beta \mathscr{F}^0_t \subset \mathscr{L}^0_t(\beta)$. Similarly it can be proved that the inverse inclusion holds. Consequently $\theta_\beta \mathscr{F}^0_t = \mathscr{L}^0_t(\beta)$.

Obviously, $\mathscr{L}^0_t(\beta) \subset \mathscr{F}'_\beta$. Now we need to prove that $\{x'(s) = i\} \in \mathscr{F}_{\beta+t}$ $(s \leqslant t, i \in E)$, that is, for arbitrary $u \geqslant 0$, $s \leqslant t$, we have

$$\{x(\beta + s) = i \qquad \beta + t < u < \sigma\} \in \mathscr{F}^0_u$$

When $u \leqslant t$, the left is empty. We can reasonably assume $u > t$. Set

$$\beta_n = \frac{k+1}{2^n}(u - t) \qquad \text{if} \qquad \frac{k}{2^n}(u - t) \leqslant \beta < \frac{k+1}{2^n}(u - t)$$

We have $P\{x(\beta_n + s) = \infty\} = 0$ by (7.6.1). From the right-continuity of X and $(\beta + t < u) \subset (\beta_n + t \leqslant u)$, it follows that

$$\{x(\beta + s) = i, \beta + t < u < \sigma\} = \left\{\lim_{n\to\infty} x(\beta_n + s) = i, \beta < u - t, u < \sigma\right\} \in \mathscr{F}^0_u$$

and the proof is completed. QED

Lemma 2. Let τ and β be two Markov times relative to X. Then $\theta_\beta\tau$ is a Markov time relative to X'_β.

Proof.

$$\{\theta_\beta \tau < u\} = \bigcup_{t \geqslant 0} \{\theta_t \tau < u\} \cap (\beta = t) = \bigcup_{t \geqslant 0} \theta_t(\tau < u) \cap (\beta = t) = \theta_\beta(\tau < u)$$

Since $(\tau < u) \in \mathscr{F}^0_u$, we have $(\theta_\beta \tau < u) \in \mathscr{L}^0_u$, by Lemma 1, and hence $(\theta_\beta \tau < u < \sigma'_\beta) \in \mathscr{L}^0_u$. QED

Lemma 3. Let $\{\tau, \beta\}$ be a canonical pair for X. Then τ_n and β_n are Markov times relative to X. For a fixed n, set

$$\tau'_0 = \beta'_0 = 0 \qquad \tau'_k = \tau_{n+k} - \beta_n \qquad \beta'_k = \beta_{n+k} - \beta_n \qquad k \geqslant 1 \tag{6}$$

Then $\{\tau', \beta'\}$ is a canonical pair for X'_{β_n}.

Proof. It is known that β_0, τ_1 and β_1 are Markov times relative to X. Assume that τ_n, β_n are Markov times relative to X. By Lemmas 1 and 2 we have

$$\begin{aligned} &(\theta_{\beta_n}\tau_1 < t < \sigma'_{\beta_n}) \in \mathscr{L}^0_t(\beta_n) \subset \mathscr{F}_{\beta_n + t} \\ &(\theta_{\beta_n}\beta_1 < t < \sigma'_{\beta_n}) \in \mathscr{L}^0_t(\beta_n) \subset \mathscr{F}_{\beta_n + t} \end{aligned} \tag{7}$$

By Theorem 7.6.3, $\tau_{n+1} = \beta_n + \theta_{\beta_n}\tau_1$ and $\beta_{n+1} = \beta_n + \theta_{\beta_n}\beta_1$ are Markov times relative to X.

According to Lemma 2, $\tau'_1 = \tau_{n+1} - \beta_n = \theta_{\beta_n}\tau_1$, $\beta'_1 = \beta_{n+1} - \beta_n = \theta_{\beta_n}\beta_1$ are Markov time relative to X'_{β_n}. Moreover,

$$\beta'_1 + \theta_{\beta'_1}\tau'_1 = \beta_{n+1} - \beta_n + \theta_{\beta_{n+1} - \beta_n}\theta_{\beta_n}\tau_1 = \beta_{n+1} + \theta_{\beta_{n+1}}\tau_1 - \beta_n = \beta_{n+2} - \beta_n = \tau'_2$$

In a similar manner, it can be shown that (3.17) holds for $\{\tau', \beta'\}$, that is, $\{\tau', \beta'\}$ is a canonical pair for X'_{β_n}. QED

Lemma 4. Let $\{\tau, \beta\}$ be a canonical pair for X. Then random variables σ_n and δ_n determined by (2.2) and (2.3) are $\mathscr{F}_{\tau_n}$ and $\mathscr{F}_{\beta_n}$ measurable, respectively.

Proof. It is required to prove that for any $u \geqslant 0$, $t \geqslant 0$, we have

$$\begin{aligned} &(\sigma_n < u, \tau_n < t < \sigma) \in \mathscr{F}^0_t \\ &(\delta_n < u, \beta_n < t < \sigma) \in \mathscr{F}^0_t \end{aligned}$$

In fact, let R denote the set of all rational numbers on $[0, \infty)$; then

$$(\sigma_n < u, \tau_n < t < \sigma) = \left\{\sum_{k=1}^{n} (\tau_k - \beta_{k-1}) < u, \tau_n < t < \sigma\right\}$$

$$= \bigcup_{r_k \in R} (\tau_k - \beta_k < r_k, \tau_k < t) \; (\tau_n < t < \sigma)$$

$$\sum_{k=1}^{n} r_k < u$$

and

$$(\delta_n < u, \beta_n < t < \sigma) = \left\{ \sum_{k=0}^{n} (\beta_k - \tau_k) < u, \beta_n < t < \sigma \right\}$$

$$= \bigcup_{r_k \in R} (\beta_k - \tau_k < r_k, \beta_k < t) \qquad (\beta_n < t < \sigma)$$

$$\sum_{k=1}^{n} r_k < u$$

From Zhen-ting Hou and Qin-fen Guo (1978, Lemma 1.4.2) the above sets belong to $\mathscr{F}_t^0$ and the proof is concluded. QED

Lemma 5. Let $\{\tau, \beta\}$ be a canonical pair for X. α_t is determined by (2.5). For each fixed $t \geqslant 0$, α_t is a Markov time relative to X.

Proof. Since $t \leqslant \alpha_t < \sigma$ we may as well assume $u > t$. By Lemma 2

$$(\alpha_t < u < \sigma) = \bigcup_{k=1}^{\infty} (\beta_{k-1} \leqslant u < \tau_k, \alpha_t < u < \sigma) + \bigcup_{k=1}^{\infty} (\tau_k \leqslant u < \beta_k, \alpha_t < u < \sigma)$$

$$= \bigcup_{k=1}^{\infty} (\beta_{k-1} \leqslant u < \tau_k, \rho(u) < u - t, u < \sigma) + \bigcup_{k=1}^{\infty} (\tau_k \leqslant u < \beta_k, \gamma(u) > t, u < \sigma)$$

$$= \bigcup_{k=1}^{\infty} (\delta_{k-1} < u - t, \beta_{k-1} \leqslant u < \sigma)(u < \tau_k) + \bigcup_{k=1}^{\infty} (\sigma_k > t, \tau_k \leqslant u < \sigma)(u < \beta_k)$$

From Lemma 4 we know the above expression belongs to $\mathscr{F}_u^0$, and the proof is finished. QED

Lemma 6. Let $\{\tau, \beta\}$ be a canonical pair for X. α_t is determined according to (2.5). If the condition (c) in Theorem 3.3 is satisfied, then the condition (b) in Theorem 3.3 holds. More precisely

$$P\{x(t) \in E \text{ for all } t \in A\} = 1 \tag{8}$$

where A is just the same as in (4.1). In particular,

$$P\{x(\alpha_t) \in E \text{ for all } t \in [0, \bar{\sigma})\} = 1 \tag{9}$$

$$P\{x(\beta_k) = \infty \text{ for some } k\} = 0 \tag{10}$$

If the conditions (a) and (c) in Theorem 3.3 are satisfied, then for almost all $\omega \in \Omega_{\beta_k} = (\beta_k < \sigma)$ and all $u \in [\beta_k(\omega), \tau_{k+1}(\omega)]$, we have

$$\begin{aligned} u + (\theta_u \tau_1)(\omega) &= \tau_{k+1}(\omega) \\ u + (\theta_u \beta_1)(\omega) &= \beta_{k+1}(\omega) \end{aligned} \tag{11}$$

Proof. (a) First suppose $P\{\beta_0 = 0\} = 1$. Put

$$\Lambda_k = \{x(u) \in E \text{ for all } u \in [\beta_k, \tau_{k+1})\} \cap \Omega_{\beta_k}$$

By (3.17) and $\beta_0 = 0$, it is evident that $\Lambda_k = \theta_{\beta_k} \Lambda_0$. From the condition (c) in Theorem 3.3 and the strong Markov property, we have

$$P\{\Lambda_1\} = P\{\theta_{\beta_1} \Lambda_0\} = \sum_{k \in E} P\{x(\beta_1) = k\} P\{\Lambda_0 | x(0) = k\} = \sum_{k \in E} P\{x(\beta_1) = k\} = P\{\Omega_{\beta_1}\} \tag{12}$$

Similarly, it can be proved that $P\{\Lambda_k\} = P\{\Omega_{\beta_k}\}$. This verifies (8).

Set

$$\Delta_k = \{u + \theta_u \tau_1 = \tau_{k+1} \text{ for all } u \in [\beta_k, \tau_{k+1})\} \cap \Omega_{\beta_k}$$

Similarly, we have $\Delta_k = \theta_{\beta_k} \Delta_0$. From conditions (a) and (c) in Theorem 3.3 and the strong Markov property, and by following the discussion in (12), we can prove that $P(\Delta_k) = P(\Omega_{\beta_k})$, which verifies the first formula in (11). The second formula can be proved in similar way.

(b) Suppose $P\{\beta_0 = 0\} \leqslant 1$. In this case consider the post-β_0 process $X' = X'_{\beta_0}$. By Lemma 2, $\{\tau', \beta'\}$ is a canonical pair for X', where

$$\tau'_0 = \beta'_0 = 0 \qquad \tau'_l = \tau_l - \beta_0 \qquad \beta'_l = \beta_l - \beta_0 \qquad l \geqslant 1 \tag{13}$$

Since X and $\{\tau, \beta\}$ satisfy the conditions (a) and (c) in Theorem 3.3, so do X' and $\{\tau', \beta'\}$. By (a) it follows that the conclusions in this lemma hold for X' and $\{\tau', \beta'\}$. On the other hand, from (13), we have

$$\{x'(t) \in E \text{ for all } t \in A'\} = \{x(t) \in E \text{ for all } t \in A\}$$

So $\Omega_{\beta'_k} = \Omega_{\beta_R} u + \theta_u \tau'_1 = \tau'_{k+1}$ for all $u \in [\beta_k, \tau'_{k+1})$ is equivalent to $u + \theta_u \tau_1 = \tau_{k+1}$ for all $u \in [\beta_k, \tau_{k+1})$. Therefore, the conclusions in this lemma hold for X and $\{\tau, \beta\}$ and the proof is terminated. QED

Lemma 7. Let $\{\tau, \beta\}$ be a canonical pair for X, $P\{\beta_0 = 0\} = 1$. The conditions (a) and (c) in Theorem 3.3 are satisfied. Then for any fixed $s, t \geqslant 0$, we have

$$\alpha_{s+t} = \alpha_s + \theta_{\alpha_s}(\alpha_t) \tag{14}$$

Proof. Put $X' = X'_{\alpha_s}$. By Lemma 2

$$\beta'_0 = 0 \qquad \tau'_1 = \theta_{\alpha_s} \tau_1 \qquad \beta'_1 = \theta_{\alpha_s} \beta_1 \tag{15}$$

are Markov times relative to X', so that we can determine the canonical pair $\{\tau', \beta'\}$ for X' and the corresponding α'_t, Since $P\{\beta_0 = 0\} = 1$, it is evident that $\alpha_t = \theta_{\alpha_s}(\alpha_t)$.

From (v) in Lemma 1, $\alpha_s \in A = \bigcup_{k=0}^{\infty} [\beta_k, \tau_{k+1})$. To be specific, we assume $\alpha_s \in [\beta_k, \tau_{k+1})$. By Lemma 6, $\alpha_s + \theta_{\alpha_s} \tau_1 = \tau_{k+1}$, $\alpha_s + \theta_{\alpha_s} \beta_1 = \beta_{k+1}$. Hence from (15) it follows that $\tau'_1 = \tau_{k+1} - \alpha_s$, $\beta'_1 = \beta_{k+1} - \alpha_s$. Following the proof of Lemma 3,

we have

$$\tau_l' = \tau_{k+1} - \alpha_s \qquad \beta_l' = \beta_{k+l} - \alpha_s \tag{16}$$

By Lemma 1 (v) $\alpha_t' = \theta_{\alpha_s}(\alpha_t) \in A' = \bigcup_{k=0}^{\infty} [\beta_k', \tau_{k+1}')$.

To be specific, now we assume $\alpha_t' \in [\beta_l', \tau_{l+1}')$. By (16), that is, $\beta_{k+l} - \alpha_s \leqslant \theta_{\alpha_s}(\alpha_t) < \tau_{k+l} - a_s$. Hence, $\beta_{k+l} \leqslant \alpha_s + \theta_{\alpha_s}(\alpha_t) < \tau_{k+l}$, that is, $\alpha_s + \theta_{\alpha_s}(\alpha_t) \in A$. Furthermore, by (v) in Lemma 1

$$\begin{aligned} t = L\{A' \cap [0, \alpha_t')\} &= L\left\{\bigcup_{l=0}^{\infty} [\beta_l', \tau_{l+1}') \cap [0, \alpha_t')\right\} \\ &= L\left\{\bigcup_{l=0}^{\infty} [\beta_{k+1} - \alpha_s, \tau_{k+1} - \alpha_s) \cap [0, \theta_{\alpha_s}(\alpha_t'))\right\} \\ &= L\left\{\bigcup_{l=0}^{\infty} [\beta_{k+l}, \tau_{k+l}) \cap [\alpha_\delta, \alpha_s + \theta_{\alpha_s}(\alpha_t))\right\} \\ &= L\{A \cap [\alpha_s, \alpha_s + \theta_{\alpha_s}(\alpha_t))\} \end{aligned}$$

Therefore, $L\{A \cap [0, \alpha_s + \theta_{\alpha_s}(\alpha_t))\} = L\{A \cap [0, \alpha_s)\} + t = s + t$. Consequently, $\alpha_s + \theta_{\alpha_s}(\alpha_t)$ is the solution of the equation

$$\gamma(u) = s + t \qquad u \in A$$

But α_{s+t} is the unique solution of the above equation. Hence (14) holds, and the proof is terminated. QED

9.6 PROOF OF THE STRONG LIMIT THEOREM

Proof of Theorem 3.3. (a) Suppose $P\{\beta_0 = 0\} = 1$. Assume $t_1 < t_2 < \cdots < t_{n+1}$, $i_1, i_2, \ldots, i_{n+1} \in E$. By Lemma 5.5 and 5.1, and by (5.2), we have

$$\Lambda_1 = \bigcap_{k=1}^{n-1} \{x(\alpha_{t_k}) = i_k\} \cap (\alpha_{t_n} < \sigma) \in \mathscr{F}_{\alpha_{t_n}}$$

$$\Lambda_2 = \{x(\alpha_{t_{n+1}}) = i_{n+1}\} \in \mathscr{F}'_{\alpha_{t_n}}$$

From Lemma 7 and the strong Markov property, we obtain

$$\begin{aligned} P\{x(\alpha_{t_{n+1}}) = i_{n+1} \mid x(\alpha_{t_k}) = i_k, 1 \leqslant k \leqslant n\} &= P\{x(\alpha_{t_{n+1}}) = i_{n+1} \mid \Lambda_1, \alpha_{t_n} < \sigma, x(\alpha_{t_n}) = i_n\} \\ &= P\{x[\alpha_{t_n} + \theta_{\alpha_{t_n}}(\alpha_{t_{n+1}} - t_n)] = i_{n+1} \mid x(\alpha_{t_n}) = i_n\} \\ &= P\{x(\alpha_{t_{n+1} - t_n}) = i_{n+1} \mid x(0) = i_n\} \end{aligned}$$

Noticing (3.19), we could prove that $\bar{X} = W_{\tau,\beta}(X)$ is a homogeneous Markov process.

(b) Suppose $P\{\beta_0 = 0\} \leqslant 1$. Now we consider $X' = X_{\beta_0}^{\wedge}$, its canonical pair $\{\tau', \beta'\}$ determined by (5.6) with $n = 0$ and the corresponding α_t'. Then (5.13)

holds and, moreover,

$$\alpha_t' = \alpha_t - \beta_0 \qquad x'(\alpha_t') = x(\alpha_t) \tag{1}$$

Since the conditions (a) and (b) in Theorem 3.3 are satisfied for X and $\{\tau, \beta\}$, they are satisfied for X' and $\{\tau', \beta'\}$, too. Consequently according to (a), $W_{\tau',\beta'}(X')$ possesses the homogeneous Markov property. But $W_{\tau,\beta}(X) = W_{\tau',\beta'}(X')$. Therefore $\bar{X} = W_{\tau,\beta}(X)$ possesses the homogeneous Markov property.

(c) Since $X \in \mathscr{H}_s$, we have $\bar{X} \in \mathscr{H}_s$. By Lemma 6, we know that the condition (c) in Theorem 3.3 implies the condition (b), and hence we prove the conclusion (iii) in Theorem 3.3. The proof is concluded. QED

Proof of Theorem 3.4. Except that it remains to prove that $\bar{X}$ in Theorem 3.4 is a process, the rest of the conclusions of Theorem 3.4 may all be deduced from Theorems 3.1 and 3.3.

Let us now prove $P\{\bar{x}(t) = \infty\} = 0$ $(t \geqslant 0)$.

By Lemma 6, for almost all ω, $x(t, \omega) \in E$ for all $t \in A^n(\omega)$, so that

$$x(t, \omega) \in E \qquad \text{for all } t \in A(\omega)$$

Therefore, by (iii) (e) in Lemma 4.4, when $t \in \gamma[C_1(A(\omega))]$, $\gamma^{-1}(t) \in C_1[A(\omega)] \subset A(\omega)$. From (2), we get $\bar{x}(t, \omega) = x(\gamma^{-1}(t), \omega) \in E$. Consequently if we set $\bar{S}_i(\omega) = \{t \mid \bar{x}(t, \omega) = i\}$ $(i \in \bar{E})$, then

$$\gamma[C_1(A(\omega))] \subset \{t \mid \bar{x}(t, \omega) \in E\} = \bigcup_{i \in E} \bar{S}_i(\omega) \subset \bigcup_{i \in E} \bar{S}_i(\omega) \cup \bar{S}_\infty(\omega) = [0, \bar{\sigma}(\omega))$$

By (ii) in Lemma 4.4 $\bar{\sigma}(\omega) = L\{\gamma[C_1(A(\omega))]\}$, hence $L\{\bar{S}_\infty(\omega)\} = 0$. By Fubini's theorem it follows that there exists a null Lebesgue-measurable set T such that $P\{\bar{x}(t) = \infty\} = 0$ when $t \notin T$.

From (3.22), $\beta_0^n \downarrow \inf_n \beta_0^n = \alpha_0$, so that (3.21) becomes $P\{\bar{x}(0) = \infty\} = 0$. Therefore $0 \notin T$.

Now let us assume $t_0 \in T$. Clearly $t_0 > 0$. Since $L(T) = 0$, we may choose $0 < t < t_0$ such that both $t_0 - t$ and $t \notin T$. Notice that for each $\omega \in \Omega$, we have $\lim_{n \to \infty} x^n(t) = \lim_{n \to \infty} x(\alpha_t^n) = x(\alpha)$ in $\bar{E}$. Accordingly we set $\bar{D} = \bar{E} - D$ for a finite set $D \subset E$. It follows that

$$P\{\bar{x}(t_0) \in \bar{D}\} = P\left\{\lim_{n \to \infty} x(\alpha_{t_0}^n) \in \bar{D}\right\} = \lim_{n \to \infty} P\{x(\alpha_{t_0}^n) \in \bar{D}\}$$

Write $X' = X'_{\beta_0^n}$, $\alpha_t' = \alpha_t^n - \beta_0^n$. From (1), Lemma 7 and the strong Markov property for X', we obtain

$$\begin{aligned} P\{x(\alpha_{t_0}^n) \in \bar{D}\} = P\{x'(\alpha_{t_0}') \in \bar{D}\} &= P\{x'[\alpha_t' + \theta_{\alpha_t'}(\alpha_{t_0 - t}')] \in \bar{D}\} \\ &= E\{P_{x'(\alpha_t')}[x'(\alpha_{t_0 - t}') \in \bar{D}]\} \\ &= E\{P_{x(\alpha_t^n)}[x(\alpha_{t_0 - t}^n) \in \bar{D}]\} \end{aligned}$$

Therefore, $P\{\bar{x}(t_0)\in\bar{D}\} = \lim_{n\to\infty} E\{P_{x(\alpha_t^n)}[x(\alpha^n_{t_0-t})\in\bar{D}]\}$. Since $t\in T$, we have $x(\alpha_t,\omega)\in E$ for almost all ω. As a result, when n is sufficiently large, we have $x(\alpha^n_t,\omega) = x(\alpha_t,\omega)$. Hence

$$\begin{aligned}P\{\bar{x}(t_0)\in\bar{D}\} &= E\left\{\lim_{n\to\infty} P_{x(\alpha_t)}[x(\alpha^n_{t_0-t})\in\bar{D}]\right\}\\ &= E\{P_{x(\alpha_t)}[x(\alpha_{t_0-t})\in\bar{D}]\}\\ &= \sum_{k\in E} P\{x(\alpha_t)=k\}P_k\{x(\alpha_{t_0-t})\in\bar{D}\}\end{aligned}$$

Letting $D\uparrow E$, we obtain

$$\begin{aligned}P\{\bar{x}(t_0)=\infty\} &= \sum_{k\in E} P\{x(\alpha_t)=k\}P_k\{x(\alpha_{t_0-t})=\infty\}\\ &= \sum_{k\in E} P\{x(\alpha_t)=k\}\cdot 0 = 0\end{aligned}$$

Since $P\{\bar{x}(t)=\infty\} = P\{x^n(t)=\infty\} = 0$ $(t\geq 0)$, by Lemma 5.3, we know that as $n\to\infty$, $\{x^n(t_k)=i_k, 1\leq k\leq l\}$ converges to $\{\bar{x}(t_k)=i_k, 1\leq k\leq l\}$ in distribution. Hence from the homogeneous Markov property for X, we obtain the same property for $\bar{X}$. The proof is completed. QED

9.7 SEVERAL KINDS OF SPECIAL STRONG LIMIT THEOREMS

Theorem 1. Let $X\in\mathscr{H}_{\mathrm{D}}$. We fix a state $i\in E$. Let β_0 be a Markov time relative to X, satisfying $P\{\beta_0<\sigma\}>0$. Set

$$\begin{aligned}\tau_1 &= \inf\{t\mid\beta_0\leq t<\sigma, x(t)=i\}\\ \beta_n &= \inf\{t\mid\tau_n\leq t<\sigma, x(t)\neq i\}\\ \tau_{n+1} &= \inf\{t\mid\beta_n\leq t<\sigma, x(t)=i\}\end{aligned}$$

We set $\inf\varnothing=\sigma$, $\varnothing$ being an empty set. Then $\bar{X} = W_{\tau,\beta}(X)\in\mathscr{H}_{\mathrm{D}}$, and $\bar{X}$ does not take the value i.

Proof. Evidently $\{\tau,\beta\}$ is a canonical pair, and the condition (a) in Theorem 3.3 is satisfied. Since $X\in\mathscr{H}_{\mathrm{D}}$, the condition (b) in the same theorem is, of course, satisfied. Hence by quoting Theorem 3.3 we obtain Theorem 1. QED

Theorem 2. Let $X\in\mathscr{H}_{\mathrm{D}}$, and the subsets M and N of E be disjoint. Put

$$\begin{aligned}\beta_0 &= \inf\{t\mid 0\leq t<\sigma, x(t)\in N\}\\ \tau_1 &= \inf\{t\mid\beta_0\leq t<\sigma, x(t)\in M\}\\ \beta_n &= \inf\{t\mid\tau_n\leq t<\sigma, x(t)\in N\}\\ \tau_{n+1} &= \inf\{t\mid\beta_n\leq t<\sigma, x(t)\in M\}\end{aligned}$$

Then $\bar{X} = W_{\tau,\beta}(X)\in\mathscr{H}_{\mathrm{D}}$, and the value of $\bar{X}$ belongs to $E-M$.

Proof. It is the same as Theorem 1. QED

Theorem 3. Let $X\in\mathscr{H}_{\mathrm{s}}$, and finite sets $D_n\uparrow E$. Put

$$\begin{aligned}\beta^n_0 &= \inf\{t\mid 0\leq t<\sigma, x(t)\in D_n\}\\ \tau^n_1 &= \inf\{t\mid\beta^n_0\leq t<\sigma, x(t)\notin D_n\}\\ \beta^n_k &= \inf\{t\mid\tau^n_k\leq t<\sigma, x(t)\in D_n\}\\ \tau^n_{k+1} &= \inf\{t\mid\beta^n_k\leq t<\sigma, x(t)\notin D_n\}\end{aligned}$$

Then $X^n = W_{\tau^n,\beta^n}(X)$ is a minimal process, whose state space is D_n, and

$$\lim_{n\to\infty} X^n = X. \tag{1}$$

Proof. Referring to Theorem 3.4, we easily see that $\{\tau^n,\beta^n\}$ is a canonical pair for X, and the conditions (a) and (c) in Theorem 3.3 hold for X and $\{\tau^n,\beta^n\}$. Since $P\{x(0)=\infty\}=0$, it follows that $\inf_n\beta^n_0=0$, so that (3.21) holds. As $D_n\uparrow$, (3.22) holds, too.

Notice

$$\begin{aligned}&x(t)\in D_n &&\text{if } t\in A^n = \bigcup_{k=0}^{\infty}[\beta^n_k,\tau^n_{k+1})\\ &x(t)\notin D_n &&\text{if } t\in B_n = \bigcup_{k=0}^{\infty}[\tau^n_k,\beta^n_k)\end{aligned} \tag{2}$$

By (1) in Theorem 3.4, we obtain $X^n = W_{\tau^n,\beta^n}(X)\in\mathscr{H}_{\mathrm{D}}$. By the definition of X^n, we know that X^n does not have any leaping point, hence X^n is a minimal process.

Let us now proceed to prove

$$\{t\mid x(t)\in E\} = A = \bigcup_{n=1}^{\infty} A^n \tag{3}$$

Assume $x(t)=i\in E$; then there exists a number n such that $i\in D_n$, and therefore $x(t)\in D_n$. By (2), we get $t\in A^n\subset A$. Hence $\{t\mid x(t)\in E\}\subset A$. On account of (2), the inverse inclusion relation holds too.

By (3), $[0,\sigma)$ and A are differ by a null Lebesgue-measurable set $S_\infty=\{t\mid x(t)=\infty\}$. So we have

$$\gamma(u) = L\{A\cap[0,u)\} = L\{[0,\sigma)\cap[0,u)\} = u$$

Therefore, $\alpha_t=\gamma^{-1}(t)=t$, $\bar{\sigma}=\sigma$. Consequently, (3.14) becomes (1). The proof is completed. QED

Theorem 4. Let $X\in\mathscr{H}_{\mathrm{s}}$ be a non-minimal process and finite sets $D_n\uparrow E$. Put $\beta^n_0=0$,

$$\begin{aligned} &\tau_1^n \text{ being the first leaping point for } X \\ &\beta_k^n = \inf\{t \mid \tau_k^n \leq t < \sigma, x(t) \in D_n\} \\ &\tau_{k+1}^n \text{ being the first leaping point for } X \text{ after } \beta_k^n \end{aligned} \tag{4}$$

Then $X^n = W_{\tau^n, \beta^n}(X) \in \mathscr{H}_1$ (collection of all first-order processes), (3.10) holds and, moreover, (1) is still valid.

Proof. Following the proof of Theorem 3 and quoting Theorem 3.4 we need only explain that, in the present case, (3) still holds. For this, let us notice that

$$\begin{aligned} &x(t) \in E \qquad \text{for all } t \in A^n \\ &x(t) \notin D_n \qquad \text{for all } t \in B^n \end{aligned} \tag{5}$$

And following the proof of Theorem 3, we know that (3) holds. QED

Remark

The transformation X to X^n in Theorem 4 is just the transformation g_n. Theorem 4 is precisely the fundamental result in Zhen-ting Hou (1975), i.e. the first construction theorem in Zhen-ting Hou and Qin-fen Guo (1978).

CHAPTER 10

Leaping Interval and Entrance Decomposition

10.1 INTRODUCTION

In this chapter we study the entrance of a process. We have introduced the concept of leaping interval, namely, U interval, and studied the relation between leaping intervals and set of leaping points, and we have found the necessary and sufficient condition which is expressed by leaping intervals and under which the Kolmogorov equations hold. We have found that it is effective to apply leaping intervals to the study of the entrance of a process. Especially, we have derived the entrance decomposition theorem of a process. Making use of this decomposition, we can investigate distinct entrances independently in the case that various kinds of entrances present themselves. By using the results of leaping intervals, we have obtained two kinds of strong limit theorems of a process, that is, transformations $_Mg_n$ and $_Mf_n$, and the strong limit theorems corresponding to them. As a simple deduction of these theorems, they are precisely the basic results in Zheng-ting Hou (1975), namely, the first construction theorem in Zheng-ting Hou (1974) and Theorem 5.3 in Zi-kun Wang (1962).

10.2 DEFINITION OF LEAPING INTERVAL

Suppose $X = \{x(t), t < \sigma\} \in \mathscr{H}_s$. Owing to (7.6.1), without loss of generality, we shall suppose henceforth for every $\omega \in \Omega$,

$$x(r, \omega) \in E \qquad \text{for all } r \in R \cap [0, \sigma(\omega))$$

Hereafter, R represents the set of rational numbers in $[0, \infty)$.

Definition 1. A point $t \in (0, \sigma(\omega))$ is called a continuity point of $X(\omega)$ if $\lim_{s \uparrow t} x(s, \omega) = x(t, \omega) \in E$. The set of continuity points is written as $G(\omega)$. $D(\omega) = [0, \sigma(\omega)] - G(\omega)$ is called the set of discontinuity points of $X(\omega)$.

Recall Definition 7.8.1, denote the set of jumping points of $X(\omega)$ by $T(\omega)$, and denote the set of leaping points by $\Gamma(\omega)$. We appoint $0 \in T(\omega) \cap \Gamma(\omega)$, but

call $t=0$ the zeroth jumping point and zeroth leaping point. $X(\omega)$ has the first discontinuity point $\tau_1(\omega)>0$, and the first leaping point $\tau(\omega)\geqslant\tau_1(\omega)>0$.

Obviously, if $\sigma(\omega)<\infty$ and $\sigma(\omega)\in T(\omega)$, then $x(\sigma(\omega)-0,\ \omega)\in H$ (non-conservative state set). It can be easily seen that $D(\omega)$ and $\Gamma(\omega)$ are closed sets.

For every $s\in(0,\sigma(\omega))$, we can define

$$\begin{aligned}\mu_s(\omega)&=\max\{D(\omega)\cap[0,s]\},\\ v_s(\omega)&=\min\{D(\omega)\cap[s,\sigma(\omega)]\}\\ \lambda_s(\omega)&=\max\{\Gamma(\omega)\cap[0,s]\}\\ \eta_s(\omega)&=\min\{\Gamma(\omega)\cap[s,\sigma(\omega)]\}\end{aligned}\tag{1}$$

For $a=0$, define

$$v_0(\omega)=\tau_1(\omega)\qquad \eta_0(\omega)=\tau(\omega)\qquad \mu_0(\omega)=0\qquad \lambda_0(\omega)=0\tag{2}$$

For $s\geqslant 0$, call $\mu_s(\omega)$, $\lambda_s(\omega)$, $v_s(\omega)$, $\eta_s(\omega)$ respectively the last discontinuity point before s, the last leaping point before s, the first discontinuity point after s, and the first leaping point after s. Suppose $s>0$. If $s\in D(\omega)$ or $s\in\Gamma(\omega)$, then obviously $\mu_s(\omega)=v_s(\omega)=s$ or $\lambda_s(\omega)=\eta_s(\omega)=s$. If $s\notin D(\omega)$ or $s\notin\Gamma(\omega)$, then we have $\mu_s(\omega)<s<v_s(\omega)$ or $\lambda_s(\omega)<s<\eta_s(\omega)$.

Definition 2. Call $[\lambda,\eta)$ a leaping interval, namely, U interval, of $X(\omega)$ if $\lambda,\eta\in\Gamma(\omega)$, and $(\lambda,\eta)\cap\Gamma(\omega)=\varnothing$ (empty set). The collection of U intervals of $X(\omega)$ is written as $\mathscr{U}(\omega)$.

It is quite plain that U intervals in $\mathscr{U}(\omega)$ have the order of 'before' and 'after' according to its place in $[0,\sigma]$

From Zi-kun Wang (1965a, Theorem 3.2.4, corollary) we know that if $[\lambda,\eta]$ is a U interval, then the discontinuity points in (λ,η) are jumping points and, moreover,

$$x(t,\omega)\in E\qquad\text{for all } t\in(\lambda,\eta)\tag{3}$$

Definition 3. suppose $[\lambda,\eta)\in\mathscr{U}(\omega)$, $M\subset\bar{E}$, $N\subset\bar{E}$. If $x(\lambda,\omega)\in M$, call $[\lambda,\eta)$ a $_MU$ interval. If $x(\eta-0,\omega)\in N$, call $[\lambda,\eta)$ a U_N interval. Similarly, we can define $_MU_N$ interval, $_iU$ interval, etc. Write the collection of $_MU$ intervals of $X(\omega)$ as $_M\mathscr{U}(\omega)$. Similar notations are $\mathscr{U}_N(\omega)$, $_i\mathscr{U}(\omega)$, etc., whose meanings are clear.

By Theorem 7.7.2, $X(\omega)$ has no U_{E-H} interval; in other words, if $X(\omega)$ has a U_E interval, it must be a U_H interval.

By Zi-kun Wang (1965a, Theorem 3.2.3), for any fixed $t>0$ for almost all $\omega\in(t<\sigma)$ we have

$$\lambda_t(\omega)\leqslant\mu_t(\omega)<t<v_t(\omega)\leqslant\eta_t(\omega)\tag{4}$$

thus we know easily that for almost all $\omega\in\Omega$

$$\lambda_r(\omega)<r<\eta_r(\omega)\qquad\text{for all } r\in R\cap[0,\sigma(\omega))\tag{5}$$

$$\{[\lambda_r(\omega),\eta_r(\omega))\,|\,r\in R\cap[0,\sigma(\omega))\}=\mathscr{U}(\omega)\tag{6}$$

Hence $\mathscr{U}(\omega)$ is a denumerable set.

Write

$$\mathscr{U}^\tau(\omega)=\{[\lambda,\eta)\,|\,[\lambda,\eta)\in\mathscr{U}(\omega),\lambda\geqslant\tau(\omega)\}.\tag{7}$$

Similar notations are $_M\mathscr{U}^\tau(\omega)$, $\mathscr{U}^\tau_N(\omega)$, etc., whose meanings are clear. Write

$$C_2(\omega)=\bigcup_{[\lambda,\eta)\in\mathscr{U}(\omega)}[\lambda,\eta)\tag{8}$$

$$C_1(\omega)=\bigcup_{[\lambda,\eta)\in\mathscr{U}(\omega)}(\lambda,\eta)\tag{9}$$

Similar notations are $_MC_2(\omega)$, $C^\tau_{1N}(\omega)$, etc., whose meanings are clear.

10.3 LEAPING POINT AND LEAPING INTERVAL

Definition 1. Suppose $t\in\Gamma(\omega)$. Call t a right isolated leaping point of $X(\omega)$ if there exists $\varepsilon>0$ such that $(t,t+\varepsilon)\cap\Gamma(\omega)=\varnothing$. The set of all the right isolated leaping points is written as $\Gamma^r(\omega)$. Similarly we define the set $\Gamma^l(\omega)$ of left isolated leaping points.

For $M\subset\bar{E}$, let

$$\Gamma^r_M(\omega)=\{t\,|\,t\in\Gamma^r(\omega),x(t,\omega)\in M\}\tag{1}$$

$$\bar{\Gamma}^{r+}_M(\omega)=\{t\,|\,\text{there exist non-increasing } t_n\in\Gamma^r_M(\omega),\text{ such that } t_n\downarrow t\}\tag{2}$$

Furthermore, call $\bar{\Gamma}^{r+}_M$ the right closure of $\Gamma^r_M(\omega)$. Similarly, let

$$\Gamma^l_M(\omega)=\{t\,|\,t\in\Gamma^l(\omega),x(t-0,\omega)\in M\}\tag{3}$$

and the left closure

$$\bar{\Gamma}^{l-}_M(\omega)=\{t\,|\,\text{there exist non-decreasing } t_n\in\Gamma^l_M(\omega),\text{ such that } t_n\uparrow t\}\tag{4}$$

Obviously, $0\in\Gamma^l(\omega)\cap\Gamma^r(\omega)$. If $\sigma(\omega)<\infty$ then $\sigma(\omega)\in\Gamma^r(\omega)$.

Theorem 1. For almost all $\omega\in\Omega$,

$$\Gamma^l(\omega)=\{\eta\,|\,[\lambda,\eta)\in\mathscr{U}(\omega)\}\cup\{0\}\tag{5}$$

$$\Gamma^r(\omega)=\{\lambda\,|\,[\lambda,\eta)\in\mathscr{U}(\omega)\}\cup\{\sigma(\omega)\,|\,\sigma(\omega)<\infty\}\tag{6}$$

Proof. It suffices to prove (6). It is obvious that the right-hand set is included in $\Gamma^r(\omega)$. Suppose $t\in\Gamma^r(\omega)$ and $t\neq\sigma(\omega)$. By the definition, there exists $\varepsilon>0$ such that $(t,t+\varepsilon)\cap\Gamma(\omega)=\varnothing$, and therefore taking arbitrarily $r\in(t,t+\varepsilon)\cap R$, $t=\lambda_r(\omega)<r<\eta_r(\omega)$. By (2.6) we know t belongs to the right-hand set in (6), and the proof is completed. QED

Theorem 2. For almost all ω,

$$\Gamma(\omega) = \bar{\Gamma}^{r+}(\omega) = \bar{\Gamma}^{l-}(\omega) \tag{7}$$

Proof. $\bar{\Gamma}^{r+}(\omega) \subset \Gamma(\omega)$ is obvious. Suppose $t \in \Gamma(\omega)$. If $t = 0$ or $t = \sigma(\omega)$, then obviously $t \in \Gamma^r(\omega) \subset \bar{\Gamma}^{r+}(\omega)$. If $t \in (0, \sigma(\omega))$ and $t \notin \Gamma^r(\omega)$, take arbitrarily strictly decreasing $r_n \downarrow t$, $r_n \in R \cap [0, \sigma(\omega))$. By the definition of $\lambda_{r_n}(\omega)$ we have $t \leqslant \lambda_{r_n}(\omega) < r_n$. Since $t \notin \Gamma^r(\omega)$ it follows that $t < \lambda_{r_n}(\omega) < r_n$. Consequently there exists a subsequence r'_n of r_n such that $\lambda_{r'_n}(\omega)$ strictly decreases. When $r'_n \downarrow t$, $\lambda_{r'_n}(\omega) \downarrow t$. By Theorem 1, $\lambda_{r'_n}(\omega) \in \Gamma^r(\omega)$. Hence $t \in \bar{\Gamma}^{r+}(\omega)$. Therefore $\Gamma(\omega) = \bar{\Gamma}^{r+}(\omega)$. Similarly we can prove $\Gamma(\omega) = \bar{\Gamma}^{l-}(\omega)$, and the proof is concluded. QED

Theorem 3. For almost all $\omega \in \Omega$,

$$S_E(\omega) = C_1(\omega) \cup \{\lambda \mid \lambda \in \Gamma^r(\omega), x(\lambda, \omega) \in E\} \subset C_2(\omega) \tag{8}$$

$$[0, \sigma(\omega)) - C_2(\omega) \subset S_\infty(\omega) \subset \Gamma(\omega) \tag{9}$$

$$L\{C_1(\omega)\} = L\{C_2(\omega)\} = \sigma(\omega) \tag{10}$$

where

$$S_M(\omega) = \{t \mid t \in [0, \sigma(\omega)), x(t, \omega) \in M\} \qquad M \subset \bar{E} \tag{11}$$

Proof. The inclusion relation in (8) is clear. $S_\infty(\omega) \subset \Gamma(\omega)$ is also clear. Suppose $t \in S_E(\omega)$, then there must exist $i \in E$, such that t belongs to some i-interval $[a, b)$ of $X(\omega)$. We take arbitrarily $r \in [a, b) \cap R$. Then $t \in [a, b) \subset [\lambda_r(\omega), \eta_r(\omega))$. Now, either $t \in (\lambda_r(\omega), \eta_r(\omega)) \subset C_1(\omega)$ or $t = \lambda_r(\omega) \in \Gamma^r(\omega)$, and $x(t, \omega) = i \in E$. Accordingly $S_E(\omega) \subset C_1(\omega) \cup \{\lambda \mid \lambda \in \Gamma^r(\omega), x(\lambda, \omega) \in E\}$. The inverse inclusion relation is clear. Thus (8) is proved and hence so is (9). Because $[0, \sigma(\omega)) = S_E(\omega) \cup S_\infty(\omega)$ and $L\{S_\infty(\omega)\} = 0$, from (8) and (9), follows (10). The proof is terminated QED

Corollary

$$L\{\Gamma(\omega)\} = 0.$$

Proof. Because $\Gamma(\omega) \subset [0, \sigma(\omega)) - C_1(\omega)$, by (10) we obtain $L\{\Gamma(\omega)\} = 0$. QED

Let

$$A(\omega) = S_E(\omega) - \Gamma(\omega) \simeq [0, \sigma(\omega)) - \Gamma(\omega) \tag{12}$$

$\mathscr{U}[A(\omega)]$ represents the whole of the composition interval in $A(\omega)$.

Theorem 4. For almost all $\omega \in \Omega$,

$$\mathscr{U}[A(\omega)] = \mathscr{U}(\omega) \tag{13}$$

Proof. Suppose $[\lambda, \eta) \in \mathscr{U}[A(\omega)]$. By definition, $(\lambda, \eta) \subset A(\omega)$. Then by (12), $(\lambda, \eta) \cap \Gamma(\omega) = \varnothing$. Moreover taking arbitrarily $t \in (\lambda, \eta)$, we have $x(t, \omega) \in E$, $(\lambda, \eta) \subset (\lambda_t(\omega), \eta_t(\omega)) \subset A(\omega)$. By the maximality of $[\lambda, \eta)$, $[\lambda, \eta) = [\lambda_t(\omega), \eta_t(\omega)) \in \mathscr{U}(\omega)$.

Suppose $[\lambda, \eta) \in \mathscr{U}(\omega)$. By definition, $(\lambda, \eta) \cap \Gamma(\omega) = \varnothing$. Noting (2.3) we have $(\lambda, \eta) \subset A(\omega)$. If it holds that $[\lambda', \eta') \supset [\lambda, \eta)$ and $(\lambda', \eta') \subset A(\omega)$, then $\lambda' \leqslant \lambda < \eta \leqslant \eta'$, and for any $s \in (\lambda', \eta')$ we have $s \notin \Gamma(\omega)$. Thus for any $t \in (\lambda, \eta)$ we have $\lambda_t(\omega) \leqslant \lambda'$, $\eta' \leqslant \eta_t(\omega)$. However, $\lambda, \eta \in \Gamma(\omega)$, so that we also have $\lambda = \lambda_t(\omega), \eta = \eta_t(\omega)$. Thus $\lambda' = \lambda = \lambda_t(\omega)$, $\eta' = \eta = \eta_t(\omega)$. The maximality of $[\lambda, \eta)$ is proved. Thereby $[\lambda, \eta) \in \mathscr{U}[A(\omega)]$. The proof is completed. QED

Theorem 5. Suppose $M \subset \bar{E}$, $N \subset \bar{E}$, and H is a non-conservative state set. Then all the following sets are $\mathscr{F}^0_\infty$-measurable sets:

$F_1 = \{\omega \mid \mathscr{U}(\omega)$ has a last U interval and it is a U_N interval$\}$

$F_2 = \{\omega \mid \mathscr{U}(\omega) = \mathscr{U}_N(\omega)\}$

$F_3 = \{\omega \mid$ there is a last U interval in $\mathscr{U}(\omega)$, and except for that last one, the remaining intervals in $\mathscr{U}(\omega)$ are all U_N intervals$\}$

$F_4 = \{\omega \mid \mathscr{U}^\tau(\omega) = {}_M\mathscr{U}^\tau(\omega)\}$

$F_5 = \{\omega \mid$ there is at most one U_H interval in $\mathscr{U}(\omega)$, and if there is one such, it is the last U interval$\}$

$F_6 = \{\omega \mid$ there is at least one U_H interval in $\mathscr{U}(\omega)$, and such a U_H interval is not the last U interval$\}$

Proof. Obviously first, for $r \geqslant 0$, $k \in \bar{E}$,

$$\{x(\eta_r - 0) = k\} \in \mathscr{F}^0_\infty \tag{14}$$

Secondly we proceed to prove

$$\{x(\lambda_r) = k\} \in \mathscr{F}^0_r \tag{15}$$

Actually, when $k \neq \infty$, by (2.1),

$$\{x(\lambda_r) = k\} = \{x(\lambda_r) = k, \lambda_r < r\} = \bigcup_{i=1}^{\infty} \bigcap_{n=i}^{\infty} \bigcup_{m=0}^{2^i-2} \left\{ x\left(\frac{m+1}{2^n} r\right) = k, \left[\frac{m+1}{2^n} r, r\right) \cap T(\omega) \right.$$

$$\left. \text{is a finite set}, \left[\frac{m-1}{2^n} r, \frac{m+1}{2^n} r\right) \cap T(\omega) \text{ is an infinite set} \right\} \in \mathscr{F}^0_r$$

$$\{x(\lambda_r) = \infty\} = (r < \sigma) - \bigcup_{k \in E} \{(\lambda_r) = k\} \in \mathscr{F}^0_r$$

It follows that

$$F_1 = \bigcap_{r \in R} \{r < \sigma, \eta_r = \sigma, x(\eta_r - 0) \in N\} \in \mathscr{F}^0_\infty$$

$$F_2 = \bigcap_{r \in R} \{(\sigma \leqslant r) \cup (r < \sigma, x(\eta_r - 0) \in N)\} \in \mathscr{F}^0_\infty$$

$$F_3 = \bigcap_{r\in R} \{(\sigma \leqslant r)\cup(r<\sigma, \eta_r=\sigma)\cup(r<\sigma, \eta_r<\sigma, x(\eta_r-0)\in N)\}\in\mathscr{F}^0_\infty$$

$$F_4 = \bigcap_{r\in R} \{(\sigma \leqslant r)\cup(r<\tau)\cup(\tau\leqslant r<\sigma, x(\lambda_r)\in M)\}\in\mathscr{F}^0_\infty$$

$$F_5 = \Omega - F_6 \in \mathscr{F}^0_\infty$$

$$F_6 = \bigcup_{r\in R} \{\eta_r<\sigma, x(\eta_r-0)\in H\}\in\mathscr{F}^0_\infty$$

The proof is over. QED

10.4 LEAPING INTERVAL AND KOLMOGOROV EQUATIONS

Suppose $M\subset \bar{E}$, $N\subset\bar{E}$, $S\subset E$. Let

$$\xi_{MS} = \inf\{t \mid \tau\leqslant t<\sigma, x(\lambda_t)\in M, x(t)\in S\} \tag{1}$$

$$\rho_{SN} = \sup\{t \mid 0\leqslant t<\sigma, x(t)\in S, t(\eta_t-0)\in N\} \tag{2}$$

$$\delta_S = \inf\{t \mid \tau\leqslant t<\sigma, t\in\Gamma, x(t)\in S\} \tag{3}$$

Here τ is the first leaping point of X and Γ the set of leaping points. For empty set $\varnothing$, set $\inf\varnothing=\sigma$ and $\sup\varnothing=0$.

Lemma 1. Suppose S is a finite set, then the 'inf' in (1) and (3) can be replaced by 'min'.

Proof. Suppose $t_n\downarrow\xi_{MS}$, $\tau\leqslant t_n<\sigma$, $x(\lambda_{t_n})\in M, x(t_n)\in S$. Since S is finite, by the right-continuity property of X,

$$x(\xi_{MS}) = \lim_{n\to 1} x(t_n)\in S$$

Therefore ξ_{MS} belongs to some i ($\in S$) interval $[a,b)$. When n is large enough, $t_n\in[a,b)$. Thus $x(\xi_{MS})=x(t_n)$, $\lambda_{\xi_{MS}}=\lambda_{t_n}$. Hence $x(\lambda_{\xi_{MS}})=x(\lambda_{t_n})\in M$. So we have proved that 'inf' in (1) can be replaced by 'min'. The remaining proof is similar, and the proof is terminated. QED

Lemma 2.

$$\xi_{MS} = \inf\{\xi_{Mj} \mid j\in S\} = \inf\{\xi_{kS} \mid k\in M\} = \inf\{\xi_{kj} \mid k\in M, j\in S\} \tag{4}$$

$$\rho_{SN} = \sup\{\rho_{Sj} \mid j\in N\} = \sup\{\rho_{kN} \mid k\in S\} = \sup\{\rho_{kj} \mid k\in S, j\in N\} \tag{5}$$

$$\delta_S = \inf\{\delta_k \mid k\in S\} \tag{6}$$

Proof. Let

$$A_{MS} = \{t \mid \tau\leqslant t<\sigma, x(\lambda_t)\in M, x(t)\in S\} \tag{7}$$

Obviously, $A_{Mj}\subset A_{MS}$ ($j\in S$) so that $\xi_{Mj}\geqslant\xi_{MS}$ ($j\in S$). On the other hand, for any $\varepsilon>0$, there exists $t\in A_{MS}=\bigcup_{j\in S}A_{Mj}$ such that $t<\xi_{MS}+\varepsilon$. Thus there exists $j\in S$ such that $t\in A_{Mj}$. Hence $\xi_{Mj}\leqslant t<\xi_{MS}+\varepsilon$. So we have proved that $\xi_{MS}=\inf\{\xi_{Mj} \mid j\in S\}$. The remainder can be proved similarly. The proof is concluded. QED

Lemma 3. ξ_{MS}, δ_S are Markov times of X, and ρ_{SN} is a random variable.

Proof. By Lemma 2, it suffices to prove, for $k\in\bar{E}$, $j\in E$, that ξ_{kj}, δ_j are Markov times and ρ_{kj} is a random variable.

Actually, noting (3.15), for any $u\geqslant 0$

$$(\xi_{kj}<u<\sigma) = \bigcup_{\substack{r<u\\ r\in R}} \{\tau\leqslant r<\sigma, x(\lambda_r)=k, x(r)=j, u<\sigma\}\in\mathscr{F}^0_u$$

Secondly,

$$(\delta_j<u<\delta) = \bigcup_{\substack{r<u\\ r\in R}} \Bigg\{(\tau\leqslant r<\sigma, x(r)=j)\cap\Bigg[\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty}\bigcup_{v=1}^{2^n-1}$$

$$\left(X \text{ has infinitely many jumping points in } \left[\frac{v-1}{2^m}r, \frac{v}{2^m}r\right)\right.$$

$$\left.\left.\left.\text{and takes a constant value } j \text{ in } \left[\frac{v}{2^m}r, r\right)\right)\right]\right\}\in\mathscr{F}^0_u$$

and

$$(\rho_{kj}>u) = \bigcup_{\substack{r\in R\\ r>u}} \{0\leqslant\sigma<0, x(r)=k, x(\eta_r-0)=j\}\in\mathscr{F}^0_\infty$$

The proof is completed. QED

Definition 1. Assume $X\in\mathscr{H}_s$. We call X pure entrance from M, if $P_i\{\mathscr{U}^\tau={}_M\mathscr{U}^\tau\}=1$ ($i\in E$); we call X pure exit to N, if $P_i\{\mathscr{U}=\mathscr{U}_N\}=1$ ($i\in E$); we call X quasi-exit to N, if $P_i(\Omega_0)=1$ ($i\in E$). Here

$\Omega_0 = (F_2-F_1)\cup F_3$

$= \{\omega \mid$ the last U interval does not exist in $\mathscr{U}(\omega)$ and, moreover, $\mathscr{U}(\omega)=\mathscr{U}_N(\omega)\}$

$\cup\{\omega \mid$ there exists the last U interval in $\mathscr{U}(\omega)$; except for the last U interval, the others are all U_N intervals$\}$

F_1, F_2 and F_3 are determined by Theorem 3.5.

Theorem 4. Suppose $X\in\mathscr{H}_s$, then the following conclusions are equivalent to each other:

(i) The process X is pure entrance from ∞, that is, $P_i\{\mathscr{U}^\tau={}_\infty\mathscr{U}^\tau\}=1$ ($i\in E$).

(ii) $P_i\{\xi_{EE} < 0\} = 0$ $(i \in E)$, ξ_{EE} is determined by (4.1).

(iii) The process X satisfies the system of forward equations.

(iv) For any $t > 0$, $P_i\{\tau < t < \sigma, x(\lambda_t) \in E\} = 0$ $(i \in E)$. If one of the above conclusions holds, then

(v) $P_i\{\xi_{\infty E} = \tau\} = 1$ $(i \in E)$. More precisely, for any $i \in E$, P_i almost all $\omega \in (\tau < \sigma)$, we have $\tau(\omega) \in \bar{\Gamma}_\infty^{r+}(\omega)$.

Proof. Clearly, (i)$\Rightarrow$(iv). Conversely by (iv) we deduce $P_i\{x(\lambda_r) \in E$ for all $r \in R \cap [\tau, \sigma)\} = 0$ $(i \in E)$. From this and noting (2.6) we obtain $P_i\{{}_E\mathscr{U}^\tau = \mathscr{U}^\tau\} = 0$ $(i \in E)$. This is just the condition (i).

(i)$\Rightarrow$(iii). Suppose $t > 0$. Since $P_i\{x(t) = \infty\} = 0$, thus for P_i almost all $\omega \in (t < \sigma)$, we have $t \in S_E(\omega)$. By Theorem 3.3, $t \in C_2(\omega)$. Hence there exists $[\lambda, \eta) \in \mathscr{U}(\omega)$ such that $t \in [\lambda, \eta)$. If $[\lambda, \eta) = [0, \tau(\omega))$, then evidently there is the last discontinuity point $\mu_t(\omega) \in T(\omega)$, before t. Or else, by conclusion (i), $[\lambda, \eta)$ is a ${}_\infty U$ interval, $x(\lambda) = \infty$. Thus $t \in (\lambda, \eta)$ so that $\lambda < \mu_t(\omega) < \eta$, $\mu_t(\omega) \in T(\omega)$. By Theorem 7.8.3, X satisfies the system of forward equations.

(iii)$\Rightarrow$(ii). By Theorem 7.8.3, if conclusion (iii) holds, then $p_k(\Omega_1) = 1$ $(k \in E)$, where

$$\Omega_1 = \{\omega \mid \text{for all } r \in R \cap [0, \sigma(\omega)),\ x(r, \omega) \in E, \mu_r(\omega) \in T(\omega)\}$$

If $\omega \in (\xi_{ij} < \sigma)$ $(i, j \in E)$ then $x(\lambda_{\xi_{ij}}, \omega) = i$. So that there exists $r \in R \cap [0, \sigma(\omega))$ such that $\lambda_{\xi_{ij}}(\omega)$ and r are in the same i-interval; thus $\mu_r(\omega) = \lambda_{\xi_{ij}(\omega)}(\omega) \notin T(\omega)$. Therefore $\omega \notin \Omega_1$, namely, $(\xi_{ij} < \sigma) \subset \Omega - \Omega_1$. Thus by Lemma 2,

$$(\xi_{EE} < \sigma) = \bigcup_{i,j \in E} (\xi_{ij} < \sigma) \subset \Omega - \Omega_1$$

Conclusion (ii) is proved.

(ii)$\Rightarrow$(i). If $\omega(\mathscr{U}^\tau \neq {}_\infty\mathscr{U}^\tau)$, then there exists ${}_EU$ interval $[\lambda, \eta) \in \mathscr{U}^\tau(\omega)$. Hence there exists some $i \in E$ such that $\xi_{Ei}(\omega) < \sigma(\omega)$. Consequently,

$$(\mathscr{U}^\tau \neq {}_\infty\mathscr{U}^\tau) \subset \bigcup_{i \in E} (\xi_{Ei} < \sigma) = (\xi_{EE} < \sigma)$$

Conclusion (i) is proved.

(i)$\Rightarrow$(v). By conclusion (i), for P_i almost all ω, $\Gamma_E^r(\omega) - \{0\} = \phi$. Moreover, by Theorem 3.2, if $\tau(\omega) < \sigma(\omega)$, then either $\tau(\omega)$ is the left end-point of some U interval, and then $x(\tau(\omega), \omega) = \infty, \tau(\omega) \in \Gamma_\infty^r(\omega)$; or there exists strictly decreasing $\lambda_n \downarrow \tau(\omega), \lambda_n$ being the left end-point of some U interval. By conclusion (i), $x(\lambda_n, \omega) = \infty$ so that $\tau(\omega) \in \bar{\Gamma}_\infty^{r+}(\omega)$, and the proof is concluded. QED

Theorem 5. Suppose $X \in \mathscr{H}_s$, then the following conclusions are equivalent to each other:

(i) The process X satisfies the system of backward equations.

(ii) The process is quasi-exit to ∞.

(iii) $P_i\{\omega \mid \mathscr{U}(\omega)$ there is at most one U_H interval in $\mathscr{U}(\omega)$; if there is one, it is the last U interval$\} = 1$ $(i \in E)$.

(iv) $P_i\{\tau_1 \in \Gamma(\omega), \tau_1 < \sigma\} = 0$ $(i \in H)$, where τ_1 is the first jumping point, and H is the non-conservative state set.

(v) For any $t \geqslant 0$, $P_i\{t < \sigma, x(\eta_t - 0) \in H, \eta_t < \sigma\} = 0$ $(i \in E)$

Proof. (ii) and (iii) have the same meaning.

(i)$\Rightarrow$(iii). By Theorem 7.8.3 $P_i(\Omega_2) = 1$, where $\Omega_2 = \{\omega \mid$ for all $r \in R \cap [0, \sigma(\omega))$, $v_r(\omega) \in T(\omega)\}$. By Theorem 7.7.2, $P_i(\Omega_3) = 1$, where $\Omega_3 = \{\omega \mid$ for all $r \in R \cap [0, \sigma(\omega))$, if $x(\eta_r - 0, \omega) \in H$ and $\eta_r(\omega) < \sigma(\omega)$, then $x(\eta_r, \omega) = \infty\}$. We are going to prove $\Omega_2 \cap \Omega_3 \subset F_5$. Actually, suppose $\omega \in \Omega_2 \cap \Omega_3$. If $X(\omega)$ has a U_H interval $[\lambda, \eta)$, namely, $x(\eta - 0, \omega) \in H$, then there exists $r \in R \cap [0, \sigma(\omega))$ such that $\eta(\omega) = v_r(\omega) = \eta_r(\omega)$. If $\eta(\omega) < \sigma(\omega)$, since $\omega \in \Omega_2$ it follows that $\eta(\omega) = v_r(\omega) \in T(\omega)$. Because $\omega \in \Omega_3$ we have $\eta = \eta_r(\omega) \in \Gamma(\omega)$. This is a contradiction. Therefore surely $\eta(\omega) = \sigma(\omega)$, that is $[\lambda, \eta)$ is the last U interal. So we have proved $\Omega_2 \cap \Omega_3 \subset F_5$ and, hence, we have derived conclusion (iii).

(iii)$\Rightarrow$(iv) is obvious. By conclusion (iv) and the strong Markov property we obtain conclusion (v) easily.

(v)$\Rightarrow$(i). By conclusion (v) we have $P_i\{\Omega_4\} = 1$ $(i \in E)$, where $\Omega_4 = \{\omega \mid$ for any $r \in R \cap [0, \sigma(\omega))$, if $x(\eta_r - 0, \omega) \in H$, then surely $\eta_r(\omega) = \sigma(\omega)\}$.

Fixing $t \geqslant 0$, because $P_i\{x(t) = \infty\} = 0$, by Theorem 3.3, for P_i almost all $\omega \in (t < \sigma) \cap \Omega_4$, we have $t < S_E(\omega) \subset C_2(\omega)$. Thus there exists $[\lambda, \eta) \in \mathscr{U}(\omega)$ such that $t \in [\lambda, \eta)$. If $x(\eta - 0, \omega) = \infty$, then $\lambda \leqslant t < v_t(\omega) < \eta$, so that $v_t(\omega) \in T(\omega)$. Otherwise $x(\eta - 0, \omega) \in H$. If $v_t(\omega) < \eta$, obviously $v_t(\omega) \in T(\omega)$. If $v_t(\omega) = \eta$, then there exists $r \in R \cap [0, \sigma(\omega))$ such that $\eta = v_t(\omega) = v_r(\omega) = \eta_r(\omega)$. Since $\omega \in \Omega_4$, it follows that $\eta = \sigma(\omega)$. Thus $v_t(\omega) = \sigma(\omega) \in T(\omega)$. Therefore there exists constantly $v_t(\omega) \in T(\omega)$. By Theorem 7.8.3, X satisfies the system of backward equations, and the proof is terminated. QED

By Theorems 4 and 5 we obtain the following two theorems immediately.

Theorem 6. Suppose $X \in \mathscr{H}_s$, then the following conclusions are equivalent to each other:

(i) X satisfies the systems of backward and forward equations simultaneously.

(ii) X makes its pure entrance from ∞ and quasi-exist to ∞.

(iii) $P_i\{\xi_{EE} < \sigma\} = 0$ $(i \in E)$, $P_i\{\tau_1 \in \Gamma, \tau_1 < \sigma\} = 0$ $(i \in H)$.

(iv) $P_i\{\mathscr{U}^\tau = {}_\infty\mathscr{U}^\tau\} = 1$, $P_i\{$there is at most one U_H interval in $\mathscr{U}$; if there is one, it is the last U interval$\} = 1$ $(i \in E)$.

(v) $P_i\{\tau < t < \sigma, x(\lambda_t) \in E\} = 0$ $(t > 0, i \in E)$
$P_i\{t < \sigma, x(\eta_t - 0) \in H, \eta_t < \sigma\} = 0$ $(t \geqslant 0, i \in E)$

Theorem 7. Suppose $X \in \mathscr{H}_s$, then the following conclusions are equivalent to each other:

(i) X satisfies neither backward nor forward equations.

(ii) There exist $i \in E$, $k \in H$ such that $P_i\{\xi_{EE} < \sigma\} > 0$, $P_k\{\tau_1 \in \Gamma, \tau_1 < \sigma\} > 0$.

(iii) There exist $i, k \in E$ such that $P_i\{\mathscr{U}^\tau = {}_E\mathscr{U}^\tau\} > 0$ and P_k {there is a U_H interval in $\mathscr{U}$, but it is not the last U interval} > 0.

(iv) There exist $t_1 > 0$, $t_2 \geq 0$, $i, k \in E$ such that

$$P_i\{\tau < t_1 < \sigma, x(\lambda_{t_1}) \in E\} > 0 \qquad P_k\{t_2 < \sigma, X(\eta_{t_2} - 0) \in H, \eta_{t_2} < \sigma\} > 0.$$

Theorem 8. Suppose $X \in \mathscr{H}_s$, $M \subset \bar{E}$. Then the following conclusions are equivalent to each other:

(i) The process X makes its pure entrance from M, that is

$$P_i\{\mathscr{U}^\tau = {}_M\mathscr{U}^\tau\} = 1\,(i \in E)$$

(ii) $P_i\{\xi_{\bar{M}E} < \sigma\} = 0\,(i \in E, \bar{M} = \bar{E} - M)$

(iii) For any $t > 0$, $P_i\{\tau < t < \sigma, x(\lambda_t) \in \bar{M}\} = 0$ $(i \in E)$. Or equivalently, for any $i \in E$, $t > 0$, for P_i almost all $\omega \in (\tau < t < \sigma)$, we have $x[\lambda_t(\omega), \omega] \in M$. If one of the above conclusions holds, then

(iv) $P_i\{\xi_{ME} = \tau\} = 1$ $(i \in E)$. More precisely, for any $i \in E$, for P_i almost all $\omega \in (\tau < \sigma)$, $\tau(\omega) \in \bar{\Gamma}^{r+}_M(\omega)$.

Proof. (i)$\Rightarrow$(iii). This will be done by following the proof in Theorem 5 that (i) and (iv) are equivalent.

(i)$\Rightarrow$(ii). Suppose $\omega \in (\xi_{\bar{M}E} < \sigma) = \bigcup_{k \in E}(\xi_{\bar{M}k} < \sigma)$. Then there exists some $k \in E$ such that $\xi_{\bar{M}E}(\omega) < \sigma(\omega)$. So $[\lambda_{\xi_{\bar{M}k}}(\omega), \eta_{\xi_{\bar{M}k}}\omega)) \in \mathscr{U}^\tau(\omega)$, and by $x[\lambda_{\xi_{\bar{M}i}}(\omega), \omega] \in \bar{M}$ we know it is a ${}_{\bar{M}}U$ interval so that $\omega \in (\mathscr{U}^\tau \neq {}_M\mathscr{U}^\tau)$.

(ii)$\Rightarrow$(i). By the homogeneity of X and the conclusion (ii) we have for all $r \in R$, $P_i\{\theta_r(\xi_{\bar{M}E} < \sigma)\} = 0$, θ being the translation operator. Let $\Omega_0 = \bigcap_{r \in R}\{\Omega - \theta_r(\xi_{\bar{M}E} < \sigma)\}$, then $P_i(\Omega_0) = 1$. When $\omega \in \Omega_0$, for all $r \in [0, \sigma(\omega))$, there is no ${}_{\bar{M}}U$ interval in $[\max(r, \tau(\omega)), \sigma(\omega))$, so that all the U intervals in $[\tau(\omega), \sigma(\omega))$ are ${}_MU$ intervals, namely, $\Omega_0 \subset (\mathscr{U}^\tau = {}_M\mathscr{U}^\tau)$.

Suppose one of the conclusions (i)–(iii) holds, then for P_i almost all ω, $\Gamma^r_{\bar{M}}(\omega) - \{0\}$ is an empty set, so that $\tau(\omega) \notin \bar{\Gamma}^{r+}_{\bar{M}}(\omega)$. By Theorem 3.2, $\tau(\omega) \in \Gamma(\omega) = \bar{\Gamma}^+(\omega) = \bar{\Gamma}^{r+}_M(\omega)$, and the proof is completed. QED

10.5 THE ${}_Mg_n$ TRANSFORMATION AND ITS STRONG LIMIT THEOREM

Suppose $X = \{x(t), t < \sigma\} \in \mathscr{H}_s$, $M \subset \bar{E}$, and finite sets $D_n \uparrow E$. Let

$${}_M\beta_0^n = 0$$

${}_M\tau_1^n$ be the first leaping point of X

$${}_M\beta_k^n = \min\{t \mid {}_M\tau_k^n \leq t < \sigma, X(\lambda_t) \in M, X(t) \in D_n\} \tag{1}$$

${}_M\tau_{k+1}^n$ be the first leaping point after ${}_M\beta_k^n$

λ_t is the last leaping point before t. By Lemma 4.1, the 'min' in (1) exists, so long as the set after it is not empty, or else let it take value σ.

By Lemma 4.3, ${}_M\tau_k^n$, ${}_M\beta_k^n$ are Markov times of X, and we can easily see that for every n, $\{{}_M\tau^n, {}_M\beta^n\}$ is the canonical pair of X, and satisfies the conditions (a), (c) in Theorem 9.3.3. Thus by theorem 9.3.3, ${}_MX^n = W_{{}_M\tau^n, {}_M\beta^n}(X) \in \mathscr{H}_D$. Actually, ${}_MX^n \in \mathscr{H}_1$, that is, ${}_MX^n$ is a first-order process.

Definition 1. The transformation $W_{{}_M\tau^n, {}_M\beta^n}$ obtained by transforming $X \in \mathscr{H}_s$ into ${}_MX^n \in \mathscr{H}_1$ in the same way as stated above is called the ${}_Mg_n$ transformation. When $M = \bar{E}$, we simply write ${}_{\bar{E}}g_n$ as g_n.

Thus,

$${}_MX^n = {}_Mg_n(X) \tag{2}$$

Obviously, when $n \uparrow \infty$,

$${}_MA^n = \bigcup_{k=0}^{\infty} [{}_M\beta_k^n, {}_M\tau_{k+1}^n) \uparrow {}_MA$$

$${}_MA_1^n = \bigcup_{k=0}^{\infty} ({}_M\beta_k^n, {}_M\tau_{k+1}^n) \uparrow {}_MA_1 \tag{3}$$

Theorem 1. For almost all $\omega \in \Omega$ we have

$$\mathscr{U}[{}_MA_1^\tau(\omega)] = {}_M\mathscr{U}^\tau(\omega) \tag{4}$$

$$C_2[{}_MA^\tau(\omega)] = C_2[{}_MA_1^\tau(\omega)] = {}_MC_2^\tau(\omega) \tag{5}$$

where ${}_MA_1^\tau = {}_MA_1 - (0, \tau)$, ${}_MA^\tau = {}_MA - [0, \tau)$, τ is the first leaping point. Here $\mathscr{U}(A)$, $C_2(A)$ are understood according to Definition 9.3.1 and ${}_M\mathscr{U}(\omega)$, ${}_MC_2^\tau(\omega)$ are understood according to Definition 2.3 and (2.8).

Proof. Note first that according to the definition, for any $t \in {}_MB^n = \bigcup_{k=1}^{\infty}[{}_M\tau_k^n, {}_M\beta_k^n)$, there must exist

$$x(t) \bar{\in} D_n \qquad or \qquad x(t) \in D_n, x(\lambda_t) \bar{\in} M \tag{6}$$

while for $\tau \leq t \in {}_MA^n$, there surely exists

$$x(\lambda_t) \in M \tag{7}$$

We proceed to prove (4) so that (5) follows from (4).

(a) Suppose $[\lambda,\eta)\in\mathscr{U}[{}_MA_1^{\tau}(\omega)]$. Since $(\lambda,\eta)\subset {}_MA_1^{\tau}(\omega)$, and ${}_MA_1^{\tau}(\omega)\cap\Gamma(\omega)=\varnothing$, it follows that $(\lambda,\eta)\cap\Gamma(\omega)=\varnothing$. Therefore take arbitrarily $t\in(\lambda,\eta)$, and we have $(\lambda_t,\eta_t)\supset(\lambda,\eta)$. If $\lambda_t<\lambda$, then we can take arbitrarily $s\in(\lambda_t,\lambda)$. Since $x(s,\omega)\in E$, there exists n such that $x(s,\omega)\in D_n$. But $\lambda_s=\lambda_t$, since $t\in(\lambda,\eta)\subset{}_MA_1^{\tau}(\omega)\subset {}_MA^{\tau}(\omega)$. So by (6) and (7), $x(\lambda_s,\omega)=x(\lambda_t,\omega)\in M$, and moreover $\eta_s=\eta_t\geqslant\eta$. On account of (6), $s\bar{\in}{}_MB^n(\omega)$, $s\in{}_MA^n(\omega)$. Namely, there exists $l\geqslant 1$ such that $s\in[{}_M\beta_l^n(\omega),{}_M\tau_{l+1}^n(\omega))$. Because $\eta_s={}_M\tau_{l+1}^n(\omega)\geqslant\eta$, $(s,\eta)\subset({}_M\beta_l^n(\omega),{}_M\tau_{l+1}^n(\omega))\subset {}_MA_1^{\tau}(\omega)$. This contradicts the fact that $[\lambda,\eta)$ is a composite interval of ${}_MA_1^{\tau}(\omega)$. Thus $\lambda_t=\lambda$. That $\eta_t=\eta$ can be proved similarly. Hence $\lambda,\eta\in\Gamma(\omega)$.

Summing up what precedes we get $[\lambda,\eta)\in{}_M\mathscr{U}^{\tau}(\omega)$.

(b) Suppose $[\lambda,\eta)\in{}_M\mathscr{U}^{\tau}(\omega)$, that is, $\lambda\geqslant\tau(\omega)$, $\lambda,\eta\in\Gamma(\omega)$, $(\lambda,\eta)\cap\Gamma(\omega)=\varnothing$, $x(\lambda,\omega)\in M$.

Take arbitrarily $s\in(\lambda,\eta)$, then $\lambda=\lambda_s$, $\eta=\eta_s$, and $x(s,\omega)\in E$. Thus there exists n such that $x(s,\omega)\in D_n$, and $x(\lambda_s,\omega)=x(\lambda,\omega)\in M$. By (6) and (7) we obtain $s\in{}_MA_n$, so that there exists $k\geqslant 1$ such that $s\in[{}_M\beta_k^n(\omega),{}_M\tau_{k+1}^n(\omega))$, thus $\eta=\eta_s={}_M\tau_{k+1}^n$. Hence $(s,\eta)\subset({}_M\beta_k^n(\omega),{}_M\tau_{k+1}^n(\omega))\subset{}_MA_1^{\tau}(\omega)$. By the arbitrariness of s, $(\lambda,\eta)\subset {}_MA_1^{\tau}(\omega)$.

Next, if $[\lambda',\eta')\supset[\lambda,\eta)$, and $(\lambda',\eta')\subset{}_MA_1^{\tau}(\omega)$, then we can take arbitrarily $t\in(\lambda,\eta)$. Since λ, $\eta\in\Gamma(\omega)$ it follows that $\lambda=\lambda_t$, $\eta=\eta_t$. On the other hand, since $(\lambda',\eta')\subset{}_MA_1^{\tau}(\omega)$ and ${}_MA_1^{\tau}(\omega)\cap\Gamma(\omega)=\varnothing$, so $\lambda_t\leqslant\lambda'$, $\eta'\leqslant\eta_t$. Thus $\lambda'=\lambda$, $\eta'=\eta$. Namely, $[\lambda',\eta')=[\lambda,\eta)$. Thus we have proved the maximal property of $[\lambda,\eta)$.

Summarizing what precedes, we obtain $[\lambda,\eta)\in\mathscr{U}[{}_MA_1^{\tau}(\omega)]$, and the proof is completed. QED

Remark

$\mathscr{U}[{}_MA^{\tau}(\omega)]={}_M\mathscr{U}^{\tau}(\omega)$ is not necessarily true. For instance, if $X\in\mathscr{H}_1$, then ${}_EA(\omega)=[0,\sigma(\omega))$, so $\mathscr{U}[{}_EA^{\tau}(\omega)]=[\tau(\omega),\sigma(\omega))$. But ${}_E\mathscr{U}^{\tau}(\omega)=\{[\lambda_n,\lambda_{n+1})|n=1,2,\ldots\}$, λ_n is the nth leaping point of $X(\omega)$.

On account of (3), according to Theorems 9.3.1 and 9.3.4,

$$ {}_MX^n={}_M\mathscr{g}_n(X)\in\mathscr{H}_1 \tag{8}$$

$$ {}_M\mathscr{g}_m({}_MX^n)={}_MX^m \qquad m<n \tag{9}$$

Furthermore the strong limit of ${}_MX^n$ exists and it is just ${}_MX\in\mathscr{H}_s$, namely,

$$ \lim_{n\to\infty}{}_MX^n={}_MX\in\mathscr{H}_s \tag{10}$$

where

$$ {}_MX=\{x({}_M\gamma^{-1}(t)),t<{}_M\sigma\} \qquad {}_M\sigma=L\{{}_MA\} \tag{11}$$

Here ${}_M\gamma$ is the transformation determined by the sequence of pairs $\{{}_M\tau^n,{}_M\beta^n\}$ $(n\geqslant 1)$, that is,

$$ {}_M\gamma(u)=L\{{}_MA\cap[0,u)\} \qquad u\in[0,\sigma) \tag{12}$$

By Theorem 1 and the conclusion (ii) of Lemma 9.4.4,

$$ {}_M\gamma(u)=L\{C_2[{}_MA(\omega)]\cap[0,u)\}=L\{{}_MC_2(\omega)\cap[0,u)\} \tag{13}$$

$$ {}_M\sigma=L\{C_2[{}_MA(\omega)]\}=L\{{}_MC_2(\omega)\} \tag{14}$$

By Lemma 9.4.4, the inverse transformation ${}_M\gamma^{-1}$ maps $[0,{}_M\sigma)$ onto $\bar{C}_2^+({}_MA)={}_M\bar{C}_2^+(\omega)$. Thus we obtain the preceding part of the following theorem.

Theorem 2. Suppose $X\in\mathscr{H}_s$. Then (8)–(10) hold, and ${}_MX$ is the process making its pure entrance from M. If X satisfies the system of backward equations, then X^n, ${}_MX$ all satisfy the system of backward equations, and have the same Q matrix as X.

Remark

The intuitive meaning of ${}_MX$ is as follows. Reserve all the parts of Xwhich correspond to $[0,\tau)$ and ${}_MU$ interval, abandon all the remaining parts,and translate the reserved parts towards the left in the original order, linking them up in a disjoint manner and, thus we obtain the process ${}_MX\in\mathscr{H}_s$.

Proof. Because among the U intervals of ${}_MX$, except for the first U interval $[0,\tau(\omega))$, the remainder are all ${}_MU$ intervals, ${}_MX$ makes its pure entrance from M. Quoting Theorem 4.5(iii) we derive the conclusion about equations in this theorem. As X, X^n, ${}_MX$ have the same first leaping point $\tau(\omega)$, their Q matrices are the same, and the proof is concluded. QED

Corollary 1

Assume $x\in\mathscr{H}_s$, and let $P\{{}_M\mathscr{U}^{\tau}=\mathscr{U}^{\tau}\}=1$. Then

$$ \lim_{n\to\infty}{}_MX^n=X \tag{15}$$

Proof. This is quite obvious, for under the assumption of Corollary 1, ${}_MX=X$. QED

Especially, when $M=\bar{E}$, the assumption in Corollary 1 is always satisfied. Thus we have

Corollary 2

Suppose $X \in \mathscr{H}_s$. Then

$$\lim_{n\to\infty} X^n = X \tag{16}$$

where

$$X^n = g_n(X) \in \mathscr{H}_1 \tag{17}$$

$$X^m = g_m(X^n) \qquad m < n \tag{18}$$

If X satisfies the system of backward equations, so does X^n; furthermore X and X^n have the same Q matrix.

Corollary 2 is just Theorem 9.7.4, that is the basic result, in Zheng-ting Hou (1974), namely the first construction theorem.

10.6 ENTRANCE DECOMPOSITION OF A PROCESS

Theorem 1. Suppose $X \in \mathscr{H}_s$, finite set $M \subset E$. If

$$P\{x(\tau) \in M \mid \tau < \sigma\} = 1 \tag{1}$$

where τ is the first leaping point, then $X \in \mathscr{H}_1$, that is, X is a one-order process. Especially when X makes its pure entrance from M, (1) holds, thus X is a one-order process.

Proof. When X makes its pure entrance from M, all U intervals of X (except the first one) are $^{+}_{M}U$ intervals. Suppose $\tau < \sigma$. By Theorem 3.2, $\tau \in \Gamma \subset \bar{\Gamma}^{r+}_{M}$. If $\tau \in \Gamma^r_M$, then $x(\tau) \in M$, or else there exist strictly decreasing $\lambda_n \in \Gamma^r_M$ such that $\lambda_n \downarrow \tau$. But $x(\lambda_n) \in M$; by the right-continuity of X and the finiteness of M, surely $x(\tau) = \lim_{n\to\infty} x(\lambda_n) \in M$. Therefore it is always true that $x(\tau) \in M$. Thus (1) is proved.

By (1) and the strong Markov property of X we can obtain the first, second, third,... leaping points $\tau^1, \tau^2, \tau^3, \ldots$, and moreover if $\tau^n < \sigma$, then $x(\tau^n) \in M$. If for $t < \sigma(\omega)$, $X(\omega)$ has infinitely many leaping points in $[0, t]$, then in $[0, t]$, $X(\omega)$ takes values in M infinitely many times. Because M is finite, there exists $i \in M$ such that $X(\omega)$ takes the value i inifnitely many times in $[0, t]$. Thus $X(\omega)$ has infinitely many i-intervals in $[0, t]$. This contradicts Theorem 7.7.1. Consequently X has only finitely many leaping points in $[0, t]$ $(t < \sigma)$. Therefore X is a first-order process, and the proof is completed. QED

Theorem 2. Assume $X \in \mathscr{H}_s$. For any $i \in \bar{E}$, let

$${}_iX = \{x({}_i\gamma^{-1}(t)), t < {}_i\sigma\} \tag{2}$$

where ${}_iC_2$ is determined by (2.8), and

$${}_i\sigma = L\{{}_iC_2\}, {}_i\gamma(u) = L\{{}_iC_2 \cap [0, u)\} \tag{3}$$

${}_i\gamma^{-1}$ is the inverse transformation of ${}_i\gamma$ and it maps $[0, {}_i\sigma)$ onto ${}_i\bar{C}_2^+$. Then:

(i) For $i \in E$, ${}_iX \in \mathscr{H}_1$ is the process making its pure entrance from i. If X satisfies the system of backward equations, then so does ${}_iX$; and what is more, X and ${}_iX$ have the same Q matrix.

(ii) ${}_\infty X \in \mathscr{H}_s$ is the process making its pure entrance from ∞, satisfying the system of forward equations. ${}_\infty X$ and X have the same Q matrix.

Proof. Letting $M = \{i\}$ $(i \in \bar{E})$ in Theorem 5.2, noting Theorem 1 and applying Theorems 4.4 and 4.5, we can derive the theorem. QED

Remark

For the intuitive meaning of Theorem 2 see the remark in Theorem 5.2.

10.7 THE ${}_Mf_n$ TRANSFORMATION AND ITS STRONG LIMIT THEOREM

Suppose $X \in \mathscr{H}_s$, Γ is a set of leaping points, $M \subset E$ and finite set $D_n \uparrow E$. Let

$$\begin{aligned} &{}_M\bar{\beta}^n_0 = 0 \\ &{}_M\bar{\tau}^n_1 \text{ be the first leaping point of } X \\ &{}_M\bar{\beta}^n_k = \min\{t \mid {}_M\bar{\tau}^n_k \leqslant t < \sigma, t \in \Gamma, x(t) \in D_n \cap M\} \\ &{}_M\bar{\tau}^n_{k+1} \text{ be the first leaping point after } {}_M\bar{\beta}^n_k \end{aligned} \tag{1}$$

We can easily prove ${}_M\bar{\tau}^n_k$, ${}_M\bar{\beta}^n_k$ are Markov times of X, for every n, $\{{}_M\bar{\tau}^n, {}_M\bar{\beta}^n\}$ is the canonical pair of X, and the conditions (a) and (c) in Theorem 9.3.3 are satisfied. Thus according to Theorem 9.3.3, ${}_M\bar{X}^n = W_{{}_M\bar{\tau}^n, {}_M\bar{\beta}^n} \in \mathscr{H}_D$. Actually ${}_M\bar{X}^n \in \mathscr{H}_1$ and is a one-order process.

Definition 1. The transformation $W_{{}_M\bar{\tau}^n, {}_M\bar{\beta}^n}$ obtained by transforming $X \in \mathscr{H}_s$ to ${}_M\bar{X}^n$ according to the above method is called the ${}_Mf_n$ transformation. When $M = E$, we simply write ${}_Ef_n$ as f_n. Thus

$${}_M\bar{X}^n = {}_Mf(X) \tag{2}$$

Obviously, when

$$
\begin{aligned}
{}_M\bar{A}^n &= \bigcup_{k=0}^{\infty} [{}_M\bar{\beta}_k^n, {}_M\bar{\tau}_{k+1}^n) \uparrow {}_M\bar{A} \\
{}_M\bar{A}_1^n &= \bigcup_{k=0}^{\infty} ({}_M\bar{\beta}_k^n, {}_M\bar{\tau}_{k+1}^n) \uparrow {}_M\bar{A}_1
\end{aligned} \tag{3}
$$

Following Theorem 5.1, we can prove:

Theorem 1. For almost all $\omega\in\Omega$,

$$\mathscr{U}[{}_M\bar{A}_1^\tau(\omega)] = {}_M\mathscr{U}^\tau(\omega) \tag{4}$$

$$C_2[{}_M\bar{A}^\tau(\omega)] = C_2[{}_M\bar{A}_1^\tau(\omega)] = {}_MC_1^\tau(\omega) \tag{5}$$

where ${}_M\bar{A}_1^\tau = {}_M\bar{A}_1 - (0,\tau)$, ${}_M\bar{A}^\tau = {}_M\bar{A} - [0,\tau)$, all the $\mathscr{U}$-quality and the C-quality are still to be understood by Definitions 9.3.1 and 2.3 and (2.8).

As a result of (3), according to Theorems 9.3.1 and 9.3.4,

$${}_M\bar{X}^n = {}_Mf_n(X)\in\mathscr{H}_1 \tag{6}$$

$${}_Mf_m({}_M\bar{X}^n) = {}_M\bar{X}^n \qquad m<n \tag{7}$$

The strong limit process of ${}_M\bar{X}^n$ exists, and by (4) and (5), the strong limit process is just the process ${}_MX\in\mathscr{H}_s$ in Theorem 5.2. Applying Theorem 4.5(iii) we derive the following theorem.

Theorem 2. Suppose $X\in\mathscr{H}_s$, then (6) and (7) hold, and

$$\lim_{n\to\infty} {}_M\bar{X}^n = {}_MX\in\mathscr{H}_s \tag{8}$$

If X satisfies the system of backward equations, then so do ${}_M\bar{X}^n$ and ${}_MX$; moreover X, ${}_M\bar{X}^n$, ${}_MX$ have the same Q matrix.

Corollary 1

Suppose $X\in\mathscr{H}_s$. If $P\{{}_M\mathscr{U}^\tau = \mathscr{U}^\tau\} = 1$, then

$$\lim_{n\to\infty} {}_M\bar{X}^n = X \tag{9}$$

Corollary 2

Assume $X\in\mathscr{H}_s$. If $P\{{}_E\mathscr{U}^\tau = \mathscr{U}^\tau\} = 1$, that is it is impossible for the process X to make its entrance from infinity, then

$$\lim_{n\to\infty} \bar{X}^n = X \tag{10}$$

where $\bar{X}^n = f_n(X)\in\mathscr{H}_1$. If X satisfies the system of backward equations, then so does $\bar{X}$, and it has the same Q matrix as X.

Remark

When X is a birth–death process, Corollary 2 is precisely the case considered in Zi-kun Wang (1962, Theorem 5.3) when $S=\infty$.

CHAPTER 11

Extension of Processes

11.1 INTRODUCTION

The construction problem of processes is basically an extension one. If the minimal Q process is a stopping process, construction of all Q processes is equivalent to that of an extension process of the minimal process, such that the extension process has the same Q matrix as the minimal process. When Q is conservative, Doob (1945) introduces for the first time extension processes of the minimal Q process, i.e. the so-called Doob processes. Chung (1967) provides a strict proof for Doob's construction. For conservative Q, under some restrictions on processes, Xiang-qun Yang (1966a) and Kunita (1962) take into account extensions which are more general than Doob processes.

In this chapter we impose no restriction on Q processes or instantaneous-return processes of order k. For example, we do not require the conservativeness of Q or the existence of 'a centre' and so on. Furthermore, we even do not require the finiteness of the Q matrix in section 11.2.

We mainly consider non-sticky extension. Because the D-type extension may change Q matrices, we introduce the D*-type extension in order that Q matrices remain unchanged. Of course, we can consider extensions of other types. For instance, for some minimal processes we can introduce the V-type extension so that for such an extended process $X = \{x(t), t < \sigma\}$ has only a finite number of leaping intervals in $[0,t]$ for any $t < \sigma(\omega)$, and all leaping intervals except the first one are ${}_{\infty}U$ intervals. Also we can introduce extensions of mixing D-type and V-type, but in this chapter we shall only give a simple and elementary discussion of it.

11.2 D-TYPE EXTENSION

Suppose $\bar{P}(t) = \{\bar{p}_{ij}(t)\}\ (i, j \in E, t \geqslant 0)$ satisfies conditions (2.2.A–C) and

$$\sum_j \bar{p}_{ij}(t) < 1 \qquad \text{for some } i \in E \tag{1}$$

Suppose that we are given a distribution $\pi = \{\pi_i, i \in E\}$ satisfying

$$0 < \sum_j \pi_j \leqslant 1 \tag{2}$$

Lemma 1. There exists a probability space $(\Omega, \mathscr{F}, P)$ on which may be defined a sequence of processes $X^n = \{x^n(t), t < \sigma^n\}\ (n \geqslant 0)$ with the following properties:

(i) $X^n \in \mathscr{H}\ (n \geqslant 0)$, and they have the same transition probability matrix $\bar{P}(t)$.

(ii) $\{\sigma^n = 0\} \cup \{\sigma^n = \infty\} \subset \{\sigma^{n+1} = 0\}$[1], $n \geqslant 0$.

(iii) $P\{x^{n+1}(0) = j \mid 0 < \sigma^n < \infty\} = \pi_j, P\{\sigma^{n+1} = 0 \mid 0 < \sigma^n < \infty\} = 1 - \sum_j \pi_j$.

(iv) Given that $(0 < \sigma^n < \infty)$ or $\{x^{n+1}(0) = i\}, X^m (m \leqslant n)$ and $X^m (m > n)$ are conditionally independent, i.e. for $0 \leqslant t_{m1} < t_{m2} < \cdots < t_{ml_m}, j_{m1}, j_{m2}, \ldots, j_{ml_m} \in E$, if we put

$$\Lambda_m = \{x^m(t_{mk}) = j_{mk}, 1 \leqslant k \leqslant l_m\} \tag{3}$$

then for any $l \geqslant 1$ and $n \geqslant 0$ we have

$$P\left\{\bigcap_{a=0}^{n+l} \Lambda_a \mid \Delta\right\} = P\left\{\bigcap_{a=0}^{n} \Lambda_a \mid \Delta\right\} P\left\{\bigcap_{a=n+1}^{n+l} \Lambda_a \mid \Delta\right\} \tag{4}$$

where $\Delta = \{0 < \sigma^n < \infty\}$ or $\Delta = \{x^{n+1}(0) = i\}$.

Proof. Utilizing the technique of making an independent product space, with no difficulty we can prove that there exists a probability space $(\Omega, \mathscr{F}, P)$ on which may be defined a sequence of processes $X^0 = \{x^0(t), t < \sigma^0\}, X_i^n = \{x_i^n(t), t < \sigma_i^n\}\ (n \geqslant 1, i \in E)$ and a family of random variables $f^n\ (n \geqslant 0)$, taking values in $\bar{E}$ with the following properties:

(1°) X^0 and $X_i^n \in \mathscr{H}\ (n \geqslant 1, i \in E)$ and they have the same transition probability $\bar{P}(t)$.

(2°) $P\{x^0(0) \in E\} = P\{\sigma^0 > 0\} = 1$

$$P\{x_i^n(0) = i\} = 1 \qquad (n \geqslant 1, i \in E)$$

$$P\{f^n = i\} = \pi_i \qquad (i \in E) \qquad P\{f^n = \infty\} = 1 - \sum_i \pi_i$$

(3°) All $X^0, X_i^n (n \geqslant 1, i \in E)$ and $f^n, n \geqslant 0$, are independent.

Let $C(t)$ be the indicator function of the set $\{t \mid 0 < t < \infty\}$. If $\omega \in \{f^0 = i, C(\sigma^0) = 1\}\ (i \in E)$, let $\sigma^1(\omega) = \sigma_i^1(\omega)$; and $x^1(t, \omega) = x_i^1(t, \omega)$ for $t < \sigma'(\omega)$; otherwise let $\sigma'(\omega) = 0$. If $\omega \in \{f^1 = i, C(\sigma^1) = 1\}\ (i \in E)$, let $\sigma^2(\omega) = \sigma_i^2(\omega)$ and $x^2(t, \omega) = x_i^2(t, \omega)$ for $t < \sigma(\omega)$; otherwise let $\sigma^2(\omega) = 0$. Continuing the procedure we can obtain a sequence of processes $X^n = \{x^n(t), t < \sigma^n\}\ (n \geqslant 0)$. It is easy to see that $X^n \in \mathscr{H}\ (n \geqslant 0)$ and has the transition probability $\bar{P}(t)$ for each $n \geqslant 0$, and so property (i) holds. It follows from the structure of X^n that they also have property (ii).

[1] Generally speaking, the initial distribution of a process has total mass 1, but in this lemma for the processes $X^n (n \geqslant 1)$, we drop the restriction, i.e. it is permitted that $P\{0 < \sigma^n\} < 1\ (n \geqslant 1)$.

Because σ^n only depends on $X^0, X_i^m (m \leqslant n, i \in E)$ and $f^m (m < n)$, by (1°) and (3°) for $j \in E$

$$P\{x^{n+1}(0) = j, 0 < \sigma^n < \infty\} = P\{f^n = j, 0 < \sigma^n < \infty\}$$
$$= P\{f^n = j\}P\{0 < \sigma^n < \infty\} = \pi_j P\{0 < \sigma^n < \infty\}$$

This is just property (iii).

For the proof of property (iv), fix n and $j \in E$ temporarily. Let $\bar{X}^0 = X_j^{n+1}$, $\bar{X}_i^m = X_i^{n+m+1}$, $\bar{f}^m = f^{n+m+1}$ As we defined $X^m (m \geqslant 0)$ by $X^0, X_i^m (m \geqslant 1, i \in E)$ and $f^m (m \geqslant 0)$ just now according to $\bar{X}^0, \bar{X}_i^m (m \geqslant 1, i \in E)$ and $\bar{f}^m (m \geqslant 0)$, we can determine $\bar{X}^m = \{\bar{x}^m(t), t < \bar{\sigma}^m\} (m \geqslant 0)$. Obviously, $\bar{X}^m (m \geqslant 0)$ only depend on $X_i^m (m > n, i \in E)$ and $f^m (m > n)$ and if we put

$$\bar{\Lambda}_{n+m+1} = \{\bar{x}^m(t_{n+m+1,k}) = j_{n+m+1,k}, 1 \leqslant k \leqslant l_{n+m+1}\}$$

$$\bar{N} = \bigcap_{a=n+1}^{n+l} \bar{\Lambda}_a \qquad M = \bigcap_{a=0}^{n} \Lambda_a \qquad N = \bigcap_{a=n+1}^{n+l} \Lambda_a$$

we can easily see that

$$\{x^{n+1}(0) = j\} \cap \bar{N} = \{x^{n+1}(0) = j\} \cap N$$

and

$$P\{\bar{N}\} = P\{N | x^{n+1}(0) = j\}$$

So from (1°), (3°), property (iii) and the fact that $\Delta \cap \{x^{n+1}(0) \in E\} = \Delta$ for $\Delta = \{0 < \sigma^n < \infty\}$ we obtain that

$$\begin{aligned}
P\{MN\Delta\} &= \sum_j P\{M\Delta, x^{n+1}(0) = j, N\} \\
&= \sum_j P\{M\Delta, f^n = j, \bar{N}\} \\
&= \sum_j P\{M\Delta\}P\{f^n = j\}P\{\bar{N}\} \\
&= \sum_j P\{M\Delta\}\pi_j P\{N | x^{n+1}(0) = j\} \\
&= P(M\Delta)\sum_j P\{x^{n+1}(0) = j | \Delta\}P\{N | x^{n+1}(0) = j\} \\
&= P\{M\Delta\}P\{N | \Delta\}
\end{aligned}$$

From this we are sure that property (iv) is true for $\Delta = (0 < \sigma^n < \infty)$. For $\Delta = \{x^{n+1}(0) = i\}$ it can be proved similarly. The proof is completed. QED

Theorem 2. Assume that the sequence of processes $X^n = \{x^n(t), t < \sigma^n\} (n \geqslant 0)$ defined on the same probability space has the properties (i)–(iv) in Lemma 1.

Let

$$\tau^0 = 0 \qquad \tau^{n+1} = \sum_{a=0}^{n} \sigma^a \qquad \sigma = \sum_{a=0}^{\infty} \sigma^a \tag{5}$$

For $0 \leqslant t \leqslant \sigma$ let

$$x(t) = x^n(t - \tau^n) \qquad \tau^n \leqslant t < \tau^{n+1} \tag{6}$$

Then:

(i) $$X = \{x(t), t \leqslant \sigma\} \in \mathscr{H}.$$

(ii) The transition probability $\{p_{ij}(t)\}$ of X is given by

$$p_{ij}(t) = \bar{p}_{ij}(t) + \int_0^t \pi_j(t-s)\,\mathrm{d}K_i(s) \tag{7}$$

where

$$\pi_j(t) = \sum_i \pi_i \bar{p}_{ij}(t) \tag{8}$$

and $K_i(t)$ is defined as follows

$$L_i(t) = 1 - \sum_j \bar{p}_{ij}(t) \qquad L = \sum_i \pi_i L_i$$

$$L^0(t) = \begin{cases} 0 & \text{if } t \leqslant 0 \\ 1 & \text{if } t > 0 \end{cases}$$

$$L^{n+1} = L^n * L$$

$$K_i = \sum_{n=0}^{\infty} L_i * L^n \tag{9}$$

where $*$ represents convolution.

(iii) X is an honest process if and only if

$$\sum_j \pi_j = 1 \tag{10}$$

Proof. Because X^n are canonical processes, if only we have shown that X is a homogeneous Markov process, we can confirm that X is also a canonical one, and so (i) is proved.

For $0 \leqslant t_1 < t_2 < \cdots < t_l < t_{l+1}, j_1, j_2, \ldots, j_{l+1} \in E$, let

$$\Lambda_k = \bigcap_{a=1}^{k} \{x(t_a) = j_a\} \tag{11}$$

$$\Delta(m_1, m_2, \ldots, m_k) = \bigcap_{a=1}^{k} \{x^{m_a}(t_a - \tau^{m_a}) = j_a, \tau^{m_a} \leqslant t_a < \tau^{m_a+1}\} \tag{12}$$

It is clear that

$$P\{\Lambda_{l+1}\} = \sum_{0 \leqslant m_1 \leqslant \cdots \leqslant m_{l+1}} P\{\Delta(m_1, m_2, \ldots, m_{l+1})\} \tag{13}$$

Abbreviate $\Delta(m_1, \ldots, m_k) = \Delta_k$.

Suppose $m_l = m_{l+1} = m$. Then there exists $k < 1$ such that $m_1 \leqslant \cdots \leqslant m_k < m_{k+1} = \cdots = m_l = m_{l+1} = m$. Hence

$$\begin{aligned} &P\{\Delta(m_1, \ldots, m_{l+1})\} \\ &= \sum_i P\{\Delta_k, \tau^m \leqslant t_{k+1}, x^m(0) = i, x^m(t_a - \tau^m) = j_a, k+1 \leqslant a \leqslant l+1\} \\ &= \sum_i \int_0^{t_{k+1}} P\{x^m(t_a - \tau^m) = j_a, k+1 \leqslant a \leqslant l+1 \mid \Delta_k, \tau^m = s, x^m(0) = i\} \, ds \\ &\quad \times P\{\Delta_k, \tau^m \leqslant s, x^m(0) = i\} \end{aligned} \tag{14}$$

Since Δ_k and τ^m only depend on $X^n (n \leqslant m-1)$ it follows that by Lemma 1(iv) the integrand in (14) is equal to

$$\begin{aligned} &P\{x^m(t_a - s) = j_a, k+1 \leqslant a \leqslant l+1 \mid \Delta_k, \tau^m = s, x^m(0) = i\} \\ &= P\{x^m(t_a - s) = j_a, k+1 \leqslant a \leqslant l+1 \mid x^m(0) = i\} \\ &= P\{x^m(t_a - s) = j_a, k+1 \leqslant a \leqslant l \mid x^m(0) = i\} \bar{p}_{j_l j_{l+1}}(t_{l+1} - t_l) \\ &= P\{x^m(t_a - s) = j_a, k+1 \leqslant a \leqslant l \mid \Delta_k, \tau^m = s, x^m(0) = i\} \bar{p}_{j_l j_{l+1}}(t_{l+1} - t_l) \end{aligned}$$

Substituting this into (14) and reversing the above operation, for $m_1 = m_{l+1} = m$ we arrive at

$$P\{\Delta(m_1, \ldots, m_{l+1})\} = P\{\Delta_l\} \bar{p}_{j_l j_{l+1}}(t_{l+1} - t_l) \tag{15}$$

Now suppose $m_l = m < m_{l+1} = \gamma$. By Lemma 1(iv),

$$\begin{aligned} &P\{\Delta(m_1, \ldots, m_{l+1})\} \\ &= \sum_i P\{\Delta_l, x^r(0) = i, x^r(t_{l+1} - \tau^r) = j_{l+1}, t_l < \tau^r \leqslant t_{l+1}\} \\ &= \sum_i \int_{t_l}^{t_{l+1}} P\{x^r(t_{l+1} - \tau^r) = j_{l+1} \mid \Delta_l, \tau^r = s, x^r(0) = i\} \, ds \\ &\quad \times P\{\Delta_l, \tau^r \leqslant s, x^r(0) = i\} \end{aligned} \tag{16}$$

Also by property (iv) in Lemma 1 the integrand is equal to

$$P\{x^r(t_{l+1} - s) = j_{l+1} \mid x^r(0) = i\} = \bar{p}_{ij_{l+1}}(t_{l+1} - s) \tag{17}$$

Again by Lemma 1(ii)–(iv),

$$\begin{aligned} P\{\Delta_l, \tau^r \leqslant s, x^r(0) = i\} &= P\{\Delta_l, \tau^r \leqslant s, 0 < \sigma^{r-1} < \infty, x^r(0) = i\} \\ &= P\{\Delta_l, \tau^r \leqslant s, 0 < \sigma^{r-1} < \infty\} \pi_i \end{aligned} \tag{18}$$

If it can be shown that for

$$P\{\Delta_l, \tau^r \leqslant s, 0 < \sigma^{r-1} < \infty\} = P\{\Delta_l\}(L_{j_l} * L^{r-m-1})(s - t_l) \tag{19}$$

then substituting (15)–(19) into (13) we obtain that

$$\begin{aligned} P\{\Lambda_{l+1}\} &= \sum_{0 \leqslant m_1 \leqslant \cdots \leqslant m_l} P\{\Delta_l\} \Big\{ \bar{p}_{j_l j_{l+1}}(t_{l+1} - t_l) \\ &\quad + \sum_{m_l + 1 = m_{l+1}}^{\infty} \int_{t_l}^{t_{l+1}} \pi_{j_{l+1}}(t_{l+1} - s) \, ds \, (L_{j_l} * L^{m_{l+1} - m_l - 1})(s - t_l) \Big\} \\ &= \sum_{0 \leqslant m_1 \leqslant \cdots \leqslant m_l} p\{\Delta_l\} \Big\{ \bar{p}_{j_l j_{l+1}}(t_{l+1} - t_l) \\ &\quad + \int_{t_l}^{t_{l+1}} \pi_{j_{l+1}}(t_{l+1} - s) \, ds \, K_{j_l}(s - t_l) \Big\} \\ &= P\{\Lambda_l\} p_{j_l j_{l+1}}(t_{l+1} - t_l) \end{aligned}$$

This demonstrates that X is a homogeneous Markov process with the transition probability $\{p_{ij}(t)\}$.

Proof of (19): We have to prove that

$$P\{\Delta_1, \tau^r \leqslant s + t_l, 0 < \sigma^{r-1} < \infty\} = P\{\Delta_l\}(L_{j_l} * L^{r-m-1})(s) \qquad s \geqslant 0 \tag{20}$$

Assume $\gamma = m + 1$. The left-hand side of the above is equal to

$$\begin{aligned} &\sum_i P\{\Delta_{l-1}, \tau^m + \sigma^m \leqslant s + t_l, x^m(0) = i, x^m(t_l - \tau^m) = j_l, \tau^m \leqslant t_l\} \\ &= \sum_i \int_0^{t_l} P\{x^m(t_1 - \tau^m) = j_l, \tau^m + \sigma^m \leqslant s + t_l \mid \Delta_{l-1}, \tau^m = u, x^m(0) = i\} \, du \\ &\quad \times P\{\Delta_{l-1}, x^m(0) = i, \tau^m \leqslant u\} \end{aligned}$$

By Lemma 1(iv) the integrand is equal to

$$\begin{aligned} &P\{x^m(t_l - u) = j_l, \sigma^m \leqslant s + t_l - u \mid x^m(0) = i\} \\ &= P\{x^m(t_l - u) = j_l \mid x^m(0) = i\} L_{j_l}(s) \end{aligned}$$

Substituting this for the original and reversing the above calculation, we obtain that the left-hand side of (20) equals $P\{\Delta_l\} L_{j_l}(s)$, i.e. (20) holds for $r = m + 1$.

We will prove (20) by induction. Suppose $r > m + 1$. Then the left-hand side

of (20) is equal to

$$\sum_i P\{\Delta_l, \tau^{r-1}+\sigma^{r-1}\leqslant s+t_l, x^{r-1}(0)=i\}$$

$$=\sum_i \int_{t_l}^{s+t_l} P\{\tau^{r-1}+\sigma^{r-1}\leqslant s+t_l|\Delta_l,\tau^{r-1}=u, x^{r-1}(0)=i\}\,du$$

$$\times P\{\Delta_l,\tau^{r-1}\leqslant u, x^{r-1}(0)=i\}$$

$$=\sum_i\int_{t_l}^{s+t_l} P\{\sigma^{r-1}\leqslant s+t_l-u|x^{r-1}(0)=i\}\,du\, P\{\Delta_l,\tau^{r-1}\leqslant u,x^{r-1}(0)=i\}$$

But $P\{\sigma^{r-1}\leqslant s+t_l-u|x^{r-1}(0)=i\}=L_i(s+t_l-u)$, and by Lemma 1(ii)–(iv) and the hypothesis of induction, for $u\geqslant t_l$ we have

$$\begin{aligned}P\{\Delta_l, \tau^{r-1}\leqslant u, x^{r-1}(0)=i\} &= P\{\Delta_l, \tau^{r-1}\leqslant u, 0<\sigma^{r-2}<\infty, x^{r-1}(0)=i\}\\ &= P\{\Delta_l, \tau^{r-1}\leqslant u, 0<\sigma^{r-2}<\infty\}\pi_i\\ &= P\{\Delta_l\}(L_{jl}*L^{r-1-m-1})(u-t_l)\pi_i\end{aligned}$$

Thus the left-hand side of (20) is equal to

$$\sum_i\int_{t_l}^{s+t_l}\pi_i L_i(s+t_l-u)P\{\Delta_l\}\,du(L_l*L^{r-1-m-1})(u-t_l)$$

$$=P\{\Delta_l\}\int_0^s L(s-v)\,dv\,(L_l*L^{r-1-m-1})(v)$$

$$=P\{\Delta_l\}(L_l*L^{r-m-1})(s)$$

Thus (20) is verified.

Summing over $j\in E$ on (7) we obtain that

$$\sum_j p_{ij}(t)=1-L_i(t)+\int_0^t\sum_j\pi_j\{1-L_j(t-s)\}\,dK_i(s)$$

$$=1-L_i(t)+\int_0^t\left(\sum_j\pi_j-L(t-s)\right)dK_i(s)$$

$$=1-L_i(t)+\left(\sum_j\pi_j\right)K_i(t)-\sum_{n=0}^{\infty}(L_i*L^{n+1})(t)$$

$$=1-K_i(t)\left(1-\sum_j\pi_j\right)$$

On account of this, there exists at least one i such that $K_i\neq 0$. Therefore X is honest if and only if (10) holds. The proof is completed. QED

Remark 1

The first part $X^0=\{x^0(t),t<\sigma^0\}$ of the process $X=\{x(t),t<\sigma\}$ in Theorem 2 is a stopping process and satisfies

$$P\{x(\tau^1)=j|\tau^1<\infty\}=\pi_j \tag{21}$$

Moreover their transition probabilities have the relation (7). We call X a $\pi=\{\pi_j,j\in E\}$ D-type extension process of the stopping process X^0.

Remark 2

Taking the Laplace transform of (7) we obtain that

$$\psi_{ij}(\lambda)=\bar\psi_i(\lambda)+\bar\xi_i(\lambda)\frac{\sum_k\pi_k\bar\psi_{kj}(\lambda)}{1-\sum_k\pi_k\bar\xi_k(\lambda)} \tag{22}$$

where $\psi(t)$ and $\bar\psi(t)$ are the Laplace transforms of $P(t)$ and $\bar P(t)$ respectively and

$$\bar\xi_i(\lambda)=1-\lambda\sum_j\bar\psi_{ij}(\lambda) \tag{23}$$

11.3 D*-TYPE EXTENSION

In the previous section we did not require the finiteness of the Q matrices. In this section we assume that all Q matrices are finite.

In the D-type extension, if the Q matrix of X^0 is conservative, from the sample paths of the process we know that X^0 and its D-type extension process X have the same Q matrix. But if the Q matrix of X^0 is non-conservative, then the Q matrices of X and X^0 may be different. In order to preserve the Q matrix we introduce the D*-type extension.

Definition 1. Suppose $X=\{x(t),t<\sigma\}\in\mathscr{H}_s$. We call $\sigma(\omega)$ the T-tail of $X(\omega)$ if $0<\sigma(\omega)<\infty$ and $\sigma(\omega)$ is a jump point (see Definition 7.8.1); otherwise we call $\sigma(\omega)$ the P-tail.

Obviously, if the transition probability of the process $X^0=\{x^0(t),t<\sigma^0\}\in\mathscr{H}_s$ is $\bar P(t)$ then the following quantities

$$M_{ij}(t)=P_i\{x^0(t)=j,\sigma^0 \text{ is a } P\text{-tail}\}$$

$$N_{ij}(t)=P_i\{x^0(t)=j,\sigma^0 \text{ is a } T\text{-tail}\}$$

$$M_i(t)=\sum_j M_{ij}(t)\qquad N_i(t)=\sum_j N_{ij}(t) \tag{1}$$

$$R_i(t)=M_i(0)-M_i(t)=P_i\{\sigma^0\leqslant t,\sigma^0 \text{ is a } P\text{-tail}\}$$

and uniquely determined by $\bar P(t)$.

Imitating Lemma 2.1, we have the following:

Lemma 1. Assume that $\bar{P}(t)$ satisfies conditions (2.2.A–C), $R_i(t) > 0$ for some $i \in E$ and $t > 0$, and the distribution π satisfies (2.2). Then there exist a probability space $(\Omega, \mathscr{F}, P)$ and defined on it a sequence of processes $X^n = \{x^n(t), 0 \leqslant t < \sigma^n\}$ $(n \geqslant 0)$ with the following properties:

(i*) $X^n \in \mathscr{H}_s (n \geqslant 0)$ and they have the same transition probability $\bar{P}(t)$.

(ii*) $\{\sigma^n = 0\} \cup \{\sigma^n = \infty\} \cup \{\sigma^n$ is the T-tail of $X^n\} \subset \{\sigma^{n+1} = 0\} (n \geqslant 0)$.

(iii*) $P\{x^{n+1}(0) = j | \Delta\} = \pi_j, P\{\sigma^{n+1} = 0 | \Delta\} = 1 - \sum_j \pi_j$, where $\Delta = (0 < \sigma^n < \infty$ and σ^n is the P-tail of $X^n)$.

(iv*) Given that $\{0 < \sigma^n < \infty$ and σ^n is the P-tail of $X^n\}$ or $\{X^{n+1}(0) = i\}$, then $X^m (m \leqslant n)$ and $X^m (m > n)$ are conditionally independent.

Theorem 2. Assume that processes $X^n = \{x^n(t), t < \sigma^n\} (n \geqslant 0)$, defined on the same probability space, have the properties (i*)–(iv*) in Lemma 1. Define $X = \{x(t), t < \sigma\}$ according to (2.5) and (2.6). Then:

(i) $X \in \mathscr{H}_s$ and $X^n (n \geqslant 0)$ have the same Q matrix.

(ii) The transition probability $P^*(t) = \{p_{ij}^*(t)\}$ is given by

$$p_{ij}^*(t) = \bar{p}_{ij}(t) + \int_0^t \pi_j(t-s) \mathrm{d}s K_i^*(t) \tag{2}$$

where $\pi_j(t)$ is determined by (2.8) and K_i^* by (2.9) but $L_i(t)$ in it should be replaced by $R_i(t)$ in (1).

(iii) X is an honest process if and only if Q is conservative and (2.10) holds.

Proof. Imitate the proof of Theorem 2.1. But the last conclusion in the theorem requires a new proof. By (2) it follows that

$$\sum_j p_{ij}^*(t) = 1 - \{N_i(0) - N_i(t)\} - \left\{1 - \sum_k \pi_k + \sum_k \pi_k N_k(0)\right\} K_i^*(t)$$

$$+ \int_0^t \sum_k \pi_k N_k(t-s) \mathrm{d}K_i^*(s) \tag{3}$$

If Q is conservative, clearly $N_i(t) = 0$, and if (2.10) holds, then by (3) $\sum_j p_{ij}^*(t) = 1$, i.e. X is an honest process. Conversely, if X is an honest process, necessarily $N_i(t) = 0$. Otherwise, for some i we have $N_i(t) = P_i\{t < \sigma^0, \sigma^0$ is the T-tail of $X^0\} > 0$. Then according to Lemma 1(ii*) on the set $\{t < \sigma^0, \sigma^0$ is the T-tail of $X^0\}$ of positive probability, $\sigma = \sigma^0 < \infty$, which contradicts the hypothesis that X is honest. So by $N_i(t) = 0$ and (3), (2.10) holds. We claim that Q is conservative. In fact, if for some $i, d_i = q_i - \sum_{j \neq i} q_{ij} > 0$ then we have

$$P_i\{0 < \sigma^0 < \infty, \sigma^0 \text{ is the } T\text{-tail of } X^0\} \geqslant d_i/q_i > 0$$

and so $N_i(t) > 0$, which is impossible. QED

Remark 1

Taking the Laplace transformation in (2), we get

$$\psi_{ij}^*(\lambda) = \bar{\psi}_{ij}(\lambda) + \xi_i(\lambda) \frac{\sum_k \pi_k \bar{\psi}_{kj}(\lambda)}{1 - \sum_k \pi_k \xi_k(\lambda)} \tag{4}$$

where $\psi^*(\lambda)$ and $\bar{\psi}(\lambda)$ are respectively the Laplace transforms of $P^*(t)$ and $\bar{P}(t)$, and $\xi_i(\lambda)$ is the Laplace transform of $R_i(t)$ in (1).

Remark 2

We call the process $X \in \mathscr{H}_s$ in Theorem 2 the πD*-type extension process of $X^0 \in \mathscr{H}_s$. The process X^0 and its D*-type extension have the same Q matrix. If X^0 satisfies the system of backward equations, so does X.

Remark 3

When Q is conservative, because there is no T-tail the D*-type extensions and D-type extension are the same.

11.4 DOOB PROCESSES

In the D*-type extension, if X^0 or $\{\bar{p}_{ij}(t)\}$ is the minimal Q process, using the representation of Martin exit boundary B_e, $R_i(t)$ in (3.1) becomes

$$L_i(t) = P_i\{\sigma^0 \leqslant t, x(\sigma^0 - 0) \in B_e\} \tag{1}$$

and Theorem 3.2 becomes:

Theorem 1. Suppose $\{f_{ij}(t)\}$ is the minimal Q process. For some i and $t > 0$, $L_i(t)$, defined according to (1), is positive. Let π satisfy (2.2). Suppose $X^n \in \mathscr{H}_s(Q) (n \geqslant 0)$ are minimal Q processes defined on the same probability space with properties (ii*)–(iv*) in Lemma 3.1. Let X be defined according to (2.5)–(2.6). Then $X \in \mathscr{H}_1(Q)$ and its transition function is given by

$$p_{ij}(t) = f_{ij}(t) + \int_0^t \pi_j(t-s) \mathrm{d}K_i(s) \tag{2}$$

where $\pi_j(t) = \sum_i \pi_i f_{ij}(t)$, and $K_i(t)$ is determined by (2.9), but $L_i(t)$ in it should be understood as (1). Moreover, X is an honest process if and only if Q is conservative and (2.10) holds.

Definition 1. A πD*-type extension process of the minimal Q process is called a (Q, π) Doob process.

For the (Q,π) Doob process $X=\{x(t), t<\sigma\}$,

$$P\{x(\tau)=i|x(\tau-0)\in B_e\}=\pi_i \qquad i\in E \tag{3}$$

$$P\{\sigma=\tau|x(\tau-0)\in B_e\}=1-\sum_i \pi_i \tag{4}$$

where τ is the first leaping point. The Laplace transform of (2) is

$$\begin{aligned}\psi_{ij}(\lambda)&=\phi_{ij}(\lambda)+\xi_i(\lambda)\frac{\sum_k \pi_k\phi_{kj}(\lambda)}{1-\sum_k\pi_k\xi_k(\lambda)}\\ &=\phi_{ij}(\lambda)+\xi_i(\lambda)\frac{\sum_k\pi_k\phi_{kj}(\lambda)}{(1-\sum_k\pi_k)+\sum_k\pi_k[1-\xi_k(\lambda)]}\end{aligned} \tag{5}$$

where $\xi_i(\lambda)$ is the Laplace transform of $L_i(t)$ in (1). Particularly, when Q is conservative,

$$\xi_i(\lambda)=1-\lambda\sum_i \phi_{ij}(\lambda) \tag{6}$$

11.5 GENERALIZED D-TYPE EXTENSION

The D-type and D*-type extensions do not involve the boundaries of processes. We will consider in this section an extension that depends on boundaries of processes.

For a finite Q matrix Q, according to Chapter 7 we can determine the essential Martin boundary B, the exit boundory B_e and the passive boundary B_p. Recall that H denotes the non-conservative state set.

Assume that the minimal Q process $X=\{x(t), t<\tau\}$ is stopping, i.e. for some $i, P_i\{\tau<\infty\}<0$ or, equivalently, $\mu\{H\cup B_e\}>0$, where measure μ is defined according to (7.12.2).

Clearly, for a Borel set $\Gamma\subset H\cup B_e$, the quantities

$$L_i(\Gamma,t)=P_i\{x(\tau-0)\in\Gamma, \tau\leqslant t\} \qquad t\geqslant 0 \tag{1}$$

$$h_i(\Gamma,\lambda)=E_i\{e^{-\lambda\tau}, x(\tau-0)\in\Gamma\}=\int_0^\infty e^{-\lambda t}\mathrm{d}t\, L_i(\Gamma,t) \qquad \lambda>0 \tag{2}$$

are uniquely determined by Q.

We are given a family of distributions $\Pi(a,\cdot)(a\in H\cup B_e)$, satisfying

$$\Pi(a,E)\leqslant 1 \qquad \int_{H\cup B_e}\Pi(a,E)\mu(\mathrm{d}a)>0 \tag{3}$$

Lemma 1. There exists a probability space $(\Omega,\mathscr{F},P)$ on which a sequence of minimal Q processes $X^n=\{x^n(t), t<\sigma^n\}\in\mathscr{H}_s(Q)(n\geqslant 0)$ can be defined, with the following properties:

(i) $\{\sigma^n=0\}\cup\{\sigma^n=\infty\}\subset\{\sigma^{n+1}=0\}$.

(ii) On $\{x^n(\sigma^n-0)\in H\cup B_e\}$, almost surely

$$P\{x^{n+1}(0)=j|x^n(\sigma^n-0)\}=\Pi(x^n(\sigma^n-0),j) \qquad j\in E$$
$$P\{\sigma^{n+1}=0|x^n(\sigma^n-0)\}=1-\Pi(x^n(\sigma^n-0),E)$$

(iii) For Λ_m determined by (2.3)

$$\begin{aligned}&P\left\{\bigcap_{a=0}^{n+l}\Lambda_a\middle|x^{n+1}(0)=i\right\}\\ &=P\left\{\bigcap_{a=0}^{n}\Lambda_a\middle|x^{n+1}(0)=i\right\}P\left\{\bigcap_{a=n+1}^{n+l}\Lambda_a\middle|x^{n+1}(0)=i\right\} \qquad (i\in E, l\geqslant 1)\end{aligned}$$

(iv) On the set $\{x^n(\sigma^n-0)\in H\cup B_e\}$, almost surely

$$P\left\{\bigcap_{a=0}^{n+l}\Lambda_a\middle|x^n(\sigma^n-0)\right\}=P\left\{\bigcap_{a=0}^{n}\Lambda_a\middle|x^n(\sigma^n-0)\right\}P\left\{\bigcap_{a=n+1}^{n+l}\Lambda_a\middle|x^n(\sigma^n-0)\right\}$$

where Λ_a are also given by (2.3).

Proof. By the technique of making an independent product space, it is not difficult to show that there exists a probability space $(\Omega,\mathscr{F},P)$ on which can be defined a sequence of minimal processes $X^0=\{x^0(t), t<\sigma^0\}\in\mathscr{H}_s$, $X_i^n=\{x_i^n(t), t<\sigma_i^n\}\in\mathscr{H}_s$ $(n\geqslant 1, i\in E)$ and an $\bar{E}$-valued random variable $f^n(a)$ $(n\geqslant 0, a\in H\cup B_e)$, with the following properties:

(1°) $P\{x^0(0)\in E\}=P\{x_i^n(0)=i\}=1$.

(2°) $P\{f^n(a)=i\}=\Pi(a,i)$ $(i\in E)$; $P\{f^n(a)=\infty\}=1-\Pi(a,E)$.

(3°) All $X^0, X_i^n(n\geqslant 1, i\in E), f^n(a)$ $(n\geqslant 0, a\in H\cup B_e)$ are independent.

If $\omega\in\{x^0(\sigma^0-0)\in H\cup B_e, f^0[x^0(\sigma^0-0)]=i\}$ $(i\in E)$, let $\sigma^1(\omega)=\sigma_i^1(\omega)$ and $x^1(t,\omega)=x_i^1(t,\omega)$, $t<\sigma^1(\omega)$; otherwise, let $\sigma^1(\omega)=0$. It is easy to see that $X^1=\{x^1(t), t<\sigma^1\}\in\mathscr{H}_s$ is a minimal Q process. Therefore $\sigma^1>0$, and one can define $x^1(\sigma^1-0)$. If $\omega\in(x^1(\sigma^1-0)\in H\cup B_e$, $f^1[x^1\sigma^1-0)]=i\}$ $(i\in E)$, let $\sigma^2(\omega)=\sigma_i^2(\omega)$ and $x^2(t,\omega)=x_i^2(t,\omega), t<\sigma^2(\omega)$. Otherwise, let $\sigma^2(\omega)=0$. Continuing with this step, we can obtain a sequence of processes X^n $(n\geqslant 0)$.

Just as done in Theorem 2.1, one can prove that $X^n\in\mathscr{H}_s(n\geqslant 0)$ are all minimal Q processes and have properties (i)–(iv) in this lemma. QED

Theorem 2. Suppose that the minimal Q processes $X^n=\{x^n(t), t<\sigma^n\}\in\mathscr{H}_s$ $(n\geqslant 0)$ defined on the same probability space possess properties (i)–(iv) in Lemma 1. Define $X=\{x(t), t<\sigma\}$ according to (2.5)–(2.6). Then:

(i) If $\int_{B_e}\Pi(a,E)\mu(\mathrm{d}a)=0$, then $X\in\mathscr{H}_s$ is a minimal process and its Q matrix

is different from that of X^0. If $\int_{B_e}\Pi(a,E)\mu(\mathrm{d}a)>0$, then $X\in\mathscr{H}_1$ is a process of order 1.

(ii) The transition probability $\{p_{ij}(t)\}$ of X is given by

$$p_{ij}(t)=f_{ij}(t)+\sum_k\int_0^t f_{kj}(t-s)\,\mathrm{d}K_{ik}(s) \tag{4}$$

where $K_{ik}(t)$ are determined by

$$T_{ij}^1(t)=\int_{H\supset B_e}\Pi(a,j)L_i(\mathrm{d}a,t)$$

$$T_{ij}^{n+1}(t)=\int_{H\cup B_e}\Pi(a,j)\sum_k[T_{ik}^n(\cdot)*L_k(\mathrm{d}a,\cdot)](t) \tag{5}$$

$$K_{ij}(t)=\sum_{n=1}^{\infty}T_{ij}^n(t)$$

Here $*$ represents convolution, i.e.

$$[T_{ik}^n(\cdot)*L_k(\Gamma,\cdot)](t)=\int_0^t T_{ik}^n(t-s)L_k(\Gamma,\mathrm{d}s) \tag{6}$$

Making some changes in (4) we obtain that

$$p_{ij}(t)=f_{ij}(t)+\int_{H\cup B_e}\int_0^t\pi_j(a,t-s)K_i(\mathrm{d}a,\mathrm{d}s) \tag{7}$$

where

$$\pi_j(a,t)=\sum_k\Pi(a,k)f_{kj}(t) \tag{8}$$

$$L_i^1(\Gamma,t)=L_i(\Gamma,t)\qquad\text{(see (1))} \tag{9}$$

$$L_i^{n+1}(\Gamma,t)=\int_0^t\int_{H\cup B_e}L_i^n(\mathrm{d}a,\mathrm{d}s)\sum_k\Pi(a,k)L_k(\Gamma,t-s) \tag{10}$$

$$K_i(\Gamma,s)=\sum_{n=1}^{\infty}L_i^n(\Gamma,s) \tag{11}$$

We can imitate the proof of Theorem 2.2 to prove Theorem 2. The only difference is that the latter is more complicated in writing. Although the proofs of the theorems in sections 11.6 and 11.7, which we have omitted, are similar to this theorem, we still give a brief clue to the proof of Theorem 2 here.

Proof. We need only verify that X is a homogeneous Markov process with transition probability (4). The rest is clear.

Determine Λ_l and $\Delta(m_1,\ldots,m_k)=\Delta_k$ according to (2.11)–(2.12) and the

formula (2.13) still holds. Following the proof of (2.15), in the present case (2.15) still holds for $m_l=m_{l+1}$.

Assume $m_l<m_{l+1}$. Abbreviate $m_l=m$, $m_{l+1}=\gamma$. Then by Lemma 1(iii),

$$\begin{aligned}P\{\Delta_{l+1}\}&=\sum_k P\{\Delta_l,t_l<\tau^r\leqslant t_{l+1}<\tau^r+\sigma^r,x^r(0)=k,\ x^r(t_{l+1}-\tau^r)=j_{l+1}\}\\&=\sum_k\int_{t_l}^{t_{l+1}}P\{x^r(t_{l+1}-\tau^r)=j_{l+1}\,|\,\Delta_l,\tau^r=s,\ x^r(0)=k\}\,\mathrm{d}s\\&\quad\times P\{\Delta_l,\tau^r\leqslant s,x^r(0)=k\}\\&=\sum_k\int_{t_l}^{t_{l+1}}f_{kj_{l+1}}(t_{l+1}-s)\,\mathrm{d}s\,P\{\Delta_l,\tau^r\leqslant s,x^r(0)=k\}\end{aligned} \tag{12}$$

If it can be proved that for $s\geqslant t_l$,

$$P\{\Delta_l,\tau^r\leqslant s,x^r(0)=k\}=T_{j_lk}^{r-m}(s-t_l) \tag{13}$$

combining (12) we obtain

$$P\{\Delta_{l+1}\}=\sum_k P\{\Delta_l\}\int_{t_l}^{t_{l+1}}f_{kj_{l+1}}(t_{l+1}-s)\,\mathrm{d}s\,T_{j_lk}^{m_{l+1}-m_l}(s-t_l)$$

$$=P\{\Delta_l\}\sum_k\int_0^{t_{l+1}-t_l}f_{kj_{l+1}}(t_{l+1}-t_l-s)\,\mathrm{d}s\,T_{j_lk}^{m_{l+1}-m_l}(s) \tag{14}$$

and substituting (2.15) and (14) into (2.13) we find that $P\{\Lambda_{l+1}\}=P\{\Lambda_l\}p_{j_lj_{l+1}}\times(t_{l+1}-t_l)$, where $p_{ij}(t)$ is determined by (4). Therefore X is a homogeneous Markov process.

To prove (13) is to verify

$$P\{\Delta_l,\tau^r\leqslant s+t_l,x^r(0)=k\}=P\{\Delta_l\}T_{j_lk}^{r-m}(s)\qquad s\geqslant 0 \tag{15}$$

By Lemma 1(iv), the left-hand side of the above is equal to

$$\int_{H\cup B_e}P\{\Delta_l,x^{r-1}(\sigma^{r-1}-0)\in\mathrm{d}a,x^r(0)=k,\tau^r\leqslant s+t_l\}$$

$$=\int_{H\cup B_e}\Pi(a,k)P\{\Delta_l,x^{r-1}(\sigma^{r-1}-0)\in\mathrm{d}a,\tau^r\leqslant s+t_l\} \tag{16}$$

First we deal with $r=m+1$. Then

$$\begin{aligned}P\{\Delta_l,&x^{r-1}(\sigma^{r-1}-0)\in\mathrm{d}a,\tau^r\leqslant s+t_l\}\\&=\sum_i P\{\Delta_{l-1},x^m(0)=i,x^m(t_l-\tau^m)\\&=j_l,\tau^m\leqslant t_l<\tau^m+\sigma^m,x^m(\sigma^m-0)\in\mathrm{d}a,\tau^m+\sigma^m\leqslant s+t_l\}\end{aligned} \tag{17}$$

By Lemma 1(iii), the summand in the above equals

$$\int_0^{t_l} P\{x^m(t_l-u)$$
$$=j_l, x^m(\sigma^m-0)\in da, \sigma^m \leqslant s+t_l-u \mid x^m(0)=i\}\, du P\{\Delta_{l-1}, x^m(0)=i, \tau^m \leqslant u\}$$
$$=\int_0^{t_l} f_{ij_l}(t_l-u)L_{j_l}(da,s)\, du P\{\Delta_{l-1}, x^m(0)=i, \tau^m\leqslant u\}$$

Substituting into (17) and reversing the above calculation we see that the left-hand side of (17) equals $P\{\Delta_l\}L_{j_l}(da,s)$. Thereby substituting into (16) we know that (13) holds for $r=m+1$. By a similar consideration and by induction we can confirm that (13) holds for all $r>m$. The proof is terminated. QED

Denote the Laplace transform of $T^n_{ij}(t)$ by $G^n_{ij}(\lambda)$. Define $h_i(\Gamma,\lambda)$ by (2). Then from (5),

$$G^n_{ij}(\lambda)=\sum_k G^n_{ik}(\lambda)G^1_{ik}(\lambda)$$

Thus it follows from (4) that

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+\sum_k\left(\sum_{n=1}^{\infty} G^n_{ik}(\lambda)\right)\phi_{kj}(\lambda) \tag{18}$$

When Q is conservative, the above expression is precisely formula (10.2.25) in Kunita (1962). We shall use matrix symbols. Write $G(\lambda)=G^1(\lambda)$. Then (18) becomes

$$\psi(\lambda)=\phi(\lambda)+\sum_{n=1}^{\infty} G^n(\lambda)\phi(\lambda) \tag{19}$$

Setting

$$h^l_i(\Gamma,\lambda)=\int_0^{\infty} e^{-\lambda t}\, dt L^n_i(\Gamma,t) \tag{20}$$

it follows from (9) and (10) that

$$h^1_i(\Gamma,\lambda)=h_i(\Gamma,\lambda)$$
$$h^{n+1}_i(\Gamma,\lambda)=\int_{H\cup B_e} h^n_i(da,\lambda)\sum_k \Pi(a,k)h_k(\Gamma,\lambda) \tag{21}$$

Hence (7) becomes

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+\int_{H\cup B_e}\left(\sum_{n=1}^{\infty} h^n_i(da,\lambda)\right)\sum_k \Pi(a,k)\phi_{kj}(\lambda) \tag{22}$$

Letting

$$_0V(a,\Gamma,\lambda)=\begin{cases}0 & a\notin\Gamma\\ 1 & a\in\Gamma\end{cases}$$
$$V^n(a,\Gamma,\lambda)=\sum_k \Pi(a,k)h^n_k(\Gamma,\lambda) \tag{23}$$

we have

$$V^{n+1}(a,\Gamma,\lambda)=\int_{H\cup B_e} V^n(a,db,\lambda)V^1(b,\Gamma,\lambda) \tag{24}$$

$$h^{n+1}_i(\Gamma,\lambda)=\int_{H\cup B_e} h_i(da,\lambda)V^n(a,\Gamma,\lambda) \tag{25}$$

Consequently (22) becomes

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+\int_{H\cup B_e} h_i(a,\lambda)\int_{H\cup B_e}\left(\sum_{n=0}^{\infty} V^n(a,db,\lambda)\right)\left(\sum_k \Pi(b,k)\phi_{kj}(\lambda)\right) \tag{26}$$

When Q is conservative, this is just formula (10) in Kunita (1962).

Definition 1. Call X in Theorem 2 the $\{\Pi(a,\cdot), a\in H\cup B_e\}$ generalized D-type extension process of X^0.

Clearly, for X^0 and X in Definition 1, on the set $\{x(\sigma^0-0)\in H\cup B_e\}$ almost surely holds

$$P\{x(\sigma^0)=j \mid x(\sigma^0-0)\}=\Pi(x(\sigma^0-0),j) \qquad j\in E$$

11.6 GENERALIZED D*-TYPE EXTENSION

When Q is conservative, the generalized D-type extension preserves the Q matrix. But when Q is not conservative, things may be different. Hence we introduce the generalized D*-type extension.

Suppose the exit boundary B_e of the minimal Q process is non-empty, i.e. $\mu\{B_e\}>0$. Given a family of distributions $\Pi(a,\cdot)$ $(a\in B_e)$ satisfying $\Pi(a,E)\leqslant 1$ and $\int_{B_e}\Pi(a,E)\mu(da)>0$.

Let

$$\Pi(a,E)=0 \qquad a\in H \tag{1}$$

Then the basic conditions in section 11.5 are satisfied and so Lemma 5.1 and Theorem 5.2 are still valid.

Definition 1. Suppose X^0 is the minimal Q process. A $\{\Pi(a,\cdot), a\in H\cup B_e\}$ generalized D-type extension of X^0 that satisfies (1) is called a generalized D*-type extension.

Clearly, when Q is conservative, the generalized D-type and generalized D*-type extensions are identical. The following theorem is obvious.

Theorem 1. The minimal Q process X^0 and its $\{\Pi(a,\cdot),\ a\in B_e\}$ generalized D*-type extension process X have the same Q matrix. More precisely, X satisfies the system of backward equations and, moreover, on $\{x(\tau-0)\in B_e\}$ almost surely holds

$$P\{x(\tau)=j\,|\,x(\tau-0)\}=\Pi\{x(\tau-0),j\} \tag{2}$$

11.7 EXTENSION OF INSTANTANEOUS-RETURN PROCESSES

In this section we shall give a brief discussion of extensions of instantaneous-return processes. Because the proofs are similar to those in sections 11.5 and 11.6, they are omitted.

Suppose $X=\{x(t),t<\sigma\}\in\mathscr{H}_s$ is a kth-order instantaneous-return process with transition probability ${}_kp_{ij}(t)$ and ith-order exit boundary ${}_iB_e$. Instead of $H\cup B_e$ in section 11.5 we consider ${}_kH\cup{}_kB_e$, where

$${}_kH=H+{}_0B_e+{}_1B_e+\cdots+{}_{k-1}B_e \tag{1}$$

The measure on ${}_kH\cup{}_kB_e$ is ${}_k\mu$ for order k. Assume that X is a stopping process, i.e. ${}_k\mu\{{}_kH\cup{}_kB_e\}>0$. We are arbitrarily given a family of distributions ${}_k\Pi(a,\cdot)(a\in{}_kH\cup{}_kB_e)$ satisfying

$${}_k\Pi(a,E)\leqslant 1\qquad \int_{{}_kH\cup{}_kB_e}{}_k\Pi(a,E){}_k\mu(\mathrm{d}a)>0 \tag{2}$$

Lemma 1. There exists a probability space $(\Omega,\mathscr{F},P)$ on which a sequence of processes $X^n=\{x^n(t),\ t<\sigma^n\}\in\mathscr{H}_k$ $(n\geqslant 0)$ can be defined, with the following properties

(i) X^n $(n\geqslant 0)$ have the same transition probability ${}_kp_{ij}(t)$.

(ii) $(\sigma^n=0)\cup(\sigma^n=\infty)\subset(\sigma^{n+1}=0)$.

(iii) On the set $\{x^n(\sigma^n-0)\in{}_kH\cup{}_kB_e\}$ almost surely hold

$$P\{x^{n+1}(0)=j\,|\,x^n(\sigma^n-0)]={}_k\Pi(x^n(\sigma^n-0),j)$$
$$P\{\sigma^{n+1}=0\,|\,x(\sigma^n-0)\}=1-{}_k\Pi(x^k(\sigma^n-0),E)$$

(iv) The same as Lemma 5.1(iii)

(v) Obtained by changing $H\cup B_e$ in Lemma 5.1(iv) to ${}_kH\cup{}_kB_e$.

Theorem 2. Suppose that processes $X^n\in\mathscr{H}_k$ $(n\geqslant 0)$, which are defined on the same probability space, have properties (i)–(v) in Lemma 1. Define $X=\{x(t),\ t<\sigma\}$ according to (2.5)–(2.6). Then:

(i) if $\int_{{}_kB_e}{}_k\Pi(a,E)_k\mu(\mathrm{d}a)=0$ then $X\in\mathscr{H}_k$ but the kth-order exit boundaries of X and X^0 are different, and if the above integral is positive,

(ii) the transition function ${}_{k+1}p_{ij}(t)$ of X is given by

$${}_{k+1}p_{ij}(t)={}_kp_{ij}(t)+\sum_l\int_0^t{}_kp_{lj}(t-s)\,\mathrm{d}s\,{}_kK_{il}(s) \tag{3}$$

or

$${}_{k+1}p_{ij}(t)={}_kp_{ij}(t)+\int_{{}_kH\cup{}_kB_e}\int_0^t{}_k\pi_j(a,t-s)_kK_i(\mathrm{d}a\,\mathrm{d}s) \tag{4}$$

where

$${}_k\pi_j(a,t)=\sum_i{}_k\Pi(a,i)_kp_{ij}(t) \tag{5}$$

and ${}_kK_{ij}(t)$, ${}_kK_i(\Gamma,t)$ are still defined according to (5.5)–(5.6) and (5.9)–(5.11), but the following changes should be made. To each of $L_i(\Gamma,t)$, $\Pi(a,\cdot)$, H, B_e, $T^n_{ij}(t)$, $K_{ij}(t)$, $L^n_i(\Gamma,t)$ and $K_i(\Gamma,t)$ should be added a left subscript k, and for $\Gamma\subset{}_kH\cup{}_kB_e$, moreover,

$${}_kL_i(\Gamma,t)=P_i\{x^0(\sigma^0-0)\in\Gamma,\sigma^0\leqslant t\} \tag{6}$$

Correspondingly, after the obvious supplements to those lower left corners in (5.18)–(5.26) we have

$${}_{k+1}\psi(\lambda)={}_k\psi(\lambda)+\sum_{n=1}^{\infty}({}_kG(\lambda))^n{}_k\psi(\lambda) \tag{7}$$

$${}_{k+1}\psi(\lambda)={}_k\psi(\lambda)+\int_{{}_kH\cup{}_kB_e}\left\{\sum_{n=1}^{\infty}{}_kh^n_i(\mathrm{d}a,\lambda)\right\}\sum_l{}_k\Pi(a,l)_k\psi_{lj}(\lambda) \tag{8}$$

$${}_{k+1}\psi_{ij}(\lambda)={}_k\psi_{ij}\lambda)+\int_{{}_kH\cup{}_kB_e}{}_kh_i(\mathrm{d}a,\lambda)$$
$$\times\int_{{}_kH\cup{}_kB_e}\left(\sum_{n=0}^{\infty}{}_kV^n(a,\mathrm{d}b,\lambda)\right)\left(\sum_l{}_k\Pi(b,l)_k\psi_{lj}(\lambda)\right) \tag{9}$$

Definition 1. Call the process X in Theorem 1 the $\{{}_k\Pi(a,\cdot),\ a\in{}_kH\cup{}_kB_e\}$ generalized D-type extension process of X^0.

For X and X^0 given in Definition 1, on $\{x(\sigma^0-0)\in{}_kH\cup{}_kB_e\}$ almost surely holds

$$P\{x(\sigma^0)=j\,|\,x(\sigma^0-0)\}={}_k\Pi(x(\sigma^0-0),j) \tag{10}$$

Note that X and X^0 do not necessarily have the same kth-order exit boundary. Of course, their Q matrices may be different. Therefore it is necessary to introduce the generalized D*-type extension.

Assume that the kth-order instantaneous-return process $X \in \mathscr{H}_k$ satisfies ${}_k\mu\{{}_kB_e\} > 0$. Suppose we are given a family of distributions ${}_k\Pi(a, \cdot)$ $(a \in {}_kB_e)$ satisfying ${}_k\Pi(a, E) \leqslant 1$ and $\int_{{}_kB_e} \Pi(a, E)_k\mu(\mathrm{d}a) > 0$.

$$ {}_k\Pi(a, E) = 0 \qquad a \in {}_kH \tag{11} $$

Then the basic conditions at the beginning of this section are satisfied and so Lemma 1 and Theorem 2 are valid.

Definition 2. Suppose $X^0 \in \mathscr{H}_k$. The $\{{}_k\Pi(a, \cdot),\ a \in {}_kH \cup {}_kB_e\}$ generalized D-type extension of X^0 satisfying (11) is called a generalized D*-type extension process.

Theorem 3. A kth-order instantaneous-return process $X^0 \in \mathscr{H}_k$ and its $\{{}_k\Pi(a, \cdot),\ a \in {}_kB_e\}$ generalized D*-type extension process have the same lth-order exit boundary ${}_lB_e$ $(l \leqslant k)$. Particuarly, they have the same Q matrix and X satisfies the system of backward equations.

11.8 ON NON-STICKY EXTENSIONS

From now on, we turn to the V-type extension. The D-type extension is of instantaneous-return type: after a process moves to infinity it returns to finite states instantly. Of course, a process may return to finite states slowly from infinity. This is just the V-type extension. We may also have the mixed DV-type extension.

When Chung (1963, 1966) studied the boundary theory of Markov chains, he analysed meticulously their sample paths, introduced the concept of sticky and non-sticky boundary points and at last under the hypothesis of finiteness of exit boundaries obtained the analytic expressions for transition probabilities of processes. But he did not directly construct their sample paths. The D-type, V-type and mixed DV-type extension processes are all non-sticky return processes, i.e. processes whose leaping points may be arranged into a sequence in order of magnitude. A left-open and right-closed interval with its adjacent end-points being leaping points is a leaping interval, i.e. U interval. The construction of sample paths on leaping intervals is the key to the construction of non-sticky return processes.

The motion of a D-type extension process in each leaping interval is the motion of a minimal Q process starting from finite states. So an imbedded chain describing its jump behaviour is an ordinary discrete-parameter Markov chain $(x_n, n \geqslant 0)$ with its starting time and an initial distribution concentrated in the state space E. But for a V-type extension process its motion on each leaping interval is the motion of a minimal Q process starting from infinity. An imbedded chain describing its jump behaviour should be $(x_n, -\infty < n < +\infty)$. However, it is not a Markov chain and nor is it a stationary sequence. It belongs to the class of approximating Markov chains introduced by Hunt (1960) for the first

time. Hence approximating Markov chains are precisely the foundation stones for our construction of non-sticky returning processes. But an approximating Markov chain is defined on a measure space which may have infinite total measure.

From now on until the end of this chapter, we will first introduce approximating Markove chains and their characteristic measures and then study the relation between characteristic measure and the initial time and life-time of an approximating Markov chain. After that we shall use approximating chains are imbedding chains to construct sample paths on leaping intervals. Thus the so-called approximating minimal Q processes arise and therefore the minimal Q processes starting from infinity can be described. We establish the correspondence between entrance families and approximating minimal Q processes. Finally, on the basis of approximating minimal Q processes, we shall construct the sample paths of a class of non-sticky returning processes, i.e. we shall obtain the DV-type extension processes of minimal Q processes. In terms of entrance families we derive the analytic expressions of transition probabilities for this class of Q processes. Utilizing the boundary theory we can also construct other non-sticky processes that are more complicated, i.e. the so-called generalized DV-type and (DV)*-type extension processes.

11.9 STOCHASTIC CHAINS AND CHARACTERISTIC MEASURES

The concept of stochastic chains was introduced by Hunt (1960). We will give a brief description of their definition and results. Let E be a denumerable set and $\Pi = (\Pi_{ij}, i, j \in E)$ a sub-stochastic matrix, i.e. a matrix that has non-negative entries and row sums that are at most 1. When the row sum is equal to 1, it is called a stochastic matrix. Let $(\Omega, \mathscr{F}, P)$ be a measure space, i.e. Ω is a non-empty set of abstract points, $\mathscr{F}$ is a Borel field composed of some subsets of Ω and P is a measure on $\mathscr{F}$. We emphatically point out that we do not require that $(\Omega, \mathscr{F}, P)$ be a totally finite measure space. Henceforth $\mathbf{N}$ will denote the set of all integers.

Definition 1. Suppose on $(\Omega, \mathscr{F}, P)$ a triple (x, α, β) is given:

(i) α is an $\mathscr{F}$-measurable function with values in $\{-\infty\} \cup \mathbf{N}$, β is an $\mathscr{F}$-measurable function with values in $\mathbf{N} \cup \{+\infty\}$, and for all $\omega \in \Omega$, $\alpha(\omega) \leqslant \beta(\omega)$;

(ii) for each $\omega \in \Omega$ and $n \in \mathbf{N}$ satisfying $\alpha(\omega) \leqslant n \leqslant \beta(\omega)$, there exists a unique $x(n, \omega) \in E$ and for $n \in \mathbf{N}$, $i \in E$,

$$ \Lambda(n, i) \equiv (x(n) = i) = (\omega : x(n, \omega) = i, \alpha(\omega) \leqslant n \leqslant \beta(\omega)) \in \mathscr{F} $$

We call (x, α, β) a quasi-stochastic chain.

For a quasi-stochastic chain (x, α, β), write $\mathscr{F}(x, \alpha, \beta)$ for the minimal Borel field which contains all sets $\Lambda(n, i)$ $(n \in \mathbf{N}, i \in E)$ and call it the Borel field generated

by (x,α,β). Write $\mathscr{F}_k(x,\alpha,\beta)$ for the minimal Borel field which contains all sets $\Lambda(n,i)$ $(n \leqslant k,\ i\in E)$.

Definition 2. Suppose that (x,α,β) is a quasi-stochastic chain. If for all $n\in\mathbf{N}$ and $i\in E$, $P[\Lambda(n,i)] < +\infty$, we call (x,α,β) a stochastic chain.

Because $\Omega = \bigcup_{n\in\mathbf{N}}\bigcup_{i\in E}\Lambda(n,i)$, it follows that for a measure space $(\Omega,\mathscr{F},P)$ on which a stochastic chain is defined, Ω must be σ-finite.

We can define $x(n,\omega)$ for any $n\in\mathbf{N}$. For example, take arbitrarily two indices $\Delta\notin E$, $\theta\notin E$ and $\Delta\neq\theta$. For $n<\alpha(\omega)$ define $x(n,\omega)=\Delta$, and for $n>\beta(\omega)$ define $x(n,\omega)=\theta$. Obviously, sets $(x(n)=\Delta)=(n<\alpha)$ and $(x(n)=\theta)=(\beta<n)$ are $\mathscr{F}$-measurable. But even for a stochastic chain (x,α,β), their measures are not necessarily finite.

Definition 3. Suppose that (x,α,β) is a stochastic chain. If given 'the present', its 'past' and 'future' are conditionally independent, i.e. for any $k,m,\ n\in\mathbf{N}$, $k<m<n$ and $i_k,\ i_{k+1},\ldots,i_n\in E$. Putting

$$\Lambda=\Lambda(m,i_m) \qquad \Lambda'=\prod_{h\leqslant j\leqslant m}(j,i_j) \qquad \Lambda''=\prod_{m\leqslant j\leqslant n}\Lambda(j,i_j)$$

$$\frac{P(\Lambda'\Lambda'')}{P(\Lambda)}=\frac{P(\Lambda')}{P(\Lambda)}\frac{P(\Lambda'')}{P(\Lambda)}$$

as long as $P(\Lambda)>0$. We call (x,α,β) a Markov chain.

Definition 4. Suppose (x,α,β) is a Markov chain. If for any $n\in\mathbf{N}$, $i\in E$, we have

$$P(x(n)=i,x(n+1)=j)=P(x(n)=i)\Pi_{ij}$$

we call (x,α,β) a Π-chain. The matrix Π is called the (one-step) transition matrix of the chain.

Definition 5. Suppose (x,α,β) is a stochastic chain and σ is an $\mathscr{F}$-measurable function which is defined on Ω and takes values in $\{-\infty\}\cup\mathbf{N}\cup\{+\infty\}$. Let

$$\Omega'=(\sigma\in\mathbf{N},\alpha\leqslant\sigma\leqslant\beta) \qquad \mathscr{F}'=\Omega'\cap\mathscr{F} \qquad P'(\Lambda')=P(\Lambda') \qquad \Lambda'\in\mathscr{F}'$$

For $\omega\in\Omega'$ let

$$\gamma(\omega)=\beta(\omega)-\sigma(\omega) \qquad y(n,\omega)=x(\sigma(\omega)+n,\omega)$$

If $(y,0,\gamma)$ is a Π-chain on $(\Omega',\mathscr{F}',P')$, we say that the random time σ leads the stochastic chain (x,α,β) to the Π-chain $(y,0,\gamma)$. Write

$$f_\sigma(x,\alpha,\beta)=(y,0,\gamma)$$

Definition 6. Suppose (x,α,β) is a stochastic chain. If there exists a sequence of random times α_n, $n\geqslant 1$, taking values in $\mathbf{N}\cup\{+\infty\}$ such that on Ω almost surely $\alpha_n\downarrow\alpha$ and each α_n leads the stochastic chain (x,α,β) to some Π-chain $f_{\alpha_n}(x,\alpha,\beta)$, we call (x,α,β) an approximating Π-chain.

Definition 7. Suppose (x,α,β) is a stochastic chain and σ is an $\mathscr{F}$-measurable function with values in $\mathbf{N}\cup\{+\infty\}$. If for any $n\in\mathbf{N}$, $(\sigma\leqslant n)\in\mathscr{F}_n(x,\alpha,\beta)$, we call σ a random time independent of the future or a wide-stopping time.

Definition 8. Suppose that (x,α,β) is an approximating Π-chain. If (x,α,β) is a Markov chain and if the sequence of random times α_n which leads (x,α,β) to a Π-chain can be chosen as a sequence of wide-stopping times of the chain, we call (x,α,β) a strong approximating Π-chain.

Write $\Pi^{(n)}=\Pi^n$, and $G=\sum_{n=0}^{+\infty}\Pi^n$. Then $G_{ij}\leqslant G_{jj}$. According to Π we can divide the states in E into recurrent and non-recurrent states. A state i is Π-recurrent if and only if $G_{ii}=+\infty$.

Definition 9. Assume that $\eta=(\eta_j,\ j\in E)$ is a measure on E, i.e. $0\leqslant\eta_j\leqslant+\infty$ $(j\in E)$. We call η finite if each η_j is finite, and totally finite if $\sum_j\eta_j<+\infty$. We say that η is a Π-excessive measure if $\eta\Pi\leqslant\eta$, i.e. $\sum_i\eta_i\Pi_{ij}\leqslant\eta_j$ $(j\in E)$, and that η is a Π-harmonic measure if $\eta\Pi=\eta$.

Theorem 1. Suppose (x,α,β) is an approximating Π-chain. Let $C(i)$ be the measure concentrated on i with unit mass, i.e. $C_i(i)=1$ and $C_j(i)=0$ for $j\neq i$. Let

$$\eta_j=\int_\Omega\sum_{\alpha\leqslant n\leqslant\beta}C_j[x(n)]\,\mathrm{d}P=\sum_{n\in N}P(x(n)=j) \tag{1}$$

Then $\eta=(\eta_j,\ j\in E)$ is a Π-excessive measure, which is called the characteristic measure of the Π-chain

Proof. Suppose $(x,0,\beta)$ is a Π-chain with the initial distribution v. Then the characteristic measure determined by $(x,0,\beta)$ is $\eta=vG$. Of course, it is a Π-excessive measure. Suppose (x,α,β) is an approximating chain. Denote by η^n the characteristic measure of the Π-chain $f_{\alpha_n}(x,\alpha,\beta)$. Then η^n is Π-excessive, i.e. $\eta^n\Pi\leqslant\eta^n$. But $\eta^n\uparrow\eta$, so that $\eta\Pi\leqslant\eta$, i.e. η is Π-excesssive. The proof is completed. QED

Theorem 2. Suppose that (x,α,β) is an approximating Π-chain with a finite characteristic measure η. Let

$$\alpha'=-\beta \qquad \beta'=-\alpha \qquad x'(n)=x(-n)$$

Then the inverse chain (x',α',β') is an approximating Q-chain, where for $\eta_j>0$

$$Q_{ji}=\eta_i\Pi_{ij}/\eta_j \tag{2}$$

and for $\eta_j = 0$, Q_{ji} may be chosen arbitrarily as long as Q_{ji} is nonnegative and the row sums of Q are not more than 1.

Proof. Suppose D is finite subset of E. Let τ be the last exit time of the chain (x, α, β) from D, i.e.

$$\tau = \begin{cases} \sup\{n: \alpha \leqslant n \leqslant \beta, x(n) \in D\} \\ -\infty \qquad \text{if the above set is empty} \end{cases} \tag{3}$$

Clearly, $-\tau$ is the hitting time of (x', α', β') at D. Because $\eta(D) < \infty$ by (1), we know that there are only a finite number of $n \in \mathbf{N}$ such that $x(n) \in D$. Hence $\tau \leqslant \beta$ and $\tau < \infty$.

First of all suppose that (x, α, β) is a Π-chain, $\alpha = 0$ and its initial distribution is ν. Determine $L_D(i)$ according to (7.3.12). Then (7.3.16) is true, i.e.

$$\begin{aligned} &P(x(\tau) = i_0, x(\tau - 1) = i_1, \ldots, x(\tau - k) = i_k) \\ &= \eta(i_k)\Pi(i_k, i_{k-1})\Pi(i_{k-1}, i_{k-2}) \cdots \Pi(i_1, i_0)L_D(i_0) \end{aligned} \tag{4}$$

where $\eta = \nu G$.

The equality (4) also holds for the approximating Π-chain (x, α, β). In fact, let

$$f_{\alpha_n}(x, \alpha, \beta) = (x_n, 0, \beta_n) \tag{5}$$

and let τ_n be the last exit time of the Π-chain $(x_n, 0, \beta_n)$ from D and η^n be the characteristic measure. Then $\eta^n \uparrow \eta$ and for almost every $\omega \in \{\tau < \infty\}$ and sufficiently large n,

$$\alpha_n(\omega) + \tau_n(\omega) = \tau(\omega) \qquad x_n(\tau_n(\omega) - j) = x(\tau(\omega) - j)$$

Since (4) holds for $(x_n, 0, \beta_n)$, passing to the limit we are sure that (4) is also true for (x, α, β).

For the inverse chain, (4) can be rewritten as

$$\begin{aligned} &P(x'(\varepsilon') = i_0, x'(\varepsilon' + 1) = i_1, \ldots, x'(\varepsilon' + k) = i_k) \\ &= \eta(i_0)L_D(i_0)Q(i_0, i_1)Q(i_1, i_2) \cdots Q(i_{k-1}, i_k) \end{aligned}$$

where ε' is the hitting time for the inverse chain (x', α', β') at D. The above equality indicates that ε' leads (x', α', β') to a Q-chain. Take $D = D_n \uparrow E$. Then $\tau(D_n) \uparrow \beta$ and $\varepsilon'(D_n) = -\tau(D_n) \downarrow \alpha'$. Therefore, (x', α', β') is an approximating Q-chain, and the proof is completed. QED

Theorem 3. Suppose that (x, α, β) is an approximating Π-chain with a finite characteristic measure η. Let $\alpha(D)$ be the hitting time of the chain (x, α, β) at a finite subset D of E, i.e.

$$\alpha(D) = \begin{cases} \inf\{n: \alpha \leqslant n \leqslant \beta, x(n) \in D\} \\ +\infty \qquad \text{if the above set is empty} \end{cases} \tag{6}$$

Then $\alpha \leqslant \alpha(D)$, $-\infty < \alpha(D)$, $\alpha(D)$ is a wide-stopping time, $\alpha(D) \downarrow \alpha$ as $D \uparrow E$ and $\alpha(D)$ leads (x, α, β) to the Π-chain

$$f_{\alpha(D)}(x, \alpha, \beta) = (x_D, 0, \beta_D) \tag{7}$$

Proof. Consider the last exit time $\tau'(D)$ of the inverse chain (x', α', β') from D. According to Theorem 2, $\tau'(D) \leqslant \beta'$, $\tau'(D) < +\infty$ and $\tau'(D) \uparrow \beta'$ as $D \uparrow E$. Obviously, $\alpha(D) = -\tau'(D)$. Hence, $\alpha \leqslant \alpha(D)$, $-\infty < \alpha(D)$ and $\alpha(D) \downarrow \alpha$ as $D \uparrow E$. It follows from Theorem 2 that $\alpha(D)$ leads (x, α, β) to a Π-chain. Finally, by the definition of $\alpha(D)$ it is easy to verify that $\alpha(D)$ is a wide-stopping time of (x, α, β) and we conclude the proof. QED

Remark

Theorem 3 demonstrates that for any approximating Π-chain (x, α, β) with a finite characteristic measure we can always choose a sequence of wide-stopping times $\alpha_n \downarrow \alpha$ such that $f_{\alpha_n}(x\, \alpha, \beta)$ are Π-chains. However, (x, α, β) is not necessarily a strong approximating chain, for (x, α, β) itself may not be a Markov chain.

Corollary

Assume that (x, α, β) is an approximating Π-chain with a finite characteristic measure η. Write ν_D for the distribution that (x, α, β) hits the finite set D, i.e.

$$\nu_D(j) = P(x(\alpha(D)) = j) \tag{8}$$

where $\alpha(D)$ is defined according to (6). Then the support of ν_D is contained in D and, moreover,

$$\nu_D(j) = \eta(j)L_D^{\eta}(j)$$

where

$$L_D^{\eta}(i) = P_i(y_0 \in D, y(n) \notin D \text{ for } 0 < n \leqslant \delta)$$

Here $(y, 0, \delta)$ is a Q-chain and Q is determined by (2).

Proof. Letting $k = 0$ in (4) we have

$$P(x(\tau) = j) = \eta(j)L_D(j)$$

Applying this to the inverse chain (x', α', β') we obtain

$$P(x'(\tau') = j) = \eta(j)L_D^{\eta}(j)$$

where τ' is the last exit time of $(x', \alpha', \beta)'$ from D, i.e. the hitting time of (x, α, β) at D, and $x'(\tau') = x(\alpha(D))$. Then the above equality is just (9). The proof is completed. QED

Theorem 4. Assume that η is a finite Π-excessive measure. Fix a sequence D_n of finite subsets of E, $D_n \uparrow E$. Then there exists a sequence of measures ν^n with the following properties:

(i) ν^n is totally finite with its support contained in D_n;

(ii) $\nu^n G \leqslant \eta$ and equality holds on D_n;

(iii) $\nu^n G \uparrow \eta$ as $n \uparrow \infty$;

(iv) suppose $n < m$; then ν^n is the hitting distribution at D of the Π-chain with the initial distribution ν^m, i.e. if $(x, 0, \beta)$ is a Π-chain with the initial distribution ν^m and the hitting times $\alpha(D_n)$ lead (x, α, β) to Π-chains $(x_n, 0, \beta_n)$, then the initial distribution of $(x_n, 0, \beta_n)$ is ν^n for each $n < m$.

(v) there exists a measure space $(\Omega, \mathscr{F}, P)$ on which an approximating Π-chain (x, α, β) is defined with its characteristic measure coincident with η and with ν_{D_n} determined by (8) coincident with ν^n.

The proof of the theorem is a bit complicated. The reader is referred to Kemeny *et al.* (1966, Chapter 10, section 12). We point out that by (9) ν^n in Theorem 4 is determined by η and Π according to

$$\nu^n(j) = \eta(j) L^n_{D_n}(j) \tag{10}$$

11.10 INITIAL TIME AND LIFETIME OF APPROXIMATING Π-CHAINS

Let (x, α, β) be a quasi-stochastic chain. We call α the initial time and β the lifetime. Note that from the statement under Definition 9.2, for all $n \in \mathbf{N}$, $x(n)$ are defined but they take values in $E \bigcup \{\Delta, \theta\}$. We agree that $C_j(\Delta) = C_j(\theta) = 0$ for $j \in E$. Then, the Π-excessive measure determined by the approximating Π-chain (x, α, β) is

$$\eta_j = \int_\Omega \sum_{n \in N} C_j[x(n)]\, \mathrm{d}P = \int_\Omega \sum_{\alpha \leqslant n < +\infty} C_j[x(n)]\, \mathrm{d}p \qquad j \in E \tag{1}$$

Theorem 1. Let (x, α, β) be an approximating Π-chain with the characteristic measure η. Then the following are true:

(i) If Π is a stochastic matrix, almost surely $\beta = +\infty$.

(ii) If $\alpha = -\infty$, almost surely η is Π-harmonic.

(iii) Suppose η is finite and its Riesz decomposition is

$$\eta = \eta^1 + \eta^2 \qquad \eta^1 = \nu G \qquad \nu = \eta - \eta\Pi \qquad \text{and} \qquad \lim_{l \to +\infty} \eta \Pi^l = \eta^2 \tag{2}$$

Then the excessive quantity ν can be expressed as

$$\nu_j = P(-\infty < \alpha, x(\alpha) = j) \tag{3}$$

the potential measure η^1 can be represented by

$$\eta^1_j = \int_{-\infty < \alpha} \sum_{n \in \mathbf{N}} C_j[x(n)]\, \mathrm{d}P \tag{4}$$

and the harmonic measure has the representation

$$\eta^2 = \int_{-\infty = \alpha} \sum_{n \in \mathbf{N}} C_j[x(n)]\, \mathrm{d}P \tag{5}$$

(iv) Suppose that η is finite. Then η is a potential if and only if $-\infty < \alpha$ almost surely, and η is harmonic if and only if $-\infty = \alpha$ almost surely.

Proof. Suppose that random times α_n, $n \geqslant 1$, lead (x, α, β) to Π-chains $(x_n, 0, \beta_n)$ on $\Omega_n = (\alpha_n \in \mathbf{N}, \alpha \leqslant \alpha_n \leqslant \beta)$ respectively.

(i) Assume that Π is a stochastic matrix. It is well known that for any Π-chain, its lifetime is almost surely equal to $+\infty$. Hence it almost goes without saying that $\beta_n = \beta - \alpha_n = +\infty$, i.e. $\beta = +\infty$ on Ω_n. So $\beta = +\infty$ almost surely on Ω since $\Omega_n \uparrow \Omega$.

(ii) Since

$$\begin{aligned}
\eta_i &= \int_\Omega \sum_{\alpha \leqslant k < +\infty} C_i[x(k)]\, \mathrm{d}P \\
&= \lim_{n \to +\infty} \int_{\Omega_n} \sum_{\alpha_n \leqslant k < +\infty} C_i[x(k)]\, \mathrm{d}P \\
&= \lim_{n \to +\infty} \int_{\Omega_n} \sum_{0 \leqslant n < +\infty} C_i[x_n(m)]\, \mathrm{d}P \\
&= \lim_{n \to +\infty} \sum_{m=0}^{+\infty} P(x_n(m) = i)
\end{aligned}$$

it follows that

$$\begin{aligned}
\eta_i \Pi_{ij} &= \lim_{n \to +\infty} \sum_{m=0}^{+\infty} P(x_n(m) = i)\Pi_{ij} \\
&= \lim_{n \to +\infty} \sum_{0 \leqslant m < +\infty} P(x_n(m) = i, x_n(m+1) = j) \\
&= \lim_{n \to +\infty} \int_{\Omega_n} \sum_{0 \leqslant m < +\infty} C_i[x_n(m)] C_j[x_n(m+1)]\, \mathrm{d}P
\end{aligned}$$

$$= \lim_{n \to +\infty} \int_{\Omega_n} \sum_{\alpha_n \leqslant k < +\infty} C_i[x(k)]C_j[x(k+1)]\,dP$$

$$= \int_{\Omega} \sum_{\alpha \leqslant k < +\infty} C_i[x(k)]C_j[x(k+1)]\,dP$$

Taking summation over $i \in E$, we have

$$\sum_i \eta_i \Pi_{ij} = \int_{\Omega} \sum_{\alpha \leqslant k < +\infty} C_j[x(k+1)]\,dP = \int_{\Omega} \sum_{\alpha+1 \leqslant m < \infty} C_j[x(m)]\,dP \qquad (6)$$

If $\alpha = -\infty$ almost surely, then the right-hand side of (6) is equal to η_j. Therefore η is harmonic.

(iii) If η is finite, by (6) we have

$$(\eta\Pi)_j = \eta_j - P(-\infty < \alpha, x(\alpha) = j) \qquad (7)$$

Then (3) follows.

Similarly to (6) we have

$$(\eta\Pi^l)_j = \int_{\Omega} \sum_{\alpha+l \leqslant m < +\infty} C_j[x(m)]\,dP$$

$$= \int_{-\infty<\alpha} \sum_{\alpha+l \leqslant m < +\infty} C_j[x(m)]\,dP + \int_{-\infty=\alpha} \sum_{n \in N} C_j[x(m)]\,dP \qquad (8)$$

Noting

$$\eta_j = \int_{-\infty<\alpha} \sum_{\alpha \leqslant m < +\infty} C_j[x(m)]\,dP + \int_{-\infty=\alpha} \sum_{m \in N} C_j[x(m)]\,dP < +\infty \qquad (9)$$

(8) becomes

$$(\eta\Pi^l)_j = \eta_j - \int_{-\infty<\alpha} \sum_{\alpha \leqslant m < \alpha+l} C_j[x(m)]\,dP \uparrow \eta_j$$

$$= \int_{-\infty<\alpha} \sum_{\alpha \leqslant m < +\infty} C_j[x(m)]\,dP \qquad l \uparrow +\infty$$

Combining with (9) we obtain (5) and so (4) follows.

(iv) When $-\infty < \alpha$ almost surely, by (5) we have $\eta^2 = 0$. Hence $\eta = \eta^1$ is a potential. Conversely, if η is a finite potential, then $\eta\Pi^l \downarrow 0$. So that the second term of the right-hand side of (8) equals zero, i.e.

$$\sum_{m \in N} P\{-\infty = \alpha, x(m) = i\} = 0$$

Sum over $i \in E$, and we obtain $-\infty < \alpha$ almost surely.

If $-\infty = \alpha$ almost surely, from (4) we have $\eta^1 = 0$. Hence $\eta = \eta^2$ is harmonic. Conversely, if η is finite and harmonic, then by (7) $P\{-\infty < \alpha, x(\alpha) = j\} = 0$. Summing up over $j \in E$, we obtain $P\{-\infty < \alpha\} = 0$. We conclude the proof.

QED

Theorem 1 shows that the approximating Π-chain (x, α, β) can be decomposed into two stochastic chains which are defined respectively on $\Omega_1 = (-\infty < \alpha)$ and $\Omega_2 = (-\infty = \alpha)$; nevertheless, they may not be approximating Π-chains. Fortunately, we can make them become approximating Π-chains very easily.

Theorem 2. Assume that η is a finite Π-excessive measure with the Riesz decomposition (2). Then there exist a measure space $(\Omega, \mathscr{F}, P)$ and an approximating Π-chain (x, α, β) defined on it with the characteristic measure η. Furthermore, let $\Omega_1 = (-\infty < \alpha)$, $\Omega_2 = (-\infty = \alpha)$ and $(x_a, \alpha_a, \beta_a) = (x, \alpha, \beta)$ on Ω_a $(a = 1, 2)$. Then (x_a, α_a, β_a) $(a = 1, 2)$ are approximating Π-chains on Ω_a and they determine Π-excessive measures η^a, respectively.

Proof. According to Theorem 9.4, for each $a = 1$ and 2 there exists a measure space $(\Omega_a, \mathscr{F}_a, P_a)$ on which an approximating Π-chain (x_a, α_a, β_a) is defined with the characteristic measure η^a. By Theorem 1 we can take $\Omega_1 = (-\infty < \alpha_1)$ and $\Omega_2 = (-\infty = \alpha_2)$.

We can consider that Ω_1 and Ω_2 have no common points. Write $\Omega = \Omega_1 \cup \Omega_2$ and $\mathscr{F} = \{A : A \subset \Omega,\ A \cap \Omega_a \in \mathscr{F}_a,\ a = 1, 2\}$. Clearly $\mathscr{F}$ is a borel field on Ω and $\mathscr{F}_a \subset \mathscr{F}$. For $A \in \mathscr{F}$, let $P(A) = P_1(A \cap \Omega_1) + P_2(A \cap \Omega_2)$. Let $\omega \in \Omega$, if $\omega \in \Omega_a$, and define $\alpha(\omega) = \alpha_a(\omega)$, $\beta(\omega) = \beta_a(\omega)$, $x(n, \omega) = x_a(n, \omega)$. Then, it is easily seen that (x, α, β) is an approximating Π-chain, which is defined on $(\Omega, \mathscr{F}, P)$ and has all the properties presented in this theorem. The proof is completed. QED

11.11 IMBEDDED CHAINS

A matrix $A = (a_{ij}, i, j \in E)$ is called a Q matrix if its entries in the diagonal are non-positive and finite, if the entries outside the diagonal are non-negative and finite, and if the row sums are non-positive and finite. When the row sums are equal to zero, the Q matrix A is called conservative. Suppose we are given a Q matrix $Q = (q_{ij}, i, j \in E)$. Then

$$0 \leqslant q_{ij} < +\infty \ (i \neq j) \qquad \sum_{j \neq i} q_{ij} \leqslant -q_{ii} < +\infty \qquad (1)$$

Put $q_i = -q_{ii}$. We call $d_i = q_i - \sum_{j \neq i} q_{ij}$ the non-conservative quantity of i. Recall that $C(i)$ is the measure concentrated on i with a unit mass. Let[1]

[1] For convenience, the imbedded matrix Π here is slightly different from that in (2.9.7).

$$\Pi_{ij} = \begin{cases} [1 - C_j(i)]q_{ij}/q_i & \text{if } q_i > 0 \\ 0 & \text{if } q_i = 0 \end{cases}$$

We call the sub-stochastic matrix $\Pi = (\Pi_{ij}, i, j \in E)$ the imbedded matrix of Q and a Π-chain the imbedded chain of Q. Recall that $\theta \notin E$. Write $E_\theta = E \cup \{\theta\}$. Let $q_{i\theta} = d_i$ $(i \in E)$ and $q_\theta = -q_{\theta\theta} = 0$. Then $Q_\theta = (q_{ij}, i, j \in E_\theta)$ is a conservative Q matrix and the imbedded matrix Π of Q is a submatrix of the imbedded matrix $\Pi_\theta = (\Pi_{ij}, i, j \in E_\theta)$. Clearly

$$\Pi_{i\theta} = \begin{cases} d_i/q_i & \text{if } q_i > 0 \\ 0 & \text{if } q_i = 0 \end{cases} \qquad \Pi_{\theta j} = 0 \qquad j \in E_\theta \tag{2}$$

For $\lambda > 0$ let

$$q_{ij}(\lambda) = q_{ij}\,(i \neq j) \qquad q_i(\lambda) = -q_{ii}(\lambda) = \lambda + q_i$$

Then $\Pi(\lambda) = (q_{ij}(\lambda), i, j \in E)$ is a non-conservative Q matrix. Its imbedded matrix is denoted by $\Pi(\lambda) = (\Pi_{ij}(\lambda), i, j \in E)$ and is called a λ-imbedded matrix of Q. Denote the λ-imbedded matrix of Q_θ by $\Pi_\theta(\lambda) = \Pi_{ij}(\lambda), i, j \in E_\theta)$. Then $\Pi(\lambda)$ is a submatrix $\Pi_\theta(\lambda)$ and, moreover,

$$\Pi_{ij}(\lambda) = \frac{[1 - C_i(j)]q_{ij}}{\lambda + q_i} \qquad i, j \in E$$

$$\Pi_{i\theta}(\lambda) = \frac{d_i}{\lambda + q_i}(j \in E) \qquad \Pi_{\theta j}(\lambda) = 0\,(j \in E_\theta) \tag{3}$$

Note that all elements in the θth row of Π_θ are equal to zero and the other row sums are equal to 1, and that all elements in the θth row of $\Pi_\theta(\lambda)$ are equal to zero and the other row sums are less than 1. Hence, for any approximating Π-chain or approximating $\Pi_\theta(\lambda)$-chain (x, α, β), if at time δ the chain visits θ, i.e. $x(\delta) = \theta$, then δ must be the lifetime, i.e. $\beta = \delta$.

Write

$$G(\lambda) = \sum_{n=0}^{\infty} [\Pi(\lambda)]^n \tag{4}$$

$$\phi_{ij}(\lambda) = G_{ij}(\lambda)(\lambda + q_j)^{-1} \tag{5}$$

Then $\phi(\lambda) = \{\phi_{ij}(\lambda), i, j \in E\}$ is the Laplace transform of the Feller minimal solution $f(t) = \{f_{ij}(t), i, j \in E\}$ $(t \geq 0)$. Because $\lambda \sum_j \phi_{ij}(\lambda) \leq 1$, it follows that $G(\lambda) < \infty$ and, therefore, all states in E are $\Pi\omega$-non-recurrent.

Lemma 1. Assume that $(y, 0, \delta)$ is a $\Pi_\theta(\lambda)$-chain with its initial distribution concentrated in E. Let $\delta(E)$ be the first exit time of the chain from E:

$$\delta(E) = \sup\{n : y(n) \notin E, 0 \leq n \leq \delta\}$$

Then

$$\delta(E) = \begin{cases} \delta & \text{if } \delta = +\infty, \text{or } \delta < +\infty,\ y(\delta) \notin E \\ \delta - 1 & \text{if } \delta < +\infty,\ y(\delta) = \theta \end{cases} \tag{6}$$

and $(y, 0, \delta(E))$ is a $\Pi(\lambda)$-chain.

Proof. Since $y(0) \notin E$, it follows that $0 \leq \delta(E)$. And since all elements in the θth row of $\Pi_\theta(\lambda)$ are equal to zero, the only possible time for the chain to reach θ is the lifetime $\delta < \infty$. Hence (6) is justified. Because $(y, 0, \delta)$ is a $\Pi_\theta(\lambda)$-chain, it is clear that $(y, 0, \delta(E))$ is a $\Pi(\lambda)$-chain. The proof is completed. QED

Lemma 2. Assume that (y, α, δ) is an approximating $\Pi_\theta(\lambda)$-chain and that $\delta(E)$ is the first exit time from E,

$$\delta(E) = \begin{cases} \sup\{n : y(n) \in E, \alpha \leq n \leq \delta\} \\ -\infty \qquad \text{if the above set is empty} \end{cases} \tag{7}$$

Then

$$\delta(E) = \begin{cases} -\infty & \text{if } \alpha = \delta,\ y(\delta) = \theta \\ \delta - 1 & \text{if } \alpha < \delta < +\infty,\ y(\delta) = \theta \\ \delta & \text{if } \delta = -\infty, \text{or } \alpha \leq \delta < +\infty,\ y(\delta) \in E \end{cases}$$

and $(y, \alpha, \delta(E))$ is on approximating $\Pi(\lambda)$-chain on $\Omega(E) = (-\infty < \delta(E)) = (-\infty = \alpha) \cup (-\infty < \alpha,\ y(a) \in E)$.

Proof. Suppose the random times $\alpha_n \downarrow \alpha$ and α_n leads (y, α, δ) to a $\Pi_\theta(\lambda)$-chain $(y_n, 0, \delta_n)$ on $\Omega_n = \{\alpha_n \in N, \alpha \leq \alpha_n \leq \delta\}$ respectively. Restricted to $\Omega(E) \cap \Omega_n$, the chain $(y_n, 0, \delta_n)$ is a $\Pi_\theta(\lambda)$-chain with its initial distribution concentrated in E. On $\Omega(E)$, let $\alpha_n(E) = \alpha_n$. Then on $\Omega(E)$, $\alpha_n(E) \downarrow \alpha$ and for each n, $\alpha_n(E)$ leads the stochastic chain $(y, \alpha, \delta(E))$ on $\Omega(E)$ to the chain $(y_n, 0, \delta(E))$ on $\Omega_n(E) \equiv \{\alpha_n(E) \in \mathrm{N}, \alpha \leq \alpha_n(E) \leq \delta\}$. Notice that $\Omega(E) \cap \Omega_n = \Omega_n(E)$ and $\delta_n(E)$ is just the first exit time from E of the $\Pi_\theta(\lambda)$-chain $(y_n, 0, \delta_n)$ on $\Omega_n(E)$. According to Lemma 1 the chains $(y_n, 0, \delta_n(E))$ on $\Omega_n(E)$ are all $\Pi(\lambda)$-chains. Therefore $(y, \alpha, \delta(E))$ is an approximating $\Pi(\lambda)$-chain on $\Omega(E)$. The lemma is proved. QED

In the following it will be pointed out that we can, if necessary, enlarge the measure space and always extend a $\Pi(\lambda)$-chain or an approximating $\Pi(\lambda)$-chain to a Π-chain or an approximating Π-chain. For this purpose, we first introduce the concept of conditional distributions and conditional independence about stochastic chains. We remark that 'distribution' in the following definition means a probability measure which is deduced from a random variable or a stochastic process on its path space. For example, the exponential distribution with the parameter q is the probability measure on the real line deduced from the

distribution function

$$F(t)=\begin{cases}0 & \text{if } t<0\\ 1-e^{-qt} & \text{if } t\geqslant 0\end{cases}$$

Another example: the distribution of the Π-chain starting from i is the probability measure on the infinite-dimensional space $E\times E\times E\times\cdots$ generated by the consistent family of finite-dimensional distributions $\{R(i,i_1,i_2,\ldots,i_n)\}$, where

$$R(i,i_1,i_2,\ldots,i_n)=\Pi_{ii_1}\Pi_{i_1i_2}\cdots\Pi_{i_{n-1}i_n} \tag{8}$$

Definition 1. Let $(\Omega,\mathscr{F},P)$ be a measure space and $\mathscr{A}$ a Borel subfield of $\mathscr{F}$. Suppose ρ_n $(n\geqslant 1)$ are $\mathscr{F}$-measurable functions and F_n $(n\geqslant 1)$ are distributions on the real line. If for any $\Lambda\in\mathscr{A}$ and Borel sets B_n $(n\geqslant 1)$ on the real line,

$$P(\Lambda,\rho_1\in B_1,\ldots,\rho_n\in B_n)=P(\Lambda)F_1(B_1)\cdots F_n(B_n)$$

we say that given $\mathscr{A}$, ρ_n $(n\geqslant 1)$ are conditionally independent and have the conditional distributions F_n, respectively.

Similarly, if a quasi-stochastic chain (z,ε,δ) or a sequence of quasi-stochastic chains $(z_n,\varepsilon_n,\delta_n)$ $(n\geqslant 1)$ are defined on $(\Omega,\mathscr{F},P)$, we can state the definition that $\mathscr{A}$, all ρ_n, $n\geqslant 1$, (z,ε,δ), $(z_n,\varepsilon_n,\delta_n)$, $n\geqslant 1$, are conditionally independent and the definition that given $\mathscr{A}$, the conditional distribution of (z,ε,δ) is some distribution, and so on.

Lemma 3. Assume that ζ is a finite $\Pi(\lambda)$-excessive measure. Then there exist a measure space $(\Omega,\mathscr{F},P)$ and an approximating $\Pi_\theta(\lambda)$-chain (y,α,β) defined on it such that if $-\infty<\alpha$ then $y(\alpha)\in E$ and, moreover,

$$\zeta_j=\int_\Omega\sum_{\alpha\leqslant n\leqslant\delta}C_j[y(n)]\,\mathrm{d}P \qquad j\in E \tag{9}$$

Proof. According to Theorem 9.4, there exist a measure space $(\Omega_1,\mathscr{F}_1,P_1)$ and an approximating $\Pi(\lambda)$-chain (y_1,α_1,δ_1) defined on it with the characteristic measure ζ. It is well known that there exist a probability space $(\Omega_2,\mathscr{F}_2,P_2)$ and a family of independent random variables $\xi_2(i)$ $(i\in E)$, defined on it such that for each i, $\xi_2(i)$ take values 0 and 1 with probabilities $\lambda/(\lambda+d_i)$ and $d_i/(\lambda+d_i)$ respectively, where d_i is the non-conservative quantity of the state i. Take the product space $(\Omega,\mathscr{F},P)=(\Omega_1\times\Omega_2,\mathscr{F}_1\times\mathscr{F}_2,P_1\times P_2)$ and for $w=(w_1,w_2)\in\Omega$ let

$$\xi(i,w)=\xi_2(i,w_2) \qquad \alpha(w)=\alpha_1(w_1)$$
$$\delta(E,w)=\delta_1(w_1)$$
$$y(n,w)=y_1(n,w_1) \qquad \text{if } \alpha(w)\leqslant n\leqslant\delta(E,w)$$
$$\delta=\begin{cases}+\infty & \text{if } \delta(E)=+\infty\\ \delta(E)+\xi\{y[\delta(E)]\} & \text{if } \delta(E)<+\infty\end{cases}$$

When $\delta(E)<+\infty$ and $\xi\{y[\delta(E)]\}=1$ put $y(\delta)=\theta$. Then it is easy to prove that (y,α,δ) is what we require, and the proof is completed. QED

Lemma 4. Assume that ξ is a finite $\Pi(\lambda)$-excessive measure. Then there exists a measure space $(\Omega,\mathscr{F},P)$ on which are defined an approximating $\Pi_\theta(\lambda)$-chain (y,α,δ), a sequence of quasi-stochastic chains $(z_i,0,q_i)$ $(i\in E)$ and a family of $\mathscr{F}$-measurable functions s_u, $u\in U$, with U being a given subset of $(-\infty,+\infty)$, such that

(i) on $(-\infty,+\infty)$, $y_\alpha\in E$ and (9) is satisfied;
(ii) given $\mathscr{A}=\mathscr{F}(y,\alpha,\delta)$ for each $u\in U$ the conditional distribution of S_u is a given distribution F_u;
(iii) given $\mathscr{A}$, for each i the conditional distribution of $(z_i,0,\varepsilon_i)$ is the distribution of the Π-chain starting from i;
(iv) given $\mathscr{A}$, all S_u, $u\in U$; $(z_i,0,\varepsilon_i)$, $i\in E$, are conditionally independent.

Proof. This can be verified by an application of Lemma 3 and the technique of making independent product spaces. QED

Definition 2. Suppose stochastic chains (y,ε,ν) and (x,α,β) are defined on the same probability space. If $\varepsilon=\alpha$, $\nu\leqslant\beta$ and for $n\in\mathbf{N}$ and $\varepsilon\leqslant n\leqslant\nu$, $y(n)=x(n)$ holds, we call (x,α,β) an extension of (y,ε,ν) and (y,ε,ν) the front part of (x,α,β).

Theorem 5. Suppose that $\lambda>0$ is fixed and assume that $\zeta(\lambda)$ is a finite $\Pi(\lambda)$-excessive measure. Then, there exist a measure space $(\Omega,\mathscr{F},P)$ and defined on it an approximating $\Pi(\lambda)$-chain (y,α,ν) and an approximating Π-chain (x,α,β) such that (x,α,β) is an extension of (y,α,ν). The characteristic measure of (y,α,ν) is $\zeta(\lambda)$ and the characteristic measure of (x,α,β) is

$$\zeta_j=\zeta_j(\lambda)+\lambda\sum_i\zeta_i(\lambda)(\lambda+q_i)^{-1}G_{ij} \tag{10}$$

Proof. It follows from Lemma 4 that there exist a measure space $(\Omega,\mathscr{F},P)$ and an approximating $\Pi_\theta(\lambda)$-chain (y,α,δ) defined on it such that

$$\zeta_j(\lambda)=\int_\Omega\sum_{\alpha\leqslant n\leqslant\delta}C_j[y(n)]\,\mathrm{d}P \qquad j\in E$$

Removing $(-\infty<\alpha,\ y(\alpha)=\theta)$ if necessary, we can always assume $\{-\infty<\alpha,\ y(\alpha)=\theta\}$ is empty. On $(\Omega,\mathscr{F},P)$ we can also define a sequence of quasi-stochastic chains $\{z_i,0,\varepsilon_i\}$ $(i\in E)$ such that given $\mathscr{F}(y,\alpha,\delta)$ all $(z_i,0,\varepsilon_i)$, $i\in E$, are conditionally independent and for each i, given $\mathscr{F}(y,\alpha,\delta)$, the conditional distribution of $(z_i,0,\varepsilon_i)$ is the distribution of the Π-chain starting from i.

Define the first exist time $\delta(E)$ of the approximating $\Pi_\theta(\lambda)$-chain from E according to (7). Abbreviate $\delta(E)=\gamma^1$. Then $\Omega=(-\infty<\gamma)=(\alpha\leqslant\gamma)$ and (y,α,γ) is an approximating $\Pi(\lambda)$-chain with the characteristic measure $\zeta(\lambda)$.

Let

$$\beta = \begin{cases} \gamma & \text{if } \gamma = +\infty \text{ or } \gamma = \delta - 1 < +\infty \\ \gamma + \varepsilon_{y(\delta)} & \text{if } \gamma = \delta < +\infty \end{cases} \tag{11}$$

For $n \in \mathbf{N}$, $\alpha \leqslant n \leqslant \beta$, define

$$x(n) = \begin{cases} y(n) & n \leqslant \gamma \\ z_{y(\delta)}(n-\delta) & \gamma < +\infty, \gamma < n \leqslant \beta \end{cases} \tag{12}$$

Obviously, (x, α, β) takes its values in E and is the extension of (y, α, γ). Next we will prove that (x, α, β) is an approximating Π-chain.

First, suppose the approximating $\Pi_\theta(\lambda)$-chain (y, α, δ) is a $\Pi_\theta(\lambda)$-chain and $\alpha = 0$. We will prove that $(x, 0, \beta)$ defined as above is a Π-chain. That is, we prove $A_n = A_{n-1}\Pi_{i_{n-1}i_n}$, where $i_0, i_1, \ldots, i_n \in E$, $n \geqslant 1$,

$$A_n \equiv P(x(0) = i_0, \ldots, x(n) = i_n) \tag{13}$$

Define $R(i_0, i_1, \ldots, i_n)$ according to (8) and similarly define

$$R(\lambda; i_0, i_1, \ldots, i_n) = \Pi_{i_0 i_1}(\lambda)\Pi_{i_1 i_2}(\lambda)\cdots\Pi_{i_{n-1}i_n}(\lambda) \tag{14}$$

Note that

$$(y(0) = i_0) \subset (0 \leqslant \gamma) \qquad (y(n) = i_n) \subset (n \leqslant \delta) \subset (n \leqslant \delta)$$

hence

$$\begin{aligned} A_n &= P(x(0) = i_0, \ldots, x(n) = i_n, n \leqslant \gamma) + P(x(0) = i_0, \ldots, x(n) = i_n, \gamma < n) \\ &= P(y(0) = i_0, \ldots, y(n) = i_n, n \leqslant \delta) \\ &\quad + P(y(0) = i_0, \ldots, y(\delta) = i_\delta, z_{y(\delta)}(1) = i_{\delta+1}, \ldots, z_{y(\delta)}(n-\delta) = i_n, \delta < n) \\ &= P(y(0) = i_0)R(\lambda, i_0, i_1, \ldots, i_n) \\ &\quad + \sum_{k=0}^{n-1} P(y(0) = i_0, y(1) = i_1, \ldots, y(k) = i_k, \delta = k, z_{i_k}(1) \\ &= i_{k+1}, \ldots, z_{i_k}(n-k) = i_n) \end{aligned}$$

By conditional independence, the summand in $\sum_{k=0}^{m-1}$ is equal to

$$\begin{aligned} &P(y(0) = i_0, y(1) = i_1, \ldots, y(k) = i_k, \delta = k)R(i_k, i_{k+1}, \ldots, i_n) \\ &= P(y(0) = i_0)R(\lambda; i_0, i_1, \ldots, i_k)\left(1 - \sum_{j \in E_\theta} \Pi i_{kj}(\lambda)\right)R(i_k, i_{k+1}, \ldots, i_n) \end{aligned}$$

Consequently,

$$\begin{aligned} A_n = P(y(0) = i_0)\Big\{ &R(\lambda; i_0, i_1, \ldots, i_n) \\ &+ \sum_{k=0}^{n-1} R(\lambda; i_0, i_1, \ldots, i_k)\frac{\lambda}{\lambda + q_{i_k}}R(i_k, i_{k+1}, \ldots, i_n)\Big\} \end{aligned}$$

$$\begin{aligned} = P(y(0) = i_0)\Big\{ &R(\lambda; i_0, i_1, \ldots, i_{n-1})\Pi_{i_{n-1}i_n}(\lambda) \\ &+ R(\lambda; i_0, i_1, \ldots, i_{n-1})\lambda(\lambda + q_{i_{n-1}})^{-1}\Pi_{i_{n-1}i_n} \\ &+ \sum_{k=0}^{n-1} R(i_0, \ldots, i_k)\lambda(\lambda + q_{i_k})^{-1}R(i_k, \ldots, i_{n-1})\Pi_{i_{n-1}i_n}\Big\}. \end{aligned}$$

Noting

$$\Pi_{i_{n-1}i_n}(\lambda) + \lambda(\lambda + q_{i_{n-1}})^{-1}\Pi_{i_{n-1}i_n} = \Pi_{i_{n-1}i_n}$$

we find that $A_n = A_{n-1}\Pi_{i_{n-1}i_n}$.

On the other hand,

$$\begin{aligned} \zeta_j &= \int_\Omega \sum_{0 \leqslant n \leqslant \beta} C_j[x(n)]\,dP = \int_\Omega \sum_{0 \leqslant n \leqslant \gamma} C_j[y(n)]\,dP + \int_{\gamma < +\infty} \sum_{\gamma < n \leqslant \beta} C_j[x(n)]\,dP \\ &= \zeta_j(\lambda) + \sum_i \int_{\substack{\delta < +\infty \\ y(\delta) = i}} \sum_{0 \leqslant n \leqslant \varepsilon_i} C_j[z_i(n)]\,dP \end{aligned}$$

By conditional independence the summand in Σ is equal to $P(\delta < +\infty, y(\delta) = i)G_{ij}$, and

$$\begin{aligned} P(\delta < +\infty, y(\delta) = i) &= \sum_{n=0}^{+\infty} P(y(n) = i, \delta = n) = \sum_{n=0}^{+\infty} P(y(n) = i)\frac{\lambda}{\lambda + q_1} \\ &= \lambda\zeta_i(\lambda)(\lambda + q_i)^{-1} \end{aligned} \tag{15}$$

Therefore (10) is verified.

Now suppose that (y, α, δ) is an approximating $\Pi_\theta(\lambda)$-chain and that $\alpha_n \downarrow \alpha$ and α_n leads (y, α, δ) to $\Pi_\theta(\lambda)$-chains $(y_n, 0, \delta_n)$ respectively. Suppose α_n leads (x, α, β) to $(x_n, 0, \beta_n)$ for each n. Let γ be the first exit time of (y, α, δ) from E. Just as we have defined the extension process (x, α, β) of (y, α, γ) from (y, α, δ) and $(z_i, 0, \varepsilon_i)$ $(i \in E)$ in the fashion of (11)–(12), for each n we define the extension process of $(y_n, 0, \gamma_n)$ from $(y_n, 0, \delta_n)$ and $(z_i, 0, \varepsilon_i)$ $(i \in E)$ in the fashion of (11)–(12). Here γ_n is the first exit time of $(y_n, 0, \delta_n)$ from E. Then this extension process is just $(x_n, 0, \beta_n)$. Since, given $\mathscr{F}(y, \alpha, \delta)$, $(z_i, 0, \varepsilon_i)$ $(i \in E)$ are conditionally independent and the conditional distribution is that of the Π-chain starting from i, it follows that given $\mathscr{F}(y_n, 0, \delta_n)$ the same assertion remains true. From what we have proved, $(x_n, 0, \beta_n)$ are Π-chains. Therefore (x, α, β) is an approximating Π-chain.

Let ζ^n and $\zeta^n(\lambda)$ be the characteristic measure determined by $(x_n, 0, \beta_n)$ and $(y_n, 0, \gamma_n)$, respectively. Then we have proved that

$$\zeta_j^n = \zeta_j^n(\lambda) + \lambda\sum_i \zeta_i^n(\lambda)(\lambda + q_i)^{-1}G_{ij}$$

Since $\zeta^n \uparrow \zeta$ and $\zeta^n(\lambda) \uparrow \zeta(\lambda)$, (10) follows. And the proof is completed. QED

11.12 Q PROCESS ON A MEASURE SPACE

We will extend the concept of denumerable Markov processes defined on a probability space to that on a measure space $(\Omega, \mathscr{F}, P)$. Write T for $[0, +\infty)$ or $(0, +\infty)$. Suppose that E has the discrete topology. Let $\bar{E} = E \cup \{\infty\}$ $(\infty \notin E)$ be the one-point compactification of E.

Definition 1. Let $(\Omega, \mathscr{F}, P)$ be a measure space and σ be an $\mathscr{F}$-measurable function valued in $[0, +\infty)$. Assume that for every fixed $t \in T$ and almost every $\omega \in \{t < \sigma\}, x(t, \omega)$ is defined an takes a value in E. If for any $t \in T$ and $i \in E$, $P(x(t) = \infty) = 0$ and, moreover,

$$\Delta(t, i) \equiv (x(t) = i) = (\omega : x(t, \omega) = i, t < \sigma(\omega)) \in \mathscr{F} \tag{1}$$

we call $X = \{x(t),\ t \in T \cap [0, \sigma)\}$ a quasi-random process. If $P(\Delta(t, i)) < +\infty$ for every $t \in T$ and $i \in E$ we say that $X = \{x(t),\ t \in T \cap [0, \sigma)\}$ is a random process and that the minimal Borel field containing all $\Delta(t, i)$ $(t \in T,\ i \in E)$, denoted by $\mathscr{F}(X)$ or $\mathscr{F}\{X(t),\ t \in T \cap [0, \sigma)\}$, is the Borel field generated by X. When $T = (0, +\infty)$, a random process $X = \{x(t),\ 0 < t < \delta\}$ is said to be open.

Definition 2. Let $P(t) = (p_{ij}(t),\ i, j \in E)$, $t \geqslant 0$, be a standard generalized transition matrix and assume $X = \{x(t),\ t \in T \cap [0, \sigma)\}$ to be a random process. If for any $n \geqslant 1$, $t_0 \in T$, $t_0 < t_1 < \cdots < t_{n+1}$ and $i_0, i_1, \ldots, i_{n+1} \in E$,

$$P(x(t_0) = i_0, \ldots, x(t_{n+1}) = i_{n+1}) = P(x(t_0) = i_0, \ldots, x(t_n) = i_n) p_{i_n i_{n+1}}(t_{n+1} - t_n) \tag{2}$$

we call X a Markov process with the transition probability matrix $P(t)$. X is called a Q process if the Q matrix of $P(t)$ is the matrix Q. X is said to be a minimal process if $P(t)$ is the minimal solution.

Theorem 1. Let $X = \{x(t),\ t \in T \cap [0, \sigma)\}$ be a Markov process with the transition probability matrix $P(t)$. Suppose S is an $\mathscr{F}$-measurable function and given $\mathscr{F}(X)$, its conditions distribution is the exponential distribution with parameter 1. For $\lambda > 0$ let $p_{ij}^{\lambda}(t) = e^{-\lambda t} p_{ij}(t)$ and $\sigma(\lambda) = \min(\sigma, S/\lambda)$. Then $X^{\lambda} = \{x(t),\ t \in T \cap [0, \sigma(\lambda))\}$ is a Markov process with the transition probability matrix $P^{\lambda}(t) = (p_{ij}^{\lambda}(t),\ i, j \in E)$. In particular, if X is a minimal Q process, then X^{λ} is a minimal $Q(\lambda)$ process.

Proof. It is easy by making use of the conditional independence. QED

Lemma 2. Given a totally finite measure $v = (v_i, i \in E)$, there exist a totally finite measure space $(\Omega, \mathscr{F}, P)$, a Π-chain $(x, 0, \beta)$ and $\mathscr{F}$-measurable functions ρ_i^n $(n \geqslant 0, i \in E)$ defined on it, such that the initial distribution of $(x, 0, \beta)$ is v; given $\mathscr{F}(x, 0, \beta)$, all ρ_i^n $(n \geqslant 0,\ i \in E)$ are conditionally independent, and for each i the conditional distribution of ρ_i^n is the exponential distribution with the parameter q_i.

Noting the above, the characteristic measure of $(x, 0, \beta)$ is $\zeta = vG$ while vG is not necessarily finite. So Lemma 2 is not a direct corollary of Lemma 11.4. Since v is totally finite, it follows that with the aid of the ordinary existence theorem for Markov processes, there exists a Π-chain that is defined on a totally finite measure space and has the initial distribution v. From this, imitating the proof of Lemma 11.4, we can prove Lemma 2.

The following theorem follows immediately from the construction of minimal processes.

Theorem 3. Assume that v, $(x, 0, \beta)$ and ρ_i^n $(n \geqslant 0,\ i \in E)$ are the same as in Lemma 2. For $0 \leqslant n \leqslant \beta$,, let $\rho^n = \rho_{x(\tau_n)}^n$, $\tau_0 = 0$, $\tau_n = \sum_{i=0}^{n-1} \rho^i$ and $\sigma = \tau_{\beta+1}$. For $0 \leqslant t < \sigma$, let

$$X(t) = x(n) \qquad \text{if } \tau_n \leqslant t < \tau_{n+1} \tag{3}$$

Then $X = \{x(t),\ 0 \leqslant t \leqslant \sigma\}$ is a minimal Q process.

Similarly the following Lemma 4 and Theorem 5 are also clear.

Lemma 4. Assume that $v = (v_i, i \in E)$ is a totally finite measure. Then there exist a totally finite measure space $(\Omega, \mathscr{F}, P)$ and defined on it a $\Pi_\theta(\lambda)$-chain $(y, 0, \delta)$ and $\mathscr{F}$-measurable functions ρ^θ, ρ_i^n $(n \geqslant 0,\ i \in E)$ such that the initial distribution of $(y, 0, \delta)$ is γ. Given $\mathscr{F}(y, 0, \delta)$ all ρ^θ, ρ_i^n $(n \geqslant 0, i \in E)$ are conditionally independent and, furthermore, the conditional distributions are exponential distributions with parameters λ and $\lambda + q_i$ respectively.

Theorem 5. Assume that v, $(y, 0, \delta)$ and ρ^θ, $\rho_i^n (n \geqslant 0,\ i \in E)$ are the same as in Lemma 4. Let $\delta(E)$ be the first exit time of the $\Pi_\theta(\lambda)$-chain $(y, 0, \delta)$ from E. For $0 \leqslant n \leqslant \delta(E)$, let

$$\rho^n = \rho_{y(n)}^n \qquad \tau_0 = 0 \qquad \tau_{n+1} = \sum_{i=0}^{n} \rho^i$$

$$\sigma = \begin{cases} \sum_{i=0}^{+\infty} \rho^i & \text{if } \delta(E) = +\infty \\ \tau_{\delta(E)+1} & \text{if } \delta(E) < +\infty \end{cases}$$

$$\sigma_\theta = \begin{cases} \sum_{i=0}^{+\infty} \rho^i & \text{if } \delta(E) = +\infty \\ \tau_{\delta(E)+1} & \text{if } \delta(E) = \delta < +\infty \\ \tau_{\delta(E)+1} + \rho^\theta & \text{if } \delta(E) = \delta - 1 < +\infty \end{cases}$$

For $0 \leqslant t < \sigma_\theta$, let

$$X(t) = y(n) \qquad \text{if } \tau_n \leqslant t < \tau_{n+1}$$

Then $X=\{x(t), 0 \leqslant t<\sigma\}$ is a minimal $Q(\lambda)$ process and $X_\theta=\{x(t), 0 \leqslant t<\sigma_\theta\}$ is a minimal $Q_\theta(\lambda)$ process. Their initial distributions are all γ.

11.13 APPROXIMATING MINIMAL Q PROCESSES

Recall that E is endowed with the discrete topology and is compactified by the one point '∞'. Let finite sets $D_n \subset E$ and $D_n \uparrow E$ as $n \uparrow \infty$.

Definition 1. Assume that $0<\sigma \leqslant+\infty$. A function $X=\{x(t), 0 \leqslant t<\sigma\}$ defined on $[0, \sigma)$ is called a jump function of type U if the following conditions are satisfied:

(i) $x(0) \in E \cup\{\infty\}$, $x(t) \in E$ $(0<t<\sigma)$ and X is right-continuous;
(ii) for any $[c, d] \subset(0, \sigma)$, X has only a finite number of jump points in $[c, d]$;
(iii) for any $i \in E$ and any $d \in(0, \sigma)$, X has only a finite number of i intervals in $(0, d)$.

Definition 2. A random process $X=\{x(t), 0 \leqslant t<\sigma\}$ is called an approximating minimal Q process defined on a measure space $(\Omega, \mathscr{F}, P)$ if for any $t \geqslant 0$ and $i \in E$,

$$(x(t)=i) \in \mathscr{F} \qquad P(x(t)=i)<+\infty$$

and the following conditions are satisfied:

(i) Equation (12.2) holds for $0 \leqslant t_0<t_1<\cdots<t_{n+1}, i_0, i_1, \ldots, i_{n+1} \in E$, where $P(t)$ is a minimal solution $\bar{P}(t)$.
(ii) All paths of X are jump functions of type U.
(iii) There exists a sequence of $[0,+\infty]$-valued random times $\tau_n \downarrow 0$ such that $X_n=\{x(\tau_n+t),\ 0 \leqslant t<\sigma-\tau_n\}$ is a minimal Q process defined on $\Omega_n=(\tau_n<\sigma)$ and that the support of the initial distribution of X is contained in D_n.

Note that a minimal Q process is an approximating minimal Q process. In this case, τ_n may be taken as 0 or as the hitting time at D_n. When we consider the restriction on $\Omega'=(\sigma=0) \cup(\sigma>0, x(0) \neq \infty)$ of the approximating minimal Q process, we see it is a minimal Q process. In particular, if $P\{x(0)=\infty\}=0$ the approximating Q process X is just a Q process. If we still work on $(\Omega, \mathscr{F}, P)$ but restrict the parameter set to $(0, \infty)$, then $X'=\{x(t), 0<t<\sigma\}$ is an open minimal Q process. However, an approximating minimal Q process may not necessarily be a minimal Q process because for an approximating minimal Q process $X=\{x(t), 0 \leqslant t<\sigma\}$, the measure of the set $\{x(0)=\infty\}$ may be infinite although it is $\mathscr{F}$-measurable. But if $P(x(0)=\infty)=0$, then an approximating minimal Q process is precisely a minimal Q process.

Definition 3. Assume that $X=\{x(t), 0 \leqslant t<\sigma\}$ is an approximating minimal Q process. Let

$$\eta_j(\lambda)=\int_\Omega \int_0^\sigma \mathrm{e}^{-\lambda t} C_j[x(t)] \mathrm{d} P \qquad \lambda>0,\quad j \in E \tag{1}$$

$$\eta_j=\int_\Omega \int_0^\sigma C_j[x(t)] \mathrm{d} P \qquad j \in E \tag{2}$$

We call η the characteristic measure of X and $\eta(\lambda)$ its λ-characteristic measure. Theorem 14.3 will prove that $(\eta(\lambda), \lambda>0)$ is an entrance family (see Definition 2.11.1). So we call it the characteristic entrance family of X.

Theorem 1. Suppose τ_n are a sequence of random times which are given in the definition of the approximating minimal Q process $X=\{x(t), 0 \leqslant t<\sigma\}$. Suppose for each n, τ_n leads X to a minimal Q process $X_n=\{x_n(t), 0 \leqslant t<\sigma_n\}$ on $\Omega_n=(\tau_n<\sigma)$. Then on Ω for any $t \in[0, \sigma)$,

$$\lim_{n \rightarrow+\infty} X_n(t)=X(t) \qquad t \in[0, \sigma) \tag{3}$$

Proof. Since $\tau_n \downarrow 0$, $\sigma_n=\sigma-\tau_n \uparrow \sigma$. Then by the right-continuity of X, for $t \in[0, \sigma)$ we have

$$\lim_{n \rightarrow+\infty} X_n(t)=\lim_{n \rightarrow+\infty} X(\tau_n+t)=X(t) \qquad \text{QED}$$

11.14 ENTRANCE FAMILIES AND APPROXIMATING MINIMAL PROCESSES

Fix $\lambda>0$. Assume that a totally finite measure $\eta(\lambda)$ satisfies the inequality

$$\lambda u-u Q \geqslant 0 \tag{1}$$

Then $\zeta_j(\lambda)=\eta_j(\lambda)(\lambda+q_j)$ is a finite $\Pi(\lambda)$-excessive measure. According to Theorem 9.4, there exist totally finite measures $v^n(\lambda)$ such that:

(i) the support of $v^n(\lambda)$ is contained in D_n;
(ii) $v^n(\lambda) G(\lambda) \uparrow \zeta(\lambda)$ as $n \uparrow \infty$ and $(v^n(\lambda) G(\lambda))_j=\zeta_j$, $j \in D_n$;
(iii) for $m<n$, the hitting distribution that the $\Pi(\lambda)$-chain with the initial distribution $v^n(\lambda)$ hits D_m is $v^m(\lambda)$.

Note that (ii) is equivalent to

$$\begin{aligned} & v^n(\lambda) \phi(\lambda) \uparrow \eta(\lambda) \qquad n \uparrow+\infty \\ & (v^n(\lambda) \phi(\lambda))_j=\eta_j(\lambda) \qquad j \in D_n \end{aligned} \tag{2}$$

Theorem 1. Fix $\lambda>0$. Assume that the totally finite measure $\eta(\lambda)$ satisfies (1).

Then there exist a measure space $(\Omega, \mathscr{F}, P)$ and an approximating minimal $Q_\theta(\lambda)$ process $X = \{x(t), 0 \leqslant t < \sigma(\theta)\}$ defined on it, such that:

(i) if we let τ_n be the hitting time of X at D_n, then τ_n leads X to the minimal $Q_\theta(\lambda)$ process with the initial distribution $v^n(\lambda)$ for each n;

(ii)

$$\eta_j(\lambda) = \int_\Omega \int_0^{\sigma(\theta)} C_j[X(t)]\,\mathrm{d}t\,\mathrm{d}P \qquad j\in E \tag{3}$$

$$P(X(0) = \theta) = 0 \qquad P(x(0) = i) = v_i(\lambda) \tag{4}$$

where

$$v(\lambda) = \lambda\eta(\lambda)\eta(\lambda)Q \tag{5}$$

Particularly, if $\eta(\lambda)$ is the solution of the equation

$$\lambda u - uQ = 0 \tag{6}$$

then $x(0) = \infty$ almost surely on Ω.

Proof. Because $\zeta_j(\lambda) = \eta_j(\lambda)(\lambda + q_j)$ $(j\in E)$ is a $\Pi(\lambda)$-excessive measure, on the basis of Lemma 11.4 there exists a measure space $(\Omega, \mathscr{F}, P)$ on which an approximating $\Pi_\theta(\lambda)$-chain (y, α, δ) can be defined such that

$$P(-\infty < \alpha, y(\alpha) = \theta) = 0 \qquad P(-\infty < \alpha, y(\alpha) = i) = v_i(\lambda)$$

and on $(-\infty = \alpha)$ almost surely

$$\lim_{n\to-\infty} y(n) = \infty$$

and

$$\zeta_j(\lambda) = \int_\Omega \sum_{\alpha\leqslant n\leqslant\delta} C_j[y(n)]\,\mathrm{d}P \qquad j\in E \tag{7}$$

and the hitting distribution that the chain (y, α, δ) hits D_n is $v^n(\lambda)$. On $(\Omega, \mathscr{F}, P)$ a family of $\mathscr{F}$-measurable functions ρ_θ and ρ_i^n $(n \geqslant 0,\ i\in E)$ can also be defined such that given $\mathscr{F}(y, \alpha, \delta)$, all ρ_θ, ρ_i^n $(n\geqslant 0,\ i\in E)$ are conditionally independent and the conditional distributions of ρ_θ and ρ_i^n are exponential distributions with parameters λ and $\lambda + q_i$ respectively. Let $\delta(E)$ be the first exit time of the chain (y, α, δ) from E. Then $(-\infty < \delta(E)) = (\alpha \leqslant \delta(E)) = \Omega$. If $\alpha \leqslant n \leqslant \delta(E)$, we let $\rho^n = \rho^n_{y(n)}$; and if $\delta(E) = \delta - 1 < +\infty$, we let $\rho^\delta = \rho_\theta$. Then we proceed to prove

$$\int_\Omega \sum_{\alpha\leqslant n\leqslant\delta} \rho^n\,\mathrm{d}P < +\infty \tag{8}$$

In fact

$$\int_\Omega \sum_{\alpha\leqslant n\leqslant\delta} \rho^n\,\mathrm{d}P = \sum_{n\in N} \int_{\alpha\leqslant n\leqslant\delta(E)} \rho^n_{y(n)}\,\mathrm{d}P + \int_{\delta(E)=\delta-1<+\infty} \rho^\delta\,\mathrm{d}P$$

By conditional independence the first part of the above formula is equal to

$$\sum_{n\in N}\sum_{i\in E}\int_{y(n)=i} \rho_i^n\,\mathrm{d}P = \sum_{n\in N}\sum_{i\in E} P(y(n)=i)(\lambda+q_i)^{-1} = \sum_{i\in E}\sum_{n\in N} P(y(n)=i)(\lambda+q_i)^{-1}$$
$$= \sum_{i\in E}\zeta_i(\lambda)(\lambda+q_i)^{-1} = \sum_{i\in E}\eta_i(\lambda) < +\infty$$

By a deduction similar to (27) the second part is equal to

$$\int_{\substack{\delta<+\infty\\ y(\delta)=\theta}} \rho_\theta\,\mathrm{d}P = P(\delta<+\infty, y(\delta)=\theta)\lambda^{-1}$$
$$= \lambda^{-1}\sum_{i\in E} P(\delta(E)<+\infty, y(\delta(E))=i, y(\delta(E)+1)=\theta)$$
$$= \lambda^{-1}\sum_{i\in E} P(\delta(E)<+\infty, y(\delta(E))=i)\Pi_{i\theta}(\lambda)$$
$$= \lambda^{-1}\sum_{i\in E} \lambda\zeta_i(\lambda)(\lambda+q_i)^{-1}\Pi_{i\theta}(\lambda)$$
$$= \sum_{i\in E}\eta_i(\lambda)\Pi_{i\theta}(\lambda) \leqslant \sum_{i\in E}\eta_i(\lambda) < +\infty$$

Now we define $X = \{x(t), 0\leqslant t<\sigma(\theta)\}$. Let

$$\tau_\alpha = 0 \qquad \tau_n = \sum_{\alpha\leqslant k<n}\rho^k \quad (\alpha<n<\delta) \qquad \sigma(\theta) = \tau_{\delta+1} \tag{9}$$

For $0\leqslant t<\sigma(\theta)$ let

$$\begin{aligned} x(t) &= y(n) && \text{if } \tau_n \leqslant t < \tau_{n+1} \\ x(0) &= \infty && \text{if } -\infty = \alpha \end{aligned} \tag{10}$$

We shall prove that $X = \{x(t), 0\leqslant t<\sigma(\theta)\}$ is what we want.

First, (4) is clear. For each path of X, (i)–(ii) in Definition 1 hold (substitute E_θ for E). From (7) and $\zeta_j(\lambda) < +\infty$ we have almost surely $\sum_{\alpha\leqslant n<\delta} C_j[y(n)] < +\infty$. Hence condition (iii) in Definition 1 also holds. Therefore the paths of X are all jump functions of type U.

Next, for $j\in E$, (3) follows from

$$\int_\Omega\int_0^{\delta(\theta)} C_j[x(t)]\,\mathrm{d}t\,\mathrm{d}P = \int_\Omega \sum_{\alpha\leqslant n\leqslant\delta} C_j[y(n)]\rho^n\,\mathrm{d}P$$
$$= \int_\Omega \sum_{\alpha\leqslant n\leqslant\delta} C_j[y(n)]\rho_j^n\,\mathrm{d}P$$
$$= \sum_{n\in E}\int_{y(n)=j}\rho_j^n\,\mathrm{d}P = \sum_{n\in N} P(y(n)=j)(\lambda+q_j)^{-1}$$
$$= \zeta_j(\lambda)(\lambda+q_j)^{-1} = \eta_j(\lambda)$$

Moreover, for each n let α_n be the hitting time of the approximating $\Pi_\theta(\lambda)$- chain (y, α, δ) at D_n. Then the hitting time of X at D_n is

$$b_n = \begin{cases} \tau_{\alpha_n} & \text{if } \alpha_n < +\infty \\ +\infty & \text{if } \alpha_n = +\infty \end{cases}$$

Suppose for each n, α_n leads (y, α, δ) to the $\Pi_\theta(\lambda)$-chain $(y_n, 0, \delta_n)$ on $\Omega_n = (\alpha_n \leqslant \delta)$. On Ω_n let $\sigma_n = \sigma(\theta) - b_n$ and $x_n(t) = x(b_n + t)$ for $0 \leqslant t < \sigma_n$. Then $X_n = \{x_n(t), 0 \leqslant t < \sigma_n\}$ is a Markov chain defined on Ω_n with the initial distribution $v^n(\lambda)$. As α_n is a random time of the chain (y, α, δ) which is independent of the future, given $\mathscr{F}(y_n, 0, \delta_n)$, ρ^θ, $\bar{\rho}_i^m \equiv \rho_i^{\alpha_n + m}$ $(m \geqslant 0, i \in E)$ are conditionally independent and given $\mathscr{F}(y_n, 0, \delta_n)$ the conditional distributions of ρ^θ and $\bar{\rho}_i^m$ are exponential distributions with parameters λ and $\lambda + q_i$ respectively. In the fashion of Theorem 12.5, from $(y_n, 0, \delta_n)$ and $\rho^\theta, \bar{\rho}_i^m$ $(m \geqslant 0, i \in E)$ we can obtain a minimal $Q_\theta(\lambda)$ process that has the same initial distribution as $(y_n, 0, \delta_n)$. This minimal $Q_\theta(\lambda)$ process is none other than X_n. But $(\alpha_n < +\infty) = (b_n < \sigma(\theta))$, hence for each n, X_n is a $Q_\theta(\lambda)$ process defined on $\Omega_n = (b_n < \sigma(\theta))$ and the support of the initial distribution of X_n is contained in D_n.

Finally we are going to prove that (12.2) holds for $0 \leqslant t_\theta < t_1 < \cdots < t_{n+1}, i_0, i_1, \ldots, i_{n+1} \in E$ and the minimal solution $\bar{P}^\lambda(t) = (\bar{p}_{ij}^\lambda(t), i, j \in E_\theta)$ to the Q matrix $Q_\theta(\lambda)$.

In fact, $\alpha_n \downarrow \alpha$ since $y(\alpha) \in E$ on $(-\infty < \alpha)$. Hence $\Omega_n = (\alpha_n < +\infty) \uparrow \Omega$. Since on Ω, $\sigma(\theta) < +\infty$, it follows that $b_n = \tau_n \downarrow 0$ and so $\sigma(\theta) - b_n \uparrow \sigma(\theta)$. Therefore for all $t \in [0, \sigma(\theta))$, $x_n(t) = x(b_n + t) \to x(t)$. Hence

$$(x(t_0) = i_0, \ldots, x(t_{m+1}) = j_{m+1}) = \lim_{n \to +\infty} \Omega_n \cap (X_n(t_0) = i_0, \ldots, X_n(t_{m+1}) = i_{m+1})$$

so that

$$\begin{aligned} P(X(t_0) &= i_0, \ldots, X(t_{m+1}) = i_{m+1}) \\ &= \lim_{n \to +\infty} P(\Omega_n, X_n(t_0) = i_0, \ldots, X_n(t_{m+1}) = i_{m+1}) \\ &= \lim_{n \to +\infty} P(\Omega_n, X_n(t_0) = i_0) \bar{p}_{i_0 i_1}^\lambda(t_1 - t_0) \cdots \bar{P}_{i_m i_{m+1}}^\lambda(t_{m+1} - t_m) \\ &= P(X(t_0) = i) \bar{P}_{i_0 i_1}^\lambda(t_1 - t_0) \cdots \bar{P}_{i_m i_{m+1}}^\lambda(t_{m+1} - t_m) \end{aligned}$$

Thus, $X = \{x(t), 0 \leqslant t < \sigma(\theta)\}$ is an approximating minimal $Q_\theta(\lambda)$ process. The proof is terminated. QED

The following theorem is obvious.

Theorem 2. Fix $\lambda > 0$. Assume that $X = \{x(t), 0 \leqslant t < \sigma(\theta)\}$ is an approximating minimal $Q_\theta(\lambda)$ process. Define $\eta_j(\lambda)$ $(j \in E)$ according to (3). Let $\sigma(E)$ be the last exit time of X from E, that is,

$$\sigma(E) = \begin{cases} \sup\{t: 0 \leqslant t < \sigma(\theta), x(t) \in E\} & \\ -\infty & \text{if the above set is empty} \end{cases} \tag{11}$$

Then $X_E = \{X(t), 0 \leqslant t < \sigma(E)\}$ is an approximating minimal $Q(\lambda)$ process which is defined on $\Omega(E) = (-\infty < \sigma(E))$ and has the λ-characteristic measure $\eta(\lambda)$.

Theorem 3. Assume that $(\eta(\lambda), \lambda > 0)$ is an entrance family and the number $\mu > 0$ is fixed. Then there exists a measure space $(\Omega, \mathscr{F}, P)$ on which are defined an approximating minimal $Q(\mu)$ process $Y = \{y(t), 0 \leqslant t < \sigma(E)\}$ and an approximating minimal Q process $X = \{x(t), 0 \leqslant t < \sigma\}$, such that:

(i) X is an extension of Y, i.e. $\sigma(E) \leqslant \sigma$ and $y(t) = x(t)$ for $t \in [0, \sigma(E))$;

(ii) $P\{y(0) = i\} = P\{x(0) = i\} \equiv v_i$, $i \in E$; in particular, if $v = 0$, then $y(0) = x(0) = \infty$ almost surely;

(iii) for each n, let τ_n be the hitting time of Y at D_n, then $\tau_n \downarrow 0$ $(n \uparrow +\infty)$ and τ_n leads Y and X respectively to a minimal $Q(\mu)$ process Y_n and to a minimal Q process X_n on $\Omega_n = (\tau_n < +\infty)$, which have the same initial distribution $v^n(\lambda)$;

(iv) the characteristic measure of Y is $\eta(\mu)$;

(v) the characteristic entrance family of X is $(\eta(\lambda), \lambda > 0)$;

(vi) the characteristic measure of X is the standard image of $(\eta(\lambda), \lambda > 0)$ (see (2.11.9)).

Proof. From Theorem 1, there exists a measure space $(\Omega, \mathscr{F}, P)$ on which can be defined an approximating $Q_\theta(\mu)$ process $Y_\theta = \{y(t), 0 \leqslant t < \sigma(\theta)\}$ such that $P\{y(0) = \theta\} = 0$, $P\{y(0) = i\} = v_i$ and, moreover,

$$\eta_j(\mu) = \int_\Omega \int_0^{\sigma(\theta)} C_j[y(t)] \, dt \, dP \qquad j \in E \tag{12}$$

By the technique of independent product spaces, we can also suppose that on $(\Omega, \mathscr{F}, P)$ a family of quasi-random processes $Z_i = \{z_i(t), 0 \leqslant t < \varepsilon_i\}$ $(i \in E)$ are defined such that given $\mathscr{F}(Y_\theta)$, all Z_i, $i \in E$, are conditionally independent and for each i the conditional distribution of Z_i is the distribution of the minimal Q process starting from i.

Let $\sigma(E)$ be the last exit time of Y_θ from E. Then, clearly $Y = \{y(t), 0 \leqslant t < \sigma(E)\}$ is an approximating minimal $Q(\lambda)$ process and the characteristic measure of Y is $\eta(\mu)$, i.e. (iv) holds.

Let

$$\sigma = \begin{cases} \sigma(E) & \text{if } \sigma(E) < \sigma(\theta) \text{ or } \sigma(E) = +\infty \\ \sigma(E) + \varepsilon_{y(\sigma(E)-0)} & \text{if } \sigma(E) = \sigma(\theta) < +\infty \end{cases} \tag{13}$$

For $0 \leqslant t < \sigma$, let

$$X(t) = \begin{cases} y(t) & \text{if } 0 \leqslant t < \sigma(E) \\ Z_{y(\sigma(E)-0)}(t - \sigma(E)) & \text{if } \sigma(E) = \sigma(\theta) < +\infty \end{cases} \tag{14}$$

$$\sigma(E) \leqslant t < \sigma.$$

Then X is an extension of Y, i.e. (i) holds and so does (ii). Suppose the characteristic measure of X is η. Then

$$\begin{aligned} \eta_j &= \int_\Omega \int_0^\sigma C_j[x(t)]\,\mathrm{d}t\,\mathrm{d}p \\ &= \int_\Omega \int_0^{\sigma(E)} C_j[y(t)]\,\mathrm{d}t\,\mathrm{d}p + \int_{\sigma(E)=\sigma(\theta)<+\infty} \int_{\sigma(E)}^{\sigma} C_j[Z_{y(\sigma(E)-0)}(t - \sigma(E))\,\mathrm{d}t\,\mathrm{d}P \\ &= \eta_j(\mu) + \sum_{i \in E} \int_{\substack{\sigma(E)=\sigma(\theta)<+\infty \\ y(\sigma(E))=i}} \int_0^{\varepsilon_i} C_j[Z_i(u)]\,\mathrm{d}u\,\mathrm{d}p \end{aligned}$$

By conditional independence and (11.5) the above is equal to

$$\begin{aligned} &\eta_j(\mu) + \sum_{i \in E} P(y(\sigma(E)) = i, \sigma(E) = \sigma(\theta) < +\infty)\Gamma_{ij} \\ &= \eta_j(\mu) + \sum_{i \in E} P(y(\sigma(\theta)) = i)\Gamma_{ij} \\ &= \eta_j(\mu) + \mu \sum_{i \in E} \eta_i(\mu)\Gamma_{ij} \end{aligned}$$

This is just the standard image (2.11.9) of $(\eta(\lambda), \lambda > 0)$ and so (vi) is verified. We now prove claim (iii). For each n, let τ_n be the hitting time of Y at D_n. Then $\tau_n \downarrow 0$. Suppose τ_n leads Y_θ, Y and X respectively to $Y_{n\theta}$, Y_n and X_n on $\Omega_n = (\tau_n < +\infty)$. By Theorem 1, Y_n is a minimal $Q(\mu)$ process with the initial distribution $v^n(\mu)$ for each n. Clearly, X_n is an extension of Y_n and, furthermore, they have the same initial distribution. Hence if we can prove that X_n $(n \geqslant 1)$ are minmal Q processes, then X is an approximating minimal Q process.

Just as we obtain X from Y_θ and Z_i $(i \in E)$ in the fashion (13)–(14), on Ω_n we can derive X_n from $Y_{n\theta}$ and Z_i $(i \in E)$ in the same fashion. Hence we need only prove that if Y_θ is a minimal $Q_\theta(\lambda)$ process, then the process X obtained in the fashion of (13)–(14) is also a minimal Q process. Notice that under the above hypothesis y is a minimal $Q(\lambda)$ process.

For $i_0, i_1, \ldots, i_{n+1} \in E$ and $0 \leqslant t_0 < t_1 < \cdots < t_{n+1}$, assume that

$$\Delta_k = \bigcap_{j=0}^{k} (x(t_j) = i_j) \qquad \Lambda_k = \bigcap_{j=0}^{k} (y(t_j) = i_j)$$

We are to prove

$$R_k(i, u) = \sum_{j=k}^{n+1} (Z_i(t_j - u) = i_j, 0 \leqslant t_j - u \leqslant \varepsilon_i)$$

$$P(\Delta_{n+1}) = P(\Delta_n)\bar{P}_{i_n i_{n+1}}(t_{n+1} - t_n) \tag{15}$$

Clearly

$$P(\Delta_{n+1}) = A_{n+1} + B_{n+1} + C_{n+1} \tag{16}$$

where

$$\begin{aligned} A_{n+1} &= P(\Delta_{n+1}, t_{n+1} < \sigma(E)) \\ B_{n+1} &= P(\Delta_{n+1}, \sigma(E) \leqslant t_0) \\ C_{n+1} &= \sum_{k=0}^{n} P(\Delta_{n+1}, t_k < \sigma(E) \leqslant t_{k+1}) \end{aligned}$$

It follows that

$$\begin{aligned} (\Delta_{n+1}, t_{n+1} < \sigma(E)) &= (\Lambda_{n+1}, t_{n+1} < \sigma(E)) = \Lambda_{n+1} \\ A_{n+1} &= P(\Lambda_{n+1}) = P(\Lambda_n)\bar{P}^{\lambda}_{i_n i_{n+1}}(t_{n+1} - t_n) \\ &= A_n \bar{P}_{i_n i_{n+1}}(t_{n+1} - t_n)\exp[-\lambda(t_{n+1} - t_n)] \end{aligned} \tag{17}$$

By conditional independence

$$\begin{aligned} B_{n+1} &= \sum_{i \in E} P(y(\sigma(E) - 0) = i, R_0(i, \sigma(E))) \\ &= \sum_i \int_0^{t_0} \bar{P}_{i_0 i_1}(t_1 - t_0) \cdots \bar{P}_{i_n i_{n+1}}(t_{n+1} - t_n)\,\mathrm{d}u\,P(y(\sigma(E) - 0) \\ &= i, \sigma(E) \leqslant u, Z_i(t_0 - \sigma(E)) = i_0) \\ &= \sum_i \int_0^{t_0} \bar{P}_{i_0 i_1}(t_1 - t_0) \cdots \bar{P}_{i_{n-1} i_n}(t_n - t_{n-1})\,\mathrm{d}u\,P(y(\sigma(E) - 0) \\ &= i, \sigma(E) \leqslant u, Z_i(t_0 - \sigma(E)) = i_0)\bar{P}_{i_n i_{n+1}}(t_{n+1} - t_n) \\ &= B_n \bar{P}_{i_n i_{n+1}}(t_{n+1} - t_n) \end{aligned}$$

By a similar calculation we have

$$P(\Delta_{n+1}, t_k < \sigma(E) \leqslant t_{k+1}) = P(\Delta_n, t_k < \sigma(E) \leqslant t_{k+1})\bar{P}_{i_n i_{n+1}}(t_{n+1} - t_n)$$

so that

$$C_{n+1} = C_n \bar{P}_{i_n i_{n+1}}(t_{n+1} - t_n) + P(\Delta_{n+1}, t_n < \sigma(E) \leqslant t_{n+1})$$

Therefore, to prove (15) it suffices to verify

$$P(\Delta_{n+1}, t_n < \sigma(E) \leqslant t_{n+1}) = A_n \bar{P}_{i_n i_{n+1}}(t_{n+1} - t_n)\{1 - \exp[-\lambda(t_{n+1} - t_n)]\} \tag{18}$$

Note that

$$L_{ki} \equiv P(y(\sigma(E)-0)=i \mid y(0)=k)$$
$$= \sum_{m=0}^{+\infty} \Pi_{ki}^{(m)}(\lambda)\left(1-\sum_{j\in E_\theta}\Pi_{ij}(\lambda)\right)$$
$$= G_{ki}(\lambda)(\lambda+q_i)^{-1}\lambda = \lambda\phi_{ki}(\lambda) = \lambda\int_0^{+\infty} e^{-\lambda t}\bar{P}_{ki}(t)\,dt$$

and

$$L_{ki}(u) \equiv P(y(\sigma(E)-0)=i, \sigma(E)\leqslant u \mid y(0)=k)$$
$$= L_{ki} - \sum_{l\in E} P(y(u)=l, u<\sigma(E), y(\sigma(E)-0)=i \mid y(0)=k)$$
$$= L_{ki} - \sum_{l\in E}\bar{P}_{kl}^{\lambda}(u)L_{li}$$

But

$$\sum_{l\in E}\bar{P}_{kl}^{\lambda}(u)L_{li} = \sum_{l\in E} e^{-\lambda t}\bar{P}_{kl}(u)\lambda\int_0^{+\infty} e^{-\lambda t}\bar{P}_{li}(t)\,dt$$
$$= \lambda\int_0^{+\infty} e^{-\lambda(u+t)}\bar{P}_{ki}(u+t)\,dt$$
$$= \lambda\int_u^{+\infty} e^{-\lambda t}\bar{P}_{ki}(t)\,dt$$

so that

$$L_{ki}(u) = \lambda\int_0^{u} e^{-\lambda t}\bar{P}_{ki}(t)\,dt$$
$$dL_{ki}(u) = \lambda e^{-\lambda u}\bar{P}_{ki}(u)\,du$$

Abbreviate $\delta = t_{n+1}-t_n$. By conditional independence,

$$P(\Delta_{n+1}, t_n<\sigma(E)\leqslant t_{n+1})$$
$$= \sum_i P(\Delta_{n+1}, y(\sigma(E)-0)=i, t_n<\sigma(E)\leqslant t_{n+1})$$
$$= \sum_i P(\Lambda_n, y(\sigma(E)-0)=i, Z_i(t_{n+1}-\sigma(E))=i_{n+1}, t_n<\sigma(E)\leqslant t_{n+1})$$
$$= \sum_i \int_{t_n}^{t_{n+1}} \bar{P}_{ii_{n+1}}(t_{n+1}-u)\,du\,P(\Lambda_1, y(\sigma(E)-0)=i, \sigma(E)\leqslant u)$$
$$= \sum_i \int_{t_n}^{t_{n+1}} \bar{P}_{ii_{n+1}}(t_{n+1}-u)\,du\,P(\Lambda_n)L_{i_ni}(u-t_n)$$

$$= A_n\sum_i\int_0^{\delta}\bar{P}_{ii_{n+1}}(\delta-u)\,du\,L_{i_ii}(u)$$
$$= A_n\sum_i\int_0^{\delta}\bar{P}_{ii_{n+1}}(\delta-u)\lambda e^{-\lambda u}\bar{P}_{i_ni}(u)\,du$$
$$= A_n\sum_i\int_0^{\delta}\lambda e^{-\lambda u}\bar{P}_{i_ni_{n+1}}(\delta)\,du$$
$$= A_n\bar{P}_{i_ni_{n+1}}(\delta)(1-e^{-\lambda\delta})$$

Hence, (18) is proved

We now prove claim (v). To begin with, suppose X is a minimal Q process with the initial distribution v. Then

$$\int_\Omega\int_0^{\sigma} e^{-\lambda t}C_j[x(t)]\,dt\,dp = \int_0^{+\infty} e^{-\lambda t}\int_{t<\sigma} C_j[x(t)]\,dP\,dt$$
$$= \int_0^{+\infty} e^{-\lambda t}\sum_i v_i\bar{P}_{ij}(t)\,dt$$
$$= \sum_i v_i\phi_{ij}(\lambda) = [v\phi(\lambda)]_j$$

Now suppose X is an approximating minimal Q process. Then

$$\int_\Omega\int_0^{\sigma} e^{-\lambda t}C_j[x(t)]\,dt\,dp = \lim_{n\to+\infty}\int_{\Omega_n}\int_0^{\sigma_n} e^{-\lambda t}C_j[x_n(t)]\,dt\,dp$$
$$= \lim_{n\to+\infty}[v^n(\mu)\phi(\lambda)]$$

But by the resolvent equation for $\phi(\lambda)$,

$$\phi(\lambda) = \phi(\mu) + (\mu-\lambda)\phi(\mu)\phi(\lambda)$$

and by (2), we have

$$v^n(\mu)\phi(\lambda) = v^n(\mu)\phi(\mu) + (\mu-\lambda)v^n(\mu)\phi(\mu)\phi(\lambda) \to \eta(\mu) + (\mu-\lambda)\eta(\mu)\phi(\lambda) = \eta(\lambda)$$

Therefore (v) is proved. Thus we conclude the proof. QED

11.15 APPROXIMATING MINIMAL Q PROCESSES ON A TOTALLY FINITE MEASURE SPACE

Recall Definition 8.5.1. Let $(\eta(\lambda), \lambda>0)$ be an entrance family. If

$$\lim_{\lambda\to+\infty}\lambda\eta(\lambda)\mathbf{1} = M < +\infty \tag{1}$$

where $\eta(\lambda)1 = \sum_j \eta_j(\lambda)$, we call $(\eta(\lambda), \lambda > 0)$ a non-sticky entrance family. When $M \leqslant 1$ the entrance family is said to be probabilistic.

Theorem 1. Assume that $X = \{x(t), 0 \leqslant t < \sigma\}$ is an approximating minimal Q process defined on the measure space $(\Omega, \mathscr{F}, P)$ and for each $\lambda > 0$ the λ-characteristic measure of X is $\eta(\lambda)$. Then $(\eta(\lambda), \lambda > 0)$ is an entrance family.

Proof. Imitate the proof of Theorem 2.4.3(v). QED

Theorem 2. Assume that $X = \{x(t), 0 \leqslant t < \sigma\}$ is an approximating minimal Q-process which is defined on the totally finite measure space $(\Omega, \mathscr{F}, P)$ and has the characteristic entrance family $(\eta(\lambda), \lambda > 0)$. Then $(\eta(\lambda), \lambda > 0)$ is a non-sticky entrance family. If $(\Omega, \mathscr{F}, P)$ is a probability space, then $(\eta(\lambda), \lambda > 0)$ is also a probabilistic entrance family.

Proof.

$$\lambda\eta(\lambda)1 = \int_\Omega \int_0^\sigma \lambda e^{-\lambda t} \mathrm{d}t\, \mathrm{d}P$$

$$= \int_{\sigma > 0} (1 - e^{-\lambda\sigma}) \mathrm{d}p \uparrow p(\sigma > 0) < +\infty \qquad \lambda \uparrow \infty \tag{2}$$

Theorem 3. Assume that $(\eta(\lambda), \lambda > 0)$ is a non-sticky entrance family. Then there exists an approximating minimal Q process $X = \{x(t), 0 \leqslant t < \sigma\}$ defined on a finite measure space $(\Omega, \mathscr{F}, P)$ and its characteristic entrance family is just $(\eta(\lambda), \lambda > 0)$.

Proof. From Theorem 14.3, there exist a measure space $(\Omega, \mathscr{F}, P)$ and defined on it an approximating minimal Q process X, the characteristic measure of which is precisely $(\eta(\lambda), \lambda > 0)$. By the non-stickiness and (2) we have $P\{\sigma > 0\} < \infty$. removing $(\sigma = 0)$, we can assume that $\Omega = (\sigma > 0)$. Therefore $(\Omega, \mathscr{F}, P)$ is a totally finite measure space. QED

Theorem 4. Assume that $(\eta(\lambda), \lambda > 0)$ is an entrance family (correspondingly, probabilistic entrance family). Then there exist a measure space (correspondingly, probability space) $(\Omega, \mathscr{F}, P)$ and defined on it an approximating minimal Q process $X = \{x(t), 0 \leqslant t < \sigma\}$, such that

$$P(\sigma > 0) = \lim_{\lambda \to \infty} \lambda\eta(\lambda)1 \tag{3}$$

$$\eta_j(\lambda) = \int_\sigma \int_0^\Omega e^{-\lambda t} C_j[x(t)] \mathrm{d}t\, \mathrm{d}P$$

$$= \int_0^{+\infty} e^{-\lambda t} H_j(t) \mathrm{d}t \qquad \lambda > 0 \tag{4}$$

where

$$H_j(t) = P(x(t) = j) \qquad t \geqslant 0 \tag{5}$$

is the inverse Laplace transform of $\eta_j(\lambda), \lambda \geqslant 0$. Suppose the Riesz decomposition of $(\eta(\lambda), \lambda > 0)$ is

$$\eta(\lambda) = \alpha\phi(\lambda) + \bar{\eta}(\lambda) \tag{6}$$

Then

$$\alpha_i = P(x(0) = i) \qquad P(x(0) = \infty) = \lim_{\lambda \to +\infty} \lambda\bar{\eta}(\lambda)\mathbf{1} \tag{7}$$

Proof. We need only to prove the second equality of (7). In fact,

$$P(x(0) = \infty) = 1 - \sum_i P(x(0) = i)$$

$$= \lim_{\lambda \to +\infty} \lambda\eta(\lambda)\mathbf{1} - \lim_{\lambda \to +\infty} \lambda v\phi(\lambda)\mathbf{1}$$

$$= \lim_{\lambda \to +\infty} \lambda\bar{\eta}(\lambda)\mathbf{1} \tag{8}$$

11.16 PATH STRUCTURE OF NON-STICKY RETURN PROCESSES: DV-TYPE AND (DV)*-TYPE EXTENSIONS

In this section we assume that the minimal solution $(\bar{p}_{ij}(t))$ determined by a Q matrix Q is stopping. Let $\bar{X} = \{\bar{x}(t), t < \bar{\sigma}\}$ be a minimal Q process defined on a probability space $(\Omega, \mathscr{F}, P)$. Then

$$L_i(t) = P_i(\bar{\sigma} \leqslant t) = 1 - \sum_i \bar{P}_{ij}(t) \tag{1}$$

$$R_i(t) = P_i(\bar{\sigma} \leqslant t,\ \bar{x}(\bar{\sigma} - 0) = \infty) \tag{2}$$

are determined uniquely by Q, where $P_i(\cdot) = P\{\cdot \mid \bar{x}(0) = i\}$.

In the following we always assume that $(\eta(\lambda), \lambda > 0)$ is a given probabilistic entrance family and $H_i(t)$ is the inverse Laplace transform of $(\eta_i(\lambda), \lambda > 0)$. Let

$$H(t) = \sum_{i \in E} H_i(t) \qquad H^0(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geqslant 0 \end{cases}$$

$$H^{n+1} = H^n * H$$

$$M_i = \sum_{n=0}^{+\infty} (L_i * H^n) \tag{3}$$

$$P_{ij}(t) = \bar{P}_{ij}(t) + \int_0^t H_j(t - s) \mathrm{d}M_i(s) \tag{4}$$

Here $*$ represents convolution.

Lemma 1. There exists a probability space $(\Omega, \mathscr{F}, P)$ on which can be defined a minimal Q process $X^0 = \{x^0(t), 0 \leqslant t < \sigma^0\}$ and a sequence of approximating minimal Q processes $X^n = \{x^n(t), 0 \leqslant t < \sigma^n\}$, every X^n $(n \geqslant 1)$ having the same characteristic entrance family $(\eta(\lambda), \lambda > 0)$. They have the following properties:

(i) $(\sigma_n = 0) \cup (\sigma_n = +\infty) \subset (\sigma_{n+1} = 0)$.

(ii) Write $\Delta_n = (0 < \sigma_n < +\infty)$, $\Omega_n = (\sigma_n > 0)$, $P(\Omega_{n+1} | \Delta_n) = M$, where M is determined by (15.1).

(iii) Given Δ_n or Ω_{n+1}, all X^m $(m \leqslant n)$ and X^m $(m \geqslant n+1)$ are conditionally independent.

Proof. Imitate the proof of Lemma 2.1.

Theorem 2. For X^0 and X^n $(n \geqslant 1)$ in Lemma 1, let

$$\tau^0 = 0 \qquad \tau^{n+1} = \sum_{m=0}^{n} \sigma^m \qquad \sigma = \sum_{m=0}^{+\infty} \sigma^m \tag{5}$$

For $0 \leqslant t < \sigma$ let

$$X(t) = X^n(t - \tau^n) \qquad \text{if } \tau^n \leqslant t < \tau^{n+1} \tag{6}$$

Then $X = \{x(t), 0 \leqslant t < \sigma\}$ is a Markov process and its transition probability is given by (3).

Proof. Imitate the proof of Theorem 11.2.2.

Remark 1

Taking Laplace transforms on both sides of (4) we obtain

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + Z_i(\lambda) \frac{\eta_j(\lambda)}{(1 - M) + \lambda\eta(\lambda)\mathbf{1}} \tag{7}$$

where

$$Z(\lambda) = 1 - \lambda\phi(\lambda)\mathbf{1}$$

Remark 2

When Q is conservative or $(\eta(\lambda), \lambda > 0)$ is a harmonic entrance family, the process X in Theorem 2 is a Q process. But in general it may not necessarily be so.

Remark 3

The Markov process X in Theorem 2 is called a DV-type extension of the minimal Q process X^0. Suppose the Riesz decomposition of the entrance family

$(\eta(\lambda), \lambda > 0)$ is (15.6). When $\bar{\eta}(\lambda) = 0$, the DV-type extension of X^0 becomes the D-type extension in section 11.2. When $\alpha = 0$, we call the DV-type extension the V-type extension. In this case, the leaping intervals of X, except the first one, are all ${}_xU$ intervals.

Lemma 3. Assume that for some $i \in E$ and $t > 0$, $R_i(t)$ is given by (2), and $R_i(t) > 0$. Then there exist a probability space $(\Omega, \mathscr{F}, P)$ and defined on it a minimal Q process $X^0 = \{x^0(t), 0 \leqslant t < \sigma^0\}$ and a sequence of approximating minimal Q processes $X^n = \{x^n(t), 0 \leqslant t < \sigma^n\}, n \geqslant 1$, every X^n $(n \geqslant 1)$ having the same characteristic entrance family $(\eta(\lambda), \lambda > 0)$. They satisfy the following conditions:

(i) $(\sigma^n = 0) \cup (\sigma^n = +\infty) \cup (X^n(\sigma^n - 0) \in E) \subset (\sigma^{n+1} = 0), n \geqslant 0$[1].

(ii) Put $\Delta_n = (0 < \sigma^n < +\infty, X^n(\sigma^n - 0) = \infty), \Omega_n = (\sigma_n > 0)$. Then

$$P(\Omega_{n+1} | \Delta_n) = M$$

where M is determined by (15.1).

(iii) Given Δ_n or Ω_{n+1}, all X^m $(m \leqslant n)$ and X^m $(m \geqslant n+1)$ are conditionally independent.

Proof. Follow the proof of Lemma 2.1. QED

Theorem 4. For processes X^0 and X^n in Lemma 3, define $X = \{x(t), 0 \leqslant t < \sigma\}$ according to (5)–(6). Then X is a Q process with transition probabilities

$$p_{ij}(t) = \bar{p}_{ij}(t) + \int_0^t H_j(t-s)\,\mathrm{d}N_i(s) \tag{8}$$

where

$$N_i = \sum_{n=0}^{+\infty} (R_i * H^n) \tag{9}$$

Proof. Imitiate the proof of Theorem 11.2.2. QED

Remark 1

Take Laplace transforms on both sides of (8) to obtain

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \zeta_i(t) \frac{\eta_j(\lambda)}{1 - M + \lambda\eta(\lambda)\mathbf{1}} \tag{10}$$

where $\zeta_i(\lambda)$ are the Laplace transforms of $R_i(t)$, i.e.

$$\zeta(\lambda) = \mathbf{1} - \lambda\phi(\lambda)\mathbf{1} - \phi(\lambda)\boldsymbol{d} \tag{11}$$

where $\boldsymbol{d} = -Q\mathbf{1}$.

[1] when $\sigma^n = 0$, $x^n(\sigma^n - 0)$ are not defined.

Remark 2

The first leaping point τ of X in Theorem 4 is just σ^0 and if α is determined by (15.6) then

$$P(X(\tau)=i\,|\,\tau<\infty, X(\tau-0)=\infty)=\begin{cases}\alpha_i & \text{if } i\in E,\\ \lim\limits_{\lambda\to+\infty}\lambda\bar{\eta}(\lambda)\mathbf{1}. & \text{if } i=\infty\end{cases} \tag{12}$$

$$P(\sigma=\tau\,|\,\tau<\infty, X(\tau-0)=\infty)=1-M$$

$$P(\sigma=\tau\,|\,X(\tau-0)\in H_e)=1$$

where H_e is the set of non-conservative states, and as long as the above conditional probabilities are meaningful.

Remark 3

The Q process X in Theorem 4 is called a (DV)*-type extension of the minimal Q process X^0. The (DV)*-type extension preserves the Q matrix. As in Remark 3 after Theorem 2, when $\bar{\eta}(\lambda)=0$, the (DV)*-type extension becomes a D*-type extension. When $\alpha=0$, the (DV)*-type extension is called the V*-type extension. The D-type and D*-type extensions, generally speaking, are different extensions, but the extensions of type V and type V* are the same. Therefore, we only use the phrase 'the V*-type extension'.

11.17 GENERALIZED DV-TYPE AND GENERALIZED (DV)*-TYPE EXTENSIONS

We keep the hypotheses and symbols used in the beginning of section 11.5 to (5.2). Write $A_e=H\cup B_e$. Denote by $\mathscr{B}_e$ the class of all Borel sets in A_e. For $L_i(\Gamma,t)$ and $h_i(\Gamma,\lambda)$ given in (5.1)–(5.2), we have $\lim_{t\to\infty}L_i(\Gamma,t)=\lim_{\lambda\downarrow 0}h_i(\Gamma,\lambda)$. Denote this limit by $h_i(\Gamma)$.

Suppose we are given a mapping $G(\cdot,\cdot)$ from $A_e\times\mathscr{B}_e$ to $[0,1]$ satisfying:

(i) for each $a\in A_e$, $G(a,\cdot)$ is a measure on $\mathscr{B}_e$ and $G(a,A_e)\leqslant 1$;

(ii) for each $\Gamma\in\mathscr{B}_e$, $G(\cdot,\Gamma)$ is a $\mathscr{B}_e$-measurable function.

Suppose we are also given a mapping from $A_e\times(0,\infty)$ to the Banach space l satisfying:

(i) for each $a\in A_e$, $(\eta(a,\lambda),\lambda>0)$ is a probabilistic entrance family;

(ii) for each $\lambda>0$, $\eta(\cdot,\lambda)$ is a measurable mapping from A_e to l.

Let $H_j(a,t)$ be the inverse Laplace transform of $\eta_j(a,\lambda)$, $\lambda>0$. Set

$$W(a,\Gamma,\lambda)=\lambda\sum_j\eta_j(a,\lambda)h_j(\Gamma) \tag{1}$$

By Lemma 2.11.4 there exist limits

$$W(a,\Gamma,\lambda)\uparrow W(a,\Gamma)\qquad \lambda\uparrow\infty$$

Let

$$C(a,\Gamma,t)=W(a,\Gamma)-\sum_j H_j(a,t)h_j(\Gamma) \tag{2}$$

The following lemma illuminates the probability meaning of $W(a,\Gamma)$ and $C(a,\Gamma,t)$.

Lemma 1. Fix some $a\in A_e$. Assume that the characteristic entrance family of the approximating minimal Q process $X=\{x(t),0\leqslant t<\sigma\}$ is $(\eta(a,\lambda),\lambda>0)$. Then

$$P(X(\sigma-0)\in\Gamma)=W(a,\Gamma) \tag{3}$$

$$P(X(\sigma-0)\in\Gamma,\sigma\leqslant t)=C(a,\Gamma,t) \tag{4}$$

Proof. According to the note after Definition 13.2 we know that $\{x(t),0<t<\sigma\}$ is a minimal Q process. Hence

$$\begin{aligned}P(t<\sigma,x(\sigma-0)\in\Gamma)&=\sum_j P(X(t)=j,t<\sigma,X(\sigma-0)\in\Gamma)\\&=\sum_j H_j(a,t)h_j(\Gamma)\end{aligned} \tag{5}$$

Then letting $t\downarrow 0$ in the above we obtain

$$\begin{aligned}P(X(\sigma-0)\in\Gamma)&=\lim_{t\downarrow 0}\sum_j H_j(a,t)h_j(\Gamma)\\&=\lim_{\lambda\to\infty}\lambda\sum_j\eta_j(a,\lambda)h_j(\Gamma)\\&=\lim_{\lambda\to\infty}W(a,\Gamma,\lambda)=W(a,\Gamma)\end{aligned}$$

Therefore (4) follows from (3) and (5) and we conclude the proof. QED

Remark

When $(\eta(a,\lambda),\lambda>0)$ is a general (i.e. probabilistic or non-probabilistic) entrance family, Lemma 1 also holds.

Let

$$F(a,\Gamma,\lambda)=\int_0^\infty e^{-\lambda t}C(a,\Gamma,dt)=\lambda\int_0^\infty e^{-\lambda t}C(a,\Gamma,t)\,dt \tag{6}$$

By (2) we have

$$F(a,\Gamma,\lambda)=W(a,\Gamma)-W(a,\Gamma,\lambda) \tag{7}$$

Define

$$L_i^1(\Gamma,t)=L_i(\Gamma,t) \qquad \text{(see (5.1))} \tag{8}$$

$$L_i^{n+1}(\Gamma,t)=\int_0^t\int_{A_e}\int_{A_e} G(a,\mathrm{d}b)C(b,\Gamma,t-s)L_i^n(\mathrm{d}a,\mathrm{d}s) \tag{9}$$

$$K_i(\Gamma,t)=\sum_{n=0}^{\infty} L_i^{n+1}(\Gamma,t) \tag{10}$$

$$p_{ij}(t)=f_{ij}(t)+\int_0^t\int_{A_e}\int_{A_e} G(a,\mathrm{d}b)H_j(b,t-s)K_i(\mathrm{d}a,\mathrm{d}s) \tag{11}$$

where $(f_{ij}(t))$ is the minimal solution. The above $P(t)=(p_{ij}(t))$ is determined completely by Q, $G(\cdot,\cdot)$ and $\eta(\cdot,\cdot)$.

Considering Theorem 15.4 and using the technique of independent product spaces, we can obtain the following lemma easily.

Lemma 2. There exists a probability space $(\Omega,\mathscr{F},P)$ on which can be defined a minimal Q process $X^0=\{x^0(t),0\leqslant t<\sigma^0\}$, a family of approximating minimal Q processes $X_a^n=\{x^n(a,t),0\leqslant t<\sigma^n(a)\}$, $a\in A_e$, $n\geqslant 1$, and a family of A_e-valued random variables $f^n(a)$, $a\in A_e$, $n\geqslant 0$, such that

(i) $P(X^0(0)\in E)=1$;

(ii) for each a, the characteristic entrance family of all X_a^n $(n\geqslant 1)$ is $(\eta(a,\lambda),\lambda>0)$;

(iii) $P(f^n(a)\in\Gamma)=G(a,\Gamma)$;

(iv) all X^a, X_a^n $(a\in A_e, n\geqslant 1)$, $f^n(a)$ $(a\in A_e, n\geqslant 0)$ are independent.

When $\sigma^0>0$ and $X^0(\sigma^0-0)=a\in A_e$, we let $\sigma^1=\sigma^1(f^0(a))$, and $X^1(t)=X^1(f^0(a),t)$ for $t<\sigma^1$; otherwise let $\sigma^1=0$. Obviously, $X^1=\{X^1(t),0\leqslant t<\sigma^1\}$ is an approximating minimal Q process. When $\sigma^1>0$ and $X^1(\sigma^1-0)=a\in A_e$, we let $\sigma^2=\sigma^2(f^1(a))$ and $X^2(t)=X^2(f^1(a),t)$ for $t<\sigma^2$; otherwise let $\sigma^2=0$. Continuing with the procedure, we can obtain a minimal Q process X^0 and a sequence of approximating minimal Q processes $X^n=\{X^n(t),0\leqslant t<\sigma^n\}$ $(n\geqslant 1)$. Define $X=\{X(t),0\leqslant t<\sigma\}$ according to (16.5)–(16.6).

Theorem 3. X is a homogeneous Markov process with transition probability matrix $P(t)=(p_{ij}(t))$ given by (11), the resolvent matrix $\psi(\lambda)=(\psi_{ij}(\lambda))$ of which is given by

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+\int_{A_e}\int_{A_e} h_i(\mathrm{d}a,\lambda)\left(\sum_{n=0}^{\infty} V^n(a,\mathrm{d}b,\lambda)\right)\eta_j(b,\lambda) \tag{12}$$

where

$$V^0(a,\Gamma,\lambda)=\begin{cases}0 & \text{if } a\notin\Gamma\\ 1 & \text{if } a\in\Gamma\end{cases}$$

$$V^1(a,\Gamma,\lambda)=\int_{A_e} G(a,\mathrm{d}b)F(b,\Gamma,\lambda) \tag{13}$$

$$V^{n+1}(a,\Gamma,\lambda)=\int_{A_e} V^n(a,\mathrm{d}b,\lambda)V^1(b,\Gamma,\lambda)$$

Proof. We can prove the first conclusion by following the proof of Theorem 5.2. Here we only point out how to calculate $p_{ij}(t)$ and how to apply Lemma 2.

Clearly

$$p_i(X(t)=j)=P_i(X^0(t)=j,t<\sigma^0)+\sum_{n=0}^{\infty} R_{ij}^{n+1}(t)$$

where the first term equals $f_{ij}(t)$ while

$$\begin{aligned}R_{ij}^{n+1}(t)&=P_i(x^{n+1}(t-\tau^{n+1})=j,\tau^{n+1}\leqslant t<\tau^{n+2})\\ &=\int_0^t\int_{A_e}\int_{A_e} P_i(\tau^{n+1}\in\mathrm{d}s, X^n(\sigma^n-0)\in\mathrm{d}a, f^n(a)\in\mathrm{d}b, X^{n+1}(b,t-s)\\ &=j,t-s<\sigma^{n+1})\end{aligned}$$

Apply Lemma 2 and the above becomes

$$R_{ij}^{n+1}(t)=\int_0^t\int_{A_e}\int_{A_e} G(a,\mathrm{d}b)H_j(b,t-s)L_i^{n+1}(\mathrm{d}a,\mathrm{d}s)$$

where

$$L_i^{n+1}(\Gamma,t)=P_i(\tau^{n+1}\leqslant t, X^n(\sigma^n-0)\in\Gamma) \tag{14}$$

Hence

$$L_i^1(\Gamma,t)=L_i(\Gamma,t)$$

$$\begin{aligned}L_i^{n+1}(\Gamma,t)=\int_0^t\int_{A_e}\int_{A_e} &P_i(\tau^n\in\mathrm{d}s, X^{n-1}(\sigma^{n-1}-0)\in\mathrm{d}a, f^{n-1}(a)\in\mathrm{d}b,\\ &X^n(b,\sigma^n(b)-0)\in\Gamma,\sigma^n(b)\leqslant t-s)\end{aligned}$$

Applying Lemma 2 again we obtain (9). Therefore $P_i\{X(t)=j\}$ equals $p_{ij}(t)$, which is determined by (11).

We now perform the calculation for the Laplace transform $\psi_{ij}(\lambda)$. Take Laplace transforms on both sides of (11) to obtain

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+\int_{A_e}\int_{A_e} G(a,\mathrm{d}b)\eta_j(b,\lambda)\sum_{n=0}^{\infty} h_i^{n+1}(\mathrm{d}a,\lambda) \tag{15}$$

where

$$h_i^{n+1}(\Gamma,\lambda)=\int_0^\infty e^{-\lambda t}\Gamma_i^{n+1}(\Gamma,dt) \tag{16}$$

Taking Laplace transforms on both sides of (9) we obtain

$$h_i^{n+1}(\Gamma,\lambda)=\int_{A_e}\int_{A_e}G(a,db)F(b,\Gamma,\lambda)h_i^n(da,\lambda) \tag{17}$$

It follows from this that

$$h_i^{n+1}(\Gamma,\lambda)=\int_{A_e}V^n(a,\Gamma,\lambda)h_i(da,\lambda) \tag{18}$$

In fact, the above holds for $n=0$ obviously. Suppose it holds for $n=m-1$. Then

$$\begin{aligned}\int_{A_e}V^m(a,\Gamma,\lambda)h_i(da,\lambda)&=\int_{A_e}\left(\int_{A_e}V^{m-1}(a,db,\lambda)V^1(b,\Gamma,\lambda)\right)h_i(da,\lambda)\\&=\int_{A_e}\left(\int_{A_e}V^{m-1}(a,db,\lambda)h_i(da,\lambda)\right)V^1(b,\Gamma,\lambda)\\&=\int_{A_e}h_i^m(db,\lambda)V^1(b,\Gamma,\lambda)\\&\qquad\text{(by the hypothesis of induction)}\\&=\int_{A_e}\int_{A_e}h_i^m(db,\lambda)G(b,dc)F(c,\Gamma,\lambda)\\&=h_i^{m+1}(\Gamma,\lambda)\qquad\text{(by (17))}\end{aligned}$$

i.e. (18) holds for $n=m$.

Substituting (18) into (15) we obtain (12). The proof is completed. QED

Remark 1

We call X in Theorem 3 a generalized (DV)-type extension process of X^0. The generalized (DV)-type extension may change the Q matrix because σ^0 is not necessarily the first leaping point of X according to the fact that the probability

$$P(X(\sigma^0)\in E\,|\,\sigma^0<\infty,\ X(\sigma^0-0)=a) \tag{19}$$

may be positive for some $a\in H$.

Remark 2

Suppose that the Riesz decomposition of $\eta(a,\lambda)$ is

$$\eta(a,\lambda)=\alpha(a)\phi(\lambda)+\bar{\eta}(a,\lambda) \tag{20}$$

Then, the generalized (DV)-type extension preserves the Q matrix Q if and only if the probability in Remark 1 is equal to zero, i.e.

$$\int_{A_e}G(a,db)\sum_i\alpha_i(b)=0\qquad a\in H_e \tag{21}$$

or equivalently,

$$\int_{A_e}G(a,db)\alpha(b)=0\qquad a\in H_e \tag{22}$$

The generalized (DV)-type extension satisfying the above conditions is called a generalized (DV)*-type extension. Obviously, when Q is conservative, generalized (DV)-type and generalized (DV)*-type extensions are identical. For the generalized (DV)*-type extension σ^0 is the first leaping point of X and, furthermore,

$$P(X(\sigma^0)=i\,|\,\sigma^0<\infty,\ X(\sigma^0-0))=\begin{cases}\displaystyle\int_{A_e}G(X(\sigma^0-0),db)\alpha_i(b) & \text{if } i\in E\\ \displaystyle\int_{A_e}G(X(\sigma^0-0),db)\bar{M}(b) & \text{if } i=\infty\end{cases} \tag{23}$$

where

$$\bar{M}(b)=\lim_{\lambda\to\infty}\lambda\|\bar{\eta}(b,\lambda)\|$$

Remark 3

Suppose $\alpha(b)=0$ for $b\in A_e$. Then (22) holds. Thus the generalized (DV)-type extensions and generalized (DV)*-type extensions are identical; hence the Q matrix remains unchanged. Therefore, when $\alpha(a)=0$, $a\in A_e$, we call the generalized (DV)*-type a generalized V*-type extension. As pointed out in Remark 3 after Theorem 16.2, we only use the phrase 'generalized V*-type extension'. For the generalized V*-type extension σ^0 is the first leaping point and (22) becomes

$$P(X(\sigma^0)=i\,|\,\sigma^0<\infty,\ X(\sigma^0-0))=\begin{cases}0 & \text{if } i\in E\\ \displaystyle\int_{A_e}G(X(\sigma^0-0),db)M(b) & \text{if } i=\infty\end{cases} \tag{24}$$

where

$$M(a)=\lim_{\lambda\to\infty}\lambda\|\eta(a,\lambda)\|$$

Remark 4

Suppose $\psi(\lambda)$ is a Q process in Theorem 8.5.1 (remark: H in this section is

denoted by H_e in Chapter 8). For $a\in A_e$, let

$$\eta(a,\lambda)=\begin{cases}\bar{\eta}^{a_i}(\lambda) & \text{if } a\in \text{some } a_i\in J\\ 0 & \text{if } a\in \text{some } a_i\in A-J\end{cases} \tag{25}$$

and

$$G(a,b)=\begin{cases}G^{a_ia_k} & \text{if } a\in \text{some } a_i\in J \text{ and } b\in \text{some } a_k\in J\\ 0 & \text{otherwise}\end{cases} \tag{26}$$

Then $G(\cdot,\cdot)$ and $\eta(\cdot,\cdot)$ satisfy the requirement presented at the beginning of this section. For these $G(\cdot,\cdot)$ and $\eta(\cdot,\cdot)$, $\psi(\lambda)$ determined by (12) and that by Theorem 8.5.1 are the same. Therefore, for both $G(\cdot,\cdot)$ and $\eta(\cdot,\cdot)$ given by (25)–(26), Lemma 2 and Theorem 3 give the paths of Q processes in Theorem 8.5.1.

PART V CONSTRUCTION THEORY OF BIRTH–DEATH PROCESSES: PROBABILITY METHOD

CHAPTER 12

Probability Structure of Birth–Death Processes

12.1 INTRODUCTION

When $E = \{0, 1, 2, \ldots\}$, and Q has the form (6.1.1), the process $X \in \mathscr{H}_s(Q)$ is called a birth–death process. In this chapter, Q as used below will always refer to a birth–death matrix. In Chapter 6 we have constructed all birth–death processes by using the analytic method.

In section 2.3, we have already pointed out that Professor Zi-kun Wang introduced the method for solving the construction problem for conservative birth–death processes in 1958, that is, the limit transition method. The advantage of this method is that the path structure of the processes constructed is quite clear and their probability meaning is obvious. In order to study the properties of the processes we may first make a clear study of simple processes and, then, it suffices to turn to the investigation of the limits. Therefore, this method has important potential value in terms, of theory and practice. For examples, see Xiang-qun Yang (1965b, 1966b) and Zhen-ting Hou (1975).

The logical foundation of this method is published in Zi-kun Wang (1965a). In the paper all honest birth–death processes are constructed. But in construction of paths, transformation g_n and transformation f_n are employed respectively to handle such cases as $S < \infty$ and $S = \infty$. Zi-kun Wang (1980) provides a unified treatment for the two cases with respect to the honest processes. Zi-kun Wang and Xiang-qun Yang (1978, 1979) consistently handle the two cases and the case that processes are allowed to be stopping processes. The content of this chapter is derived from these two papers.

12.2 PROBABILITY EXPLANATION OF THE CHARACTERISTIC NUMBERS

We shall adopt the characteristic numbers and notations in section 6.2. In this section, we may assume $a_0 \geq 0$. Let $X = \{x(t), t < \sigma\} \in \mathscr{H}_s(Q)$, τ be the first leaping point and τ_n be the nth leaping point. Obviously

$$x(\tau - 0) = 0 \text{ or } \infty \tag{1}$$

and we have $|x(\tau_n) - x(\tau_n - 0)| = 1$ for the imbedded chain $X_T = \{x(\tau_n), \tau_n < \tau\}$ $(\tau_n < \tau)$.

Theorem 1

$$X_i^1 \equiv P_i\{x(\tau - 0) = 0\} = \frac{a_0(z - z_i)}{a_0(z - z_0) + 1} \tag{2}$$

$$X_i^2 = P_i\{x(\tau - 0) = \infty\} = \frac{a_0(z_i - z_0) + 1}{a_0(z - z_0) + 1} \tag{3}$$

We introduce the convention $\infty/\infty = 1$ and $0 \cdot \infty = 0$.

Proof. By Theorem 7.8.2, X^1 satisfies equation (6.2.19) where $f_0 = -a_0$ and $f_i = 0$ $(i > 0)$. Therefore by Lemma 6.2.5, we have

$$X_i^1 = [a_0(z_i - z_0) + 1]X_0^1 - a_0(z_i - z_0) \tag{4}$$

Similarly, X^2 satisfies equation (6.2.19) where $f_i \geqslant 0$ $(i \geqslant 0)$. Hence by Lemma 6.2.5, we have

$$X_i^2 = [a_0(z_i - z_0) + 1]X_0^2 \tag{5}$$

Noticing $X^1 + X^2 = 1$, it follows from the above expression that if $X_0^2 = 0$ then $X^2 = 0$, thus $X^1 = 1$; if $X_0^2 > 0$, then $X^2 > 0$, and consequently by the Martingale convergence theorem, we have

$$X^2_{x(\tau_n)} = P\{x(\tau - 0) = \infty \mid x(\tau_0), x(\tau_1), \ldots, x(\tau_n)\} \to 1$$

on the positive probability set $\{x(\tau - 0) = \infty\}$. Therefore, $X_i^2 \to 1$ $(i \to \infty)$, so that $X_i^1 \to 0$ $(i \to \infty)$. Hence by (4) and (5), it follows that

$$0 = [a_0(z - z_0) + 1]X_0^1 - a_0(z - z_0) \qquad 1 = [a_0(z - z_0) + 1]X_0^2$$

Hence when $a_0 z < \infty$,

$$X_0^1 = \frac{a_0(z - z_0)}{a_0(z - z_0) + 1} \qquad X_0^2 = \frac{1}{a_0(z - z_0) + 1}$$

Substituting the above formula into (4) and (5), we obtain that (2) and (3) hold for $a_0 z < \infty$. By convention $\infty/\infty = 1$, (2) and (3) likewise hold for $a_0 z = \infty$ and the proof is completed. QED

Set

$$\xi_i = \begin{cases} \inf\{t \mid x(t) = i, t < \tau\} \\ \infty \qquad \text{if the set above is empty} \end{cases} \tag{6}$$

Obviously,

$$P_i\{\xi_n \uparrow \tau (i \leqslant n \uparrow \infty) \mid x(\tau - 0) = \infty\} = 1 \tag{7}$$

Theorem 2. For $i \leqslant k \leqslant n$,

$$P_k\{\xi_i < \xi_n\} = \frac{z_n - z_k}{z_n - z_i} \qquad P_k\{\xi_n < \xi_i\} = \frac{z_k - z_i}{z_n - z_i} \tag{8}$$

Proof. By Theorem 7.8.2, $u_k = P_k\{\xi_i < \xi_n\}$ satisfies equation (5.4.11) where $f_i = 1$, $f_k = 0$ $(i < k \leqslant n)$. By Theorem 6.2.3 and (5.4.12) we obtain the first expression in (8). The second expression follows from $P_k\{\xi_n < \xi_i\} + P_k\{\xi_i < \xi_n\} = 1$ QED

Theorem 3. For $i \leqslant k$,

$$P_k\{\xi_i < \tau\} = \frac{z - z_k}{z - z_i} \qquad P_i\{\xi_k < \tau\} = \frac{a_0(z_i - z_0) + 1}{a_0(z_k - z_0) + 1} \tag{9}$$

$$P_k\{\tau \leqslant \xi_i, x(\tau - 0) = \infty\} = \frac{z_k - z_i}{z - z_i} \tag{10}$$

Proof. Since on $\{x(0) = k\}$,

$$(\xi_i < \xi_n) \uparrow \bigcup_{n=k+1}^{\infty} (\xi_i < \xi_n) = (\xi_i < \tau)$$

$$(\xi_n < \xi_i) \downarrow \bigcap_{n=k+1}^{\infty} (\xi_n < \xi_i) = \{\tau \leqslant \xi_i, x(\tau - 0) = \infty\} \tag{11}$$

Taking the limit in (8), we obtain (10) and the first formula in (9). Secondly, since $u_i = P_i(\xi_k < \tau)$ $(0 \leqslant i \leqslant k)$ satisfies equation (6.2.13) for $n = k$, $f_i = 0$ $(i < k)$, and $f_k = 1$, the second formula in (9) follows from Lemma 6.2.4. QED

Theorem 4. If $i \leqslant k \leqslant n$, then

$$E_k\{\xi_i, \xi_i < \xi_n\} = \frac{z_n - z_k}{z_n - z_i} \sum_{j=i+1}^{k-1} \frac{z_n - z_j}{z_n - z_i}(z_j - z_i)\mu_j + \frac{z_k - z_i}{z_n - z_i} \sum_{j=k}^{n-1} \frac{z_n - z_j}{z_n - z_i}(z_n - z_j)\mu_j \tag{12}$$

$$E_k\{\xi_n, \xi_n < \xi_i\} = \frac{z_n - z_k}{z_n - z_i} \sum_{j=i+1}^{k-1} \frac{z_j - z_i}{z_n - z_i}(z_j - z_i)\mu_j + \frac{z_k - z_i}{z_n - z_i} \sum_{j=k}^{n-1} \frac{z_j - z_i}{z_n - z_i}(z_n - z_j)\mu_j \tag{13}$$

Proof. By Theorem 7.8.2 and Theorem 2, $u_k = E_k\{\xi_i, \xi_i < \xi_n\}$ satisfies the equation (5.4.11) for $f_i = f_n = 0$, $f_k = (z_n - z_k)/(z_n - z_i)$ $(i < k < n)$. By Theorem 6.2.3 and (5.4.12) we obtain (12), and (13) can be derived similarly. QED

Theorem 5

$$E_k\{\tau, x(\tau-0)=\infty\} = \frac{z-z_k}{[a_0(z-z_0)+1]^2} + \frac{z-z_k}{a_0(z-z_0)+1}\sum_{j=1}^{k-1}\frac{a_0(z_j-z_0)+1}{a_0(z-z_0)+1}\mu_j + \frac{a_0(z_k-z_0)+1}{a_0(z-z_0)+1}\sum_{j=k}^{\infty}\frac{a_0(z_j-z_0)+1}{a_0(z-z_0)+1}(z-z_j)\mu_j \tag{14}$$

Proof. Let $u_k = E_k\{\xi_n, x(\tau-0)=\infty\}$ $(k \leqslant n)$. Then u_k satisfies the equation (6.2.13) for $f_k = P_k\{x(\tau-0)=\infty\}$ $(k<n)$, and $f_n=0$. By Lemma 6.2.4,

$$E_k\{\xi_n, x(\tau-0)=\infty\} = \frac{z_n-z_k}{a_0(z_n-z_0)+1}\frac{1}{a_0(z-z_0)+1} + \frac{z_n-z_k}{a_0(z_n-z_0)+1}\sum_{j=1}^{k-1}\frac{a_0(z_j-z_0)+1}{a_0(z-z_0)+1}\mu_j + \frac{a_0(z_k-z_0)+1}{a_0(z_n-z_0)+1}\sum_{j=k}^{n-1}\frac{a_0(z_j-z_0)+1}{a_0(z-z_0)+1}(z_n-z_j)\mu_j \tag{15}$$

Noticing (7) and taking the limit in the above expression, we obtain (14). The proof is terminated. QED

Theorem 6. Let $a_0=0$.

(i) Let c_{kj} be the probability of going from k to j in finitely many ($\geqslant 0$) steps; then

$$c_{kj} = P_k\{\xi_j<\tau\} = \begin{cases} 1 & \text{if } k \leqslant j \\ (z-z_k)/(z-z_j), & \text{if } k>j \end{cases} \tag{16}$$

(ii) For the minimal Q process X to be recurrent it is necessary and sufficient that $z=\infty$. If $z=\infty$, then for it to be ergodic it is necessary and sufficient that $\sum_{i=0}^{\infty}\mu_i<\infty$.

(iii) m_i, N_i and R and determined by (6.2.4)–(6.2.7), so that

$$m_i = E_i\xi_{i+1} \qquad N_i = E_i\tau \qquad R = E_0\tau \tag{17}$$

(iv) The necessary and sufficient condition for $P_k\{\tau<\infty\}=1$ $(k\in E)$ is $R<\infty$.

Proof. (i) follows from (9).

(ii) The probability starting from 0 and coming back in finitely many steps is

$$f_0^* = \frac{b_0}{a_0+b_0}P_1(\xi_0<\tau) = \frac{z-z_1}{z-z_0}$$

From this we know that $f_0^*=1$, that is, for the minimal process to be recurrent it is necessary and sufficient that $z=\infty$.

(iii) When $a_0=0$, choosing $k=n-1$ in (15) we obtain $E_{n-1}\xi_n=m_{n-1}$. Equation (14) $a_0=0$ is just the second and third formulae in (17).

(iv) By (17) and $N_k \leqslant R$ we know that if $R<\infty$ then $P_k\{\tau<\infty\}=1$ $(k\in E)$. Conversely, if $P_k\{\tau<\infty\}=1$ for some (hence all) k then by Theorem 7.8.2, $u_k(\lambda)=E_k(e^{-\lambda\tau})\neq 0$ $(\lambda>0)$ satisfies $D_\mu u^+(\lambda)=\lambda u(\lambda)$, and so by Lemma 6.2.5

$$u_k(\lambda) = u_0(\lambda) + \lambda\sum_{j=0}^{i-1}(z_i-z_j)u_j(\lambda)\mu_j$$

Moreover, obviously, $u_i(\lambda)\uparrow$, $(i\uparrow)$, and therefore $1 \geqslant u_i(\lambda) \geqslant \lambda u_0(\lambda)\sum_{j=0}^{i-1}(z_i-z_j)\mu_j$, $1 \geqslant \lambda u_0(\lambda)R$, and hence $R<\infty$. The proof is completed. QED

Theorem 7. Let $a_0=0$, $S=\infty$ (see (6.2.7)). Suppose that $X=\{x(t), t<\sigma\}\in\mathscr{H}_s(Q)$ and that ξ_0 is determined by (6). Then for $\lambda>0$, $E_i\{e^{-\lambda\xi_0}\}\downarrow 0$ $(i\uparrow\infty)$.

Proof. Since X must visit $i-1$ while going from i to 0, therefore, $u_i(\lambda)=E_i\{e^{-\lambda\xi_0}\}\downarrow\alpha\geqslant 0$ $(i\uparrow\infty)$. By Theorem 7.8.2, we obtain $u_0(\lambda)=1$, $D_u u_i^+(\lambda)=\lambda u_i(\lambda)$ $(i>0)$, that is,

$$u_{i-1}(\lambda)-u_i(\lambda) = \frac{b_i}{a_i}[u_i(\lambda)-u_{i+1}(\lambda)] + \frac{\lambda}{a_i}u_i(\lambda) \qquad i>0$$

Applying the above formula again, we have

$$u_{i-1}(\lambda)-u_i(\lambda) = \frac{b_ib_{i+1}\cdots b_{i+j+1}}{a_ia_{i+1}\cdots a_{i+j+1}}[u_{i+j+1}(\lambda)-u_{i+j+2}(\lambda)] + \lambda\left(\frac{u_i(\lambda)}{a_i} + \sum_{l=0}^{j}\frac{b_ib_{i+1}\cdots b_{i+l}u_{i+l+1}(\lambda)}{a_ia_{i+1}\cdots a_{i+l}a_{i+l+1}}\right) \geqslant \lambda\left(\frac{1}{a_i} + \sum_{l=0}^{j}\frac{b_ib_{i+1}\cdots b_{i+l}}{a_ia_{i+1}\cdots a_{i+l}a_{i+l+1}}\right)\alpha$$

Letting $j\to\infty$ we obtain $u_{i-1}(\lambda)-u_i(\lambda)\geqslant\lambda e_i\alpha$ $(i>0)$, so that

$$1 \geqslant u_0(\lambda) \geqslant u_0(\lambda)-u_j(\lambda) \geqslant \lambda\left(\sum_{k=1}^{j}e_k\right)\alpha$$

Letting $j\to\infty$ we obtain $1\geqslant\lambda S\alpha$. Since $S=\infty$, it follows that $\alpha=0$. The proof is over. QED

Theorem 8. If $a_0=0$, $S=\infty$, then $P\{\xi_{\infty k}<\sigma\}=0$, where $\xi_{\infty k}$ is determined by (10.4.1). In other words, each Q process $X\in\mathscr{H}_s(Q)$ is pure entrance from E, or we may say X cannot be entrance from '∞'.

Proof. First we prove $P\{\xi_{\infty 0} < \sigma\} = 0$. We shall use reduction to absurdity. Put $\Omega_i = (\xi_{\infty i} < \sigma)$, and suppose $P(\Omega_0) > 0$. Obviously, $\xi_{\infty i} \downarrow (i \uparrow)$. We write $\Omega_i \uparrow \Omega_\infty$ $(i \uparrow \infty)$. By Lemma 10.4.3 and the strong Markov property, we may consider the process $X_i = \{X(\xi_{\infty i} + t), t < \sigma - \xi_{\infty i}\}$ on the probability space $(\Omega_i, \Omega_i \mathscr{F}, P(\cdot|\Omega_i))$. X_i and X have the same transition probability matrix, and $P\{x_i(0) = i|\Omega_i\} = 1$. We denote by ξ_i^j the random variable defined by (6) for X_i. Since

$$u_i(\lambda) = E\{\exp(-\lambda \xi_0^i)|\Omega_i\} \tag{18}$$

is determined only by the initial distribution and transition probability, therefore, the quantity given by (18) and the quantity in theorem 7 are equal, so that $u_i(\lambda) \downarrow 0$ $(i \uparrow \infty)$.

On the other hand,

$$u_i(\lambda) = \frac{1}{P(\Omega_i)} \int_{\Omega_i} \exp(-\lambda \xi_0^i)\, dP \geqslant \frac{1}{P(\Omega_\infty)} \int_{\Omega_0} \exp(-\lambda \xi_0^i)\, dP$$

Since $\xi_0^i \leqslant \xi_{\infty i} < \infty$ on Ω_0, it follows that

$$u_i(\lambda) \geqslant \frac{1}{P(\Omega_\infty)} \int_{\Omega_0} \exp(-\lambda \xi_{\infty 0})\, dP > 0$$

This contradicts $u_i(\lambda) \downarrow 0$. Therefore $P(\xi_{\infty 0} < \sigma) = 0$. For $i > 0$,

$$P\{\xi_{\infty 0} < \sigma\} \geqslant P\{\xi_{\infty i} < \sigma\} \prod_{k=1}^{i} \frac{a_k}{a_k + b_k}$$

Consequently, $P(\xi_{\infty i} < \sigma) = 0$. The proof is completed. QED

12.3 AN EXTENDED DYNKIN LEMMA

We shall prove the following extended Dynkin lemma, where the processes may not necessarily be birth–death processes.

Lemma 1. Let $X = \{x(t), t < \sigma\} \in \mathscr{H}$, ξ be a non-negative random variable and θ be a translation operator satisfying the following conditions:

(i) For arbitrary $s \geqslant 0$ and $t \geqslant 0$, the set $A_s = \{\xi > s\} \in \mathscr{F}_s^0$ and $A_{s+t} \subseteq A_s \cap \theta_s A_t$.

(ii) There exist positive numbers $T > 0$ and $\alpha > 0$ such that for all $i \in E$ we have $P_i(A_T) \leqslant 1 - \alpha$.

Then for any initial distribution, all moments $E\xi^l$ of ξ are finite, and the distribution function $P\{\xi \leqslant t\}$ is uniquely determined by its moments $E\xi^l$ $(l = 0, 1, 2, \ldots)$.

Proof. Since $A_s \in \mathscr{F}_s^0, \theta_s A_T \in \mathscr{F}_\infty^s$, by the Markov property, we obtain

$$P_i\{A_{s+T}\} \leqslant P_i\{A_s \cap \theta_s A_T\} = \int_{A_s} P_{x(s)}(A_T)\, dP_i$$
$$\leqslant (1-\alpha) P_i(A_s)$$

Therefore $P_i(A_{nT}) \leqslant (1-\alpha)^n$. From this, we get $P(A_{nT}) \leqslant (1-\alpha)^n$ and $P(\xi < \infty) = 1$. Thus,

$$E\{\xi^l\} = \sum_{n=0}^{\infty} \int_{nT < \xi \leqslant (n+1)T} \xi^l\, dP \leqslant \sum_{n=0}^{\infty} \{(n+1)T\}^l P\{\xi > nT\}$$
$$\leqslant \sum_{n=0}^{\infty} \{(n+1)T\}^l (1-\alpha)^n < \infty$$

Next take a positive number r such that $e^{Tr}(1-\alpha) < 1$, then

$$\sum_{l=0}^{\infty} \frac{E\{\xi\}^l}{l!} r^l \leqslant \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} \frac{\{(n+1)T\}^l r^l}{l!} (1-\alpha)^n$$
$$= \sum_{n=0}^{\infty} e^{(n+1)Tr} (1-\alpha)^n = \frac{e^{Tr}}{1 - e^{Tr}(1-\alpha)} < \infty$$

By a theorem in Cramer (1946, section 15.4), the distribution function of ξ is uniquely determined by its moments. The proof is concluded. QED

12.4 RECURRENCE AND ERGODIC PROPERTY OF THE HONEST PROCESS

From now on, we shall always consider conservative birth–death processes, that is, $a_0 = 0$. By Theorem 2.6, the minimal process is honest if and only if $R < \infty$, hence we further suppose $R < \infty$. Then $P\{\tau < \infty\} = 1$.

Theorem 1. Let $X = \{x(t), t < \sigma\} \in \mathscr{H}_s(Q)$ be honest, and τ be the first leaping point. Put

$$\beta_1^n = \inf\{t | \tau \leqslant t < \sigma, x(t) \leqslant n\} \tag{1}$$

Then for any initial distribution, all moments $E[\beta_1^n]^l$ $(l \geqslant 0)$ of β_1^n are finite, and its distribution $P\{\beta_1^n \leqslant t\}$ is uniquely determined by its moments $E[\beta_1^n]^l$ $(l \geqslant 0)$.

Proof. Since $P_i(\tau < \infty) = 1$ and X is honest, it follows that there exists $s > 0$ such that

$$0 < P_0(\tau < s) = P_0\{\tau < s, x(s) \in E\} = \sum_j P_0\{\tau < s, x(s) = j\}$$

So there exists $j \in E$ such that $P_0(\tau < s, x(s) = j) > 0$, we have

$$\alpha \equiv P_0\{\beta_1^0 \leqslant s + t\} \geqslant P_0\{\tau < s, x(s+t) = 0\} \geqslant P_0\{\tau < s, x(s) = j\} p_{j0}(t) > 0$$

However, on the set $\{x(0) = i, \beta_1^n \leqslant s + t\}$, for η_i^* defined by (6.7.7), we have $\eta_{i+1}^* < \infty$, and $\beta_1^0 = \eta_{i+1}^* + \theta_{\eta_{i+1}^*} \beta_1^0$. Therefore

$$\{x(0) = i, \beta_1^0 \leqslant s + t\} \subset \{x(0) = i, \eta_{i+1}^* < \infty, \theta_{\eta_{i+1}^*}(\beta_1^0 \leqslant s + t)\}$$

Hence $P_i\{\beta_1^0 \leqslant s+t\}$ increases with i. If we choose $T = s+t$, then $P_i(\beta_1^0 \leqslant T) \geqslant \alpha > 0$ $(i \in E)$. Thus the condition (ii) in Lemma 3.1 is satisfied; obviously, the condition (i) in Lemma 3.1 is satisfied. By Lemma 3.1, this theorem holds for $n=0$. Again noting $\beta_1^n \leqslant \beta_1^0$, we obtain the proof of the theorem. The proof is terminated. QED

Theorem 2. All honest processes $X \in \mathscr{H}_s(Q)$ are recurrent and ergodic. Furthermore, all order moments $E[\eta_i^*]^l$ of η_i^* defined by (7.7.7) are finite, and the distribution function $P(\eta_i^* \leqslant t)$ is uniquely determined by its moments $E[\eta_i^*]^l$ $(l \geqslant 0)$.

Proof. Since $\eta_i^* \leqslant \beta_1^n$ $(n \geqslant i)$, quoting Theorem 1, we can prove this theorem. QED

12.5 TWO LEMMAS

Let $X = \{x(t), t < \sigma\} \in \mathscr{H}_s(Q)$. If X is stopping, we may take $\Delta = -1$, and transform X into the honest process $\tilde{X} = \{\tilde{x}(t), t < \infty\} \in \mathscr{H}_s(\tilde{Q})$, in the same way as (7.6.6.), where $\tilde{Q} = (\tilde{q}_{ij})$ $(i, j \in (-1) \cup E)$, and $\tilde{q}_i = q_i$, $\tilde{q}_{i,-1} = \tilde{q}_{-1,-i} = \tilde{q}_{-1,j} = 0$ $(i, j \in E)$. Henceforth we shall adopt the convention above.

Let τ still be the first leaping point, and τ_n be the nth jumping point, and ξ_i be defined by (2.6). For arbitrary $0 \leqslant \varepsilon \leqslant \infty$, put

$$f_\varepsilon(x) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant \varepsilon \\ \varepsilon & \text{if } x > \varepsilon \end{cases} \tag{1}$$

Lemma 1. For $k \geqslant i \geqslant 0$, put

$$H_{ki}^\varepsilon = E_k\left\{ \sum_{0 \leqslant \tau_j < \min(\xi_i, \tau)} f_\varepsilon(\tau_{i+1} - \tau) \right\} \tag{2}$$

In particular,

$$H_{ki}^\infty = E_k\{\min(\xi_i, \tau)\} \tag{3}$$

Then when $R < \infty$, we have

$$\begin{aligned} H_{ki}^\varepsilon = {} & \frac{z - z_k}{z - z_i} \sum_{j=i+1}^{k-1} (z - z_i)\{1 - \exp[-(a_j + b_j)\varepsilon]\}\mu_j \\ & + \frac{z_k - z_i}{z - z_i} \sum_{j=k}^{\infty} (z - z_i)\{1 - \exp[-(a_j + b_j)\varepsilon]\}\mu_j \leqslant N_k \end{aligned} \tag{4}$$

and

$$\lim_{\varepsilon \downarrow 0} H_{ki}^\varepsilon = 0 \tag{5}$$

If we also have $s < \infty$, then

$$\lim_{k \to \infty} \frac{H_{ki}^\varepsilon}{C_{k0}} = \frac{1}{C_{i0}} \sum_{j=i+1}^{\infty} (z_j - z_i)\{1 - \exp[-(a_j + b_j)\varepsilon]\} \tag{6}$$

$$\lim_{i \to \infty} \lim_{k \to \infty} \frac{H_{ki}^\varepsilon}{C_{k0}} = 0 \tag{7}$$

$$\lim_{\varepsilon \downarrow 0} \lim_{k \to \infty} \frac{H_{ki}^\varepsilon}{C_{k0}} = 0 \tag{8}$$

where C_{ki} is determined by (2.16).

Proof. Let $i \leqslant k \leqslant n$. Put

$$H_{kin}^\varepsilon = E_k\left\{ \sum_{0 \leqslant \tau_j < \min(\xi_i, \xi_n)} f_\varepsilon(\tau_{i+1} - \tau) \right\} \tag{9}$$

Obviously, $H_{kin}^\varepsilon \uparrow H_{ki}^\varepsilon$ $(n \uparrow \infty)$. We see easily that

$$E_k f_\varepsilon(\tau_1) = \frac{1}{a_k + b_k}\{1 - \exp[-(a_k + b_k)\varepsilon]\}$$

By an application of Theorem 7.8.2 it follows that $u_k = H_{kin}^\varepsilon$ satisfies equation (5.4.11) for $f_i = f_n = 0$, $f_k = 1 - \exp[-(a_k + b_k)\varepsilon]$ $(i < k < n)$. From Theorem 6.2.3, we obtain

$$\begin{aligned} H_{kin}^\varepsilon = {} & \frac{z_n - z_k}{z_n - z_i} \sum_{j=i+1}^{k-1} (z - z_i)\{1 - \exp[-(a_j + b_j)\varepsilon]\}\mu_j \\ & + \frac{z_k - z_i}{z_n - z_i} \sum_{j=k}^{n-1} (z_n - z_j)\{1 - \exp[-(a_j + b_j)\varepsilon]\}\mu_j \end{aligned} \tag{10}$$

Letting $n \to \infty$, we obtain the equality in (4). In comparison with (6.2.6), the inequality in (4) follows.

Since $R < \infty$, by using the dominated convergence theorem, (5) follows from (4).

When $s < \infty$, we have $\sum_{i=0}^\infty \mu_i < \infty$, whereas

$$\frac{1}{C_{k0}} \frac{z_k - z_i}{z - z_i} \sum_{j=k}^{\infty} (z - z_j)\{1 - \exp[-(a_j + b_j)\varepsilon\}\mu_j \leqslant z \sum_{j=k}^{\infty} \mu \to 0 \qquad (k \to \infty)$$

$$\frac{1}{C_{i0}} \sum_{j=i+1}^{\infty} (z - z_i)\{1 - \exp[-(a_j + b_j)\varepsilon]\}\mu_j \leqslant z \sum_{j=i+1}^{\infty} \mu_j \to 0 \qquad (i \to \infty)$$

Therefore, (6) follows from (4) and, thus, we obtain (7). Using the dominated convergence theorem, we derive (8) from (6). The proof is completed. QED

Lemma 2. Let $X \in \mathscr{H}_s(Q)$ be a non-minimal Q-process, τ be the first leaping point, and β_1^n be determined by (4.1). Then

$$P\left\{\lim_{n\to\infty} \beta_1^n = \tau\right\} = 1 \tag{11}$$

Proof. Obviously, $\beta_1^n \downarrow (n\uparrow)$, hence $\lim_{n\to\infty}\beta_1^n \geqslant \tau$. On the other hand, for arbitrary positive $\varepsilon_m \downarrow 0$, by (ii) in Theorem 7.6.2, it follows that $P\{x(\tau+\varepsilon_m)=\infty\}=0$. If $\tau=\sigma$, then, certainly, $\lim_{n\to\infty}\beta_1^n=\tau$. If $\tau<\sigma$ then when m is sufficiently large, we have $\tau+\varepsilon_m<\sigma$ and $x(\tau+\varepsilon_m)\in E$. Therefore when $n\geqslant x(\tau+\varepsilon_m)$, we have $\beta_1^n\leqslant\tau+\varepsilon_m$, and $\lim_{n\to\infty}\beta_1^n\leqslant\tau+\varepsilon_m$. Since m is arbitrary, it follows that $\lim_{n\to\infty}\beta_1^n\leqslant\tau$. The proof is concluded. QED

12.6 CHARACTERISTIC SEQUENCE

Let $X=\{x(t), t<\sigma\}\in\mathscr{H}_s(Q)$ be a non-minimal process. We consider the transformation g_n in Definition 10.5.1. Let $\beta_0^n=0$, τ_1^n be the first leaping point for X, and

$$\beta_1^n = \inf\{t \mid \tau_1^n \leqslant t<\sigma, x(t)\leqslant n\} \tag{1}$$

If the above set is empty, we take σ as 'inf'. Suppose that $\tau_{m-1}^n, \beta_{m-1}^n$ are already defined. If $\beta_{m-1}^n=\sigma$, then we define $\tau_m^n=\beta_m^n=\sigma$; otherwise, we define τ_m^n as the first leaping point after β_{m-1}^n, and

$$\beta_m^n = \inf\{t \mid \tau_m^n \leqslant t<\sigma, x(t)\leqslant n\}. \tag{2}$$

Transformation W_{τ^n,β^n} becomes transformation g_n and $X^n=W_{\tau^n,\beta^n}(X)$ becomes

$$X^n = g_n(X) \tag{3}$$

We have

$$g_n(X^{n+1}) = X^n \tag{4}$$

In particular, if X is a (Q,π) Doob process and $\pi_j=0$ $(j>n)$, then $g_n(X)=X$.

Theorem 1. Let $X\in\mathscr{H}_s(Q)$ be a non-minimal process. Then $X^n=\{x^n(t), t<\sigma^n\}\in\mathscr{H}_s(Q)$ given by (3) is a (Q,V^n) Doob process, satisfying (4), where $V^n=(v_j^n, 0\leqslant j\leqslant n)$, with

$$v_j^n = P\{x(\beta_1^n)=j\} \qquad (-1\leqslant j\leqslant n) \tag{5}$$

satisfies

$$v_j^n = v_j^{n+1}\left(\sum_{i=-1}^{n} v_i^{n+1} + v_{n+1}^{n+1}C_{n+1,n}\right)^{-1} \qquad -1\leqslant j<n$$

$$v_n^n = (v_n^{n+1}+v_{n+1}^{n+1}C_{n+1,n})\left(\sum_{i=-1}^{n} v_1^{n+1} + v_{n+1}^{n+1}C_{n+1,n}\right)^{-1}$$

$$\sum_{j=-1}^{n} v_j^n = 1 \qquad \sum_{i=0}^{n} v_j^n > 0 \tag{6}$$

For X to be honest it is necessary and sufficient that $v_{-1}^n=0$ $(n\geqslant 0)$.

Proof. Noticing that β_k^n is a Markov time and $P\{x(\beta_k^n)>n\}=0$, we now proceed to prove that V^n defined by (5) satisfies (6).

Let the quantity $\bar\beta_1^n$ be determined by the process $\bar X=\{x(\beta_1^{n+1}+t), t<\sigma-\beta_1^{n+1}\}$, in fashion (1). Suppose that Λ_n is the event that the process $\bar X$ goes from $n+1$ to n in finite $(\geqslant 0)$ steps, and $\bar\Lambda_n$ is the complementary event of Λ_n. We can easily identify $\{x(\beta_1^{n+1})=n+1\}$, Λ_n, $\bar\Lambda_n$ all belong to $\mathscr{F}_{\tau_2^{n+1}}$, and $\{x(\bar\beta_1^n)=j\}\in\mathscr{F}'_{\tau_2^{n+1}}$. By the corollary to Theorem 7.17.1 (note that the conditional independence in this corollary becomes independence in the case of birth–death processes), for $-1\leqslant j\leqslant n$,

$$\begin{aligned}
\Delta_j^n &\equiv P\{x(\beta_1^{n+1})=n+1, \bar\Lambda_n, \bar x(\bar\beta_1^n)=j\}\\
&= P\{x(\beta_1^{n+1})=n+1, \bar\Lambda_n\}P\{\bar x(\beta_1^n)=j \mid x(\beta_1^{n+1})=n+1\}\\
&= P\{x(\beta_1^{n+1})=n+1\}P\{\bar\Lambda_n \mid x(\beta_1^{n+1})=n+1\}v_j^n\\
&= v_{n+1}^{n+1}(1-C_{n+1,n})v_j^n
\end{aligned}$$

Consequently,

$$\begin{aligned}
v_n^n &= P\{x(\beta_1^{n+1})=n\}+P\{x(\beta_1^{n+1})=n+1,\Lambda_n\}+\Delta_n^n\\
&= v_n^{n+1}+v_{n+1}^{n+1}C_{n+1,n}+v_{n+1}^{n+1}(1-C_{n+1,n})v_n^n\\
v_j^n &= P\{x(\beta_1^{n+1})=j\}+\Delta_j^n = v_j^{n+1}+v_{n+1}^{n+1}(1-C_{n+1,n})v_j^n \qquad (-1\leqslant j<n)
\end{aligned}$$

Hence

$$\begin{aligned}
v_j^n &= v_j^{n+1}/[1-v_{n+1}^{n+1}(1-C_{n+1,n})] \qquad -1\leqslant j<n\\
v_n^n &= (v_n^{n+1}+v_{n+1}^{n+1}C_{n+1,n})/[1-v_{n+1}^{n+1}(1-C_{n+1,n})]
\end{aligned} \tag{7}$$

From the above expression, we know that either $P\{\beta_1^n<\sigma\}>0$, for all n, or $P\{\beta_1^n=\sigma\}=0$ for all n. If the latter holds, then by Lemma 5.2 we have $P\{\tau=\sigma\}=1$, that is, X is a minimal process, which is contradictory to the hypothesis of this theorem. Hence

$$\sum_{j=0}^{n} v_j^n = P\{\beta_1^n<\sigma\}>0 \qquad (n\geqslant 0)$$

Next we shall prove $\sum_{j=-1}^{n} v_j^n=1$. From this and (7) follows (6). When X is honest, by Theorem 4.1, we have $P\{\beta_1^n<\infty\}=1$. Therefore $v_{-1}^n=0$, and $\sum_{j=0}^{n} v_j^n=1$. When X is stopping, we taking $\pi_0=1$, $\pi_j=0$ $(j>0)$. By Theorem 11.2.2, we may take into account the πD-type extension process

$\bar{X} = \{\bar{X}(t), t < \infty\}$ of X, that is,

$$\begin{aligned} \bar{X}(t) &= x(t) \qquad t < \sigma \\ P\{\bar{x}(\sigma) &= 0 | \sigma < \infty\} = 1 \end{aligned} \tag{8}$$

Letting $\bar{\beta}_1^n$ be the quantity for $\bar{X}$, obviously, $\beta_1^n = \bar{\beta}_1^n$. Since $\bar{X}$ is honest, and $P(\bar{\beta}_1^n < \infty) = 1$, it follows that $\sum_{j=-1}^n v_j^n = P\{\beta_1^n < \infty\} = 1$.

Suppose $v_{-1}^n = 0$, that is, $P\{\beta_1^n < \sigma\} = \sum_{j=0}^n v_j^n = 1$. By the strong Markov property, we have $P\{\beta_1^n < \beta_2^n < \cdots < \sigma\} = 1$. This shows that there exist infinitely many $j = j(\omega)$ $(\leqslant n)$ intervals in $[0, \sigma(\omega))$. By Theorem 7.7.1, we have $P\{\sigma = \infty\} = 1$, that is, X is honest.

We now prove that $X^n = g_n(X)$ is a (Q, V^n) Doob process. Set $X_m = \{x(\beta_m^n + t), t < \tau_{m+1}^n - \beta_m^n\}$ $(m \geqslant 0)$. We can easily see that $X_m \in \mathscr{H}_s(Q)$ is a minimal Q process, satisfying (ii*) in Lemma 11.3.1. By $R < \infty$, i.e. $P(\tau < \infty) = 1$, we know that $\beta_m^n < \sigma$ if and only if $0 < \tau_{m+1}^n - \beta_m^n < \infty$. By the corollary to Theorem 7.17.1, we have

$$P\{x^{m+1}(0) = j | 0 < \tau_{m+1}^n - \beta_m^n < \infty\} = P\{x(\beta_{m+1}^n) = j | \beta_m^n < \sigma\} = v_j^n \qquad 0 \leqslant j \leqslant n$$

Therefore (iii*) in Lemma 11.3.1 is satisfied, too. By the strong Markov property of X, we may directly deduce that (iv*) in Lemma 11.3.1 is satisfied for $\Delta = \{x^{m+1}(0) = i\}$. It is not difficult to verify that X_m $(m \leqslant k)$ is $\mathscr{F}_{\tau_{k+1}^n}$-measurable, and X_m $(m > k)$ is $\mathscr{F}'_{\tau_{k+1}^n}$-measurable. We have already pointed out $\{0 < \tau_{k+1}^n - \beta_k^n < \infty\} = (\beta_k^n < \sigma)$. By the corollary to Theorem 7.17.1, (iv*) in Lemma 11.3.1 is satisfied. According to Theorem 11.4.2, the process determined by (11.2.5) and (11.2.6) for X_m $(m \geqslant 0)$ is a (Q, V^n) Doob process. But this process is precisely $X^n = g_n(X)$. The proof is completed. QED

Theorem 2. For any $X = \{x(t), t < \sigma\} \in \mathscr{H}_s(Q)$, we have

$$P\{\sigma < \infty\} = 0 \text{ or } 1$$

Proof. The conclusion of this theorem is valid for each minimal process or honest process X. Assume x is stopping and non-minimal.

By Theorem 1, we have

$$P\{\beta_1^n = \sigma = \infty\} = 1 - \sum_{j=-1}^n v_j^n = 0$$

that is,

$$P\{\beta_1^n < \sigma = \infty\} + P\{\sigma < \infty\} = 1 \tag{9}$$

By the strong Markov property, we have

$$P\{\beta_1^n < \sigma = \infty\} = \sum_{j=0}^n v_j^n P_j\{\sigma = \infty\}$$

But $u_j \equiv P_j\{\sigma = \infty\}$ satisfies the equation $Qu = 0$, so that u_j is a constant; that is, $P_j(\sigma = \infty) = P(\sigma = \infty)$ is independent of j. Therefore, (9) becomes

$$\left(\sum_{j=0}^n v_j^n\right) P\{\sigma = \infty\} + P\{\sigma < \infty\} = 1$$

By Theorem 1, we have $0 < \sum_{j=0}^n v_j^n < 1$. Therefore, $P\{\sigma = \infty\} = 0$ or $P\{\sigma < \infty\} = 1$ must follow from the above formula. The proof is concluded. QED

Theorem 3. Let $X = \{x(t), t < \sigma\} \in \mathscr{H}_s(Q)$ be a non-minimal Q process. Then there exists a non-negative sequence of numbers p, q and r_n $(n \geqslant -1)$ satisfying

$$\begin{aligned} &p + q = 1 \\ &q = 0 && \text{if } S = \infty \\ &r_n = 0\,(n \geqslant 0) && \text{if } p = 0 \\ &0 < \sum_{n=0}^\infty r_n N_n < \infty && \text{if } p > 0 \end{aligned} \tag{10}$$

such that

$$v_j^n = P\{x(\beta_1^n) = j\} \qquad -1 \leqslant j \leqslant n \tag{11}$$

which may be expressed as

$$\begin{aligned} v_i^n &= (X_n/A_n) r_j \qquad -1 \leqslant j < n \\ v_n^n &= Y_n + (X_n/A_n) \sum_{l=n}^\infty r_l C_{ln} \end{aligned} \tag{12}$$

where

$$A_n = \sum_{l=0}^\infty r_l C_{ln} \qquad d = \begin{cases} (q/p) A_0 & \text{if } p > 0 \\ 1 & \text{if } p = 0 \end{cases}$$

$$\begin{aligned} X_n &= \frac{A_n C_{n0}}{(r_{-1} + A_n) C_{n0} + d} \\ Y_n &= \frac{d}{(r_{-1} + A_n) C_{n0} + d} \\ \frac{X_n}{A_n} &= \frac{C_{n0}}{r_{-1} C_{n0} + 1} \qquad \text{if } p = 0 \end{aligned} \tag{13}$$

If we let η be the last leaping point before β_1^0, then

$$P\{x(\eta) = j\} = \begin{cases} r_{-1}/(r_{-1} + A_0 + d) & \text{if } j = -1 \\ r_j C_{j0}/(r_{-1} + A_0 + d) & \text{if } 0 \leqslant j < \infty \\ d/(r_{-1} + A_0 + d) & \text{if } j = \infty \end{cases} \tag{14}$$

Above, p and q are uniquely determined by X. If $p>0$, then γ_n $(n \geqslant -1)$ is uniquely determined by X except for a constant factor. If $p=0$, then γ_n $(n \geqslant -1)$ is uniquely determined by X

Proof. The proof is to be established in several steps.

(a) Put

$$R_n = \sum_{i=0}^{n-1} v_i^n C_{i0} \qquad S_n = v_n^n C_{n0} \qquad \Delta_n = R_n + S_n \tag{15}$$

Then by (6), we have

$$0 < \Delta_n = \frac{\Delta_{n+1}}{\delta_{n+1}} \qquad S_n = \frac{v_n^{n+1} C_{n0} + S_{n+1}}{\delta_{n+1}} \tag{16}$$

$$\delta_{n+1} = \sum_{j=-1}^{n} v_j^{n+1} + v_{n+1}^{n+1} C_{n+1,n} \tag{17}$$

Therefore, v_j^n/Δ_n is independent of $n > j$ $(j \geqslant -1)$, and there exist limits

$$\frac{S_n}{\Delta_n} \downarrow p \geqslant 0 \qquad \frac{R_n}{\Delta_n} \uparrow q \geqslant 0 \tag{18}$$

If $p=0$, we take $r=1$; if $p>0$, we arbitrarily take $\gamma > 0$. Put

$$r_j = \frac{v_j^n}{\Delta_n} r \qquad n > j \geqslant -1 \tag{19}$$

Thus we obtain the non-negative sequence of numbers p, q and r_n $(n \geqslant -1)$.

(b) Obviously, $p+q=1$. Since when $p=0$, we have $r_n=0$ $(n \geqslant 0)$. If $p>0$, then there exists a $r_k > 0$ $(k \geqslant 0)$ at least. Therefore, we need only prove

$$\sum_{n=0}^{\infty} r_n N_n < \infty \tag{20}$$

Note that $E\beta_1^0 < \infty$. When X is honest, (20) follows from Theorem 4.1. When X is stopping, there exists an honest Q process $\bar{X}$ satisfying (8). Hence we also have $E\beta_1^0 = E\bar{\beta}_1^0 < \infty$. Whence

$$E\tau_2^0 = E\beta_1^0 + E\{\tau_2^0 - \beta_1^0\} = E\beta_1^0 + v_0^0 E_0 \tau = E\beta_1^0 + v_0^0 R < \infty$$

Put

$$M_i^n = \begin{cases} \tau_{i+1}^n - \beta_i^n & \text{if } \beta_i^n < \beta_1^0 \\ 0 & \text{otherwise} \end{cases}$$

Obviously,

$$\tau_2^0 \geqslant \sum_{i=1}^{\infty} M_i^n$$

$$\infty > E\tau_2^0 \geqslant \sum_{i=1}^{\infty} EM_i^n = \sum_{i=1}^{\infty} E\{\tau_{i+1}^n - \beta_i^n, \beta_i^n < \beta_1^0\}$$

But

$$E\{\tau_2^n - \beta_1^n, \beta_1^n < \beta_1^0\} = \sum_{j=1}^{n} v_j^n E_j \tau_1^n = \sum_{j=1}^{n} v_j^n N_j$$

$$E\{\tau_{i+1}^n - \beta_i^n, \beta_i^n < \beta_1^0\} = \sum_{j=1}^{n} P\{\beta_{i-1}^n < \beta_1^0, x(\beta_i^n) = j\} E_j \tau_1^n$$

$$= \sum_{j=1}^{n} \left\{ \sum_{k=1}^{n} v_k^n (1 - C_{k0}) \right\}^{i-1} v_j^n N_j$$

Therefore,

$$\infty > E\tau_2^0 \geqslant \frac{\sum_{j=1}^{n} v_j^n N_j}{1 - \sum_{k=1}^{n} v_k^n (1 - C_{k0})} \geqslant \frac{\sum_{j=1}^{n-1} v_j^n N_j}{v_{-1}^n + \Delta_n}$$

By (19) it follows that

$$\frac{\sum_{j=1}^{n-1} r_j N_j}{r_{-1} + r} \leqslant E\tau_2^0 < \infty$$

Letting $n \to \infty$, we obtain (20).

(c) We now proceed to prove (12). By (18) and (19) we have

$$p = \lim_{n\to\infty} \frac{R_n}{\Delta_n} = \lim_{n\to\infty} \frac{1}{r} \sum_{j=0}^{n-1} r_i C_{i0} = \frac{A_0}{r}$$

so that

$$r = \begin{cases} A_0/p & \text{if } p > 0 \\ 1 & \text{if } p = 0 \end{cases} \tag{21}$$

Using induction on (6) we easily obtain that: $m > n$,

$$\begin{aligned} & v_j^n = v_j^m / \delta_{m,n} \qquad -1 \leqslant j < n \\ & v_n^n = \left(\sum_{j=n}^{m} v_j^m C_{jn} \right) \Big/ \delta_{mn} \\ & 0 < \Delta_n = \Delta_m / \delta_{mn} \\ & \delta_{mn} = \sum_{j=-1}^{m} v_j^m + \sum_{j=n+1}^{m} v_j^m C_{jn} \end{aligned} \tag{22}$$

From the above, (18) and (19), we have

$$\frac{v_n^n}{\Delta_n} = \frac{\sum_{j=n}^{m} v_j^m C_{jm}}{\Delta_m} = \frac{1}{r} \sum_{j=n}^{m-1} r_j C_{jn} + \frac{S_m}{\Delta_m C_{n0}}$$

so that when $m \to \infty$, we have

$$\frac{v_n^n}{\Delta_n} = \frac{1}{r}\sum_{j=n}^{\infty} r_j C_{jn} + \frac{q}{C_{n0}} \tag{23}$$

By (19) and (23), we obtain

$$1 = \sum_{j=-1}^{n} v_j^n = \Delta_n\left(\frac{\sum_{j=-1}^{n-1} r_j}{r} + \frac{\sum_{j=n}^{\infty} r_j C_{jn}}{r} + \frac{q}{C_{n0}}\right)$$

hence,

$$\Delta_n = \frac{rC_{n0}}{(r_{-1} + A_n)C_{n0} + qr}$$

Substituting the above formula into (19) and (23) and noticing (21), we have (12).

(d) We now prove (14). If X is honest, then $1 = P\{\beta_1^n < \infty\}$. By the strong Markov property, we have $P(\beta_1^n < \beta_2^n < \cdots) = 1$; therefore, for almost all $\omega \in \Omega$, there exist infinitely many $j(\omega)(\leqslant n)$ intervals in $[0, \lim_{l\to\infty}\beta_l^n(\omega))$ for X, whence $P(\lim_{l\to\infty}\beta_l^n = \infty) = 0$. If X is stopping, by Theorem 2, we have $P(\sigma < \infty) = 1$. Since if $\beta_l^n(\omega) < \sigma(\omega)$, then $\beta_l^n(\omega) < \tau_{l+1}^n(\omega) \leqslant \beta_{l+1}^n(\omega)$, by Theorem 7.7.1, we have $P\{$there exists l such that $\beta_l^n = \sigma\} = 1$. Thus we always have

$$P\left\{\lim_{l\to\infty}\beta_l^n = \sigma\right\} = 1 \tag{24}$$

whether X is stopping or honest. Moreover, for almost all ω, there exists unique l such that $\beta_{l-1}^n < \eta \leqslant \beta_l^n \leqslant \beta_1^0$. Write l as l_n, that is,

$$l_n = \min\{l \mid \beta_l^n \geqslant \eta\} \tag{25}$$

$$\beta_{l_n}^n = \inf\{t \mid \eta \leqslant t < \sigma, x(t) \leqslant n\} \tag{26}$$

If $x(\beta_{l_n}^n) = j$ for some $n > j \geqslant -1$, then it necessarily follows that $\beta_{l_n}^n = \eta$. Since if $\eta < \beta_{l_n}^n$ by definition we have $n < x(t) < \infty$ for $t \in (\eta, \beta_{l_n}^n)$, while $x(\beta_{l_n}^n) = j < n$. As the jump of birth–death processes at the jump point is 1, this is not possible. Hence for $n > j \geqslant -1$,

$$\{x(\beta_{l_n}^n) = j\} \subset \{\beta_{l_n}^n = \eta, x(\eta) = j\} \subset \{x(\eta) = j\} \tag{27}$$

Again it is quite clear that, if $x(\eta) = j$, then for $n > j$ we have $\beta_{l_n}^n = \eta$, so that

$$\{x(\eta) = j\} = \lim_{n\to\infty}\{x(\beta_{l_n}^n) = j\} \qquad (-1 \leqslant j < \infty) \tag{28}$$

We shall now prove

$$\{x(\eta) = \infty\} = \lim_{n\to\infty}\{x(\beta_{l_n}^n) = n\} \tag{29}$$

In fact, assuming that ω belongs to the right-hand side in (29), we surely find

$\beta_{l_n}^n \downarrow \bar\eta \geqslant \eta$. Hence by the right-continuity, we have $x(\bar\eta) = \lim_{n\to\infty} x(\beta_{l_n}^n) = \lim_{n\to\infty} n = \infty$. But by the definition of η it follows that for any $\eta < t < \beta_1^0$, $x(t) \in E$. Hence $\bar\eta = \eta$, whence $x(\eta) = \infty$, i.e. ω belongs to the left-hand side in (29). Letting ω belong to the left-hand side in (29) for arbitrary n, we certainly have $x(\beta_{l_n}^n) = n$. Otherwise, (27) will yield $x(\eta) = j \neq \infty$. Thus (29) holds.

For $0 \leqslant j \leqslant n$, we have

$$\begin{aligned} P\{x(\beta_{l_n}^n) = j\} &= \sum_{l=1}^{\infty} P\{x(\beta_l^n) = j, l_n = l\} \\ &= \sum_{l=1}^{\infty} P\{\beta_{l-1}^n < \beta_1^0, x(\beta_l^n) = j, \beta_l^n \leqslant \beta_1^0 < \tau_{l+1}^n\} \\ &= \sum_{l=1}^{\infty}\left\{\sum_{k=1}^{n} v_k^n(1 - C_{k0})\right\}^{l-1} v_j^n C_{j0} = \frac{v_j^n C_{j0}}{v_{-1}^n + \Delta_n} \end{aligned} \tag{30}$$

and

$$\begin{aligned} P\{x(\beta_{l_n}^n) = -1\} &= \sum_{l=1}^{\infty} P\{x(\beta_l^n) = -1, l_n = l\} \\ &= \sum_{l=1}^{\infty} P\{\beta_{l-1}^n < \beta_1^0, x(\beta_l^n) = -1\} \\ &= \sum_{l=1}^{\infty}\left\{\sum_{k=1}^{n} v_k^n(1 - C_{k0})\right\}^{l-1} v_{-1}^n = \frac{v_{-1}^n}{v_{-1}^n + \Delta_n} \end{aligned} \tag{31}$$

Substituting (30) and (31) into (28) and (29) and, moreover, noticing (18) and (19), we obtain

$$P\{x(\eta) = j\} = \begin{cases} r_{-1}/(r_{-1} + r) & \text{if } j = -1 \\ r_j C_{j0}/(r_{-1} + r) & \text{if } 0 \leqslant j < \infty \\ rq/(r_{-1} + r) & \text{if } r = \infty \end{cases}$$

Again noticing (21), we find that the above expression is (14).

(e) Suppose $S = \infty$. By Theorem 2.8, we have $P\{\xi_{\infty 0} < \sigma\} = 0$. But $\{x(\eta) = \infty\} \subset \{\xi_{\infty 0} < \sigma\}$. Hence $P\{x(\eta) = \infty\} = 0$. Then $q = 0$ follows from (13) and (14).

(f) Suppose that there exists a non-negative sequence of numbers $\bar p, \bar q$ and $\bar r_n (n \geqslant -1)$ such that (10)–(14) hold. By (12) we know the following: Either both p and $\bar p$ are zero; hence $\bar r_n = r_n = 0\,(n \geqslant 0)$ and $q = \bar q = 1$; and again from

$$v_{-1}^n = \frac{C_{n0}}{r_{-1}C_{n0} + 1} r_{-1} = \frac{C_{n0}}{\bar r_{-1}C_{n0} + 1}\bar r_{-1}$$

it follows that $r_{-1} = \bar r_{-1}$. Or both p and $\bar p$ are positive, and (18) follows from (11) and (12), so that $p = \bar p$ and $q = \bar q$; and again by (19) we have $r_j/r_k = v_j^n/v_k^n = \bar r_j/\bar r_k$. The proof is completed. QED

Definition 1. The non-negative sequence of numbers p, q and $r_n (n \geqslant -1)$ is called the characteristic sequence of the Q process X.

12.7 PROBABILITY STRUCTURE OF PROCESSES

Suppose that a sequence of numbers p, q and $r_n (n \geqslant -1)$ satisfying (6.10) is given.

Lemma 1. $v_j^n(-1 \leqslant j \leqslant n)$ defined by (6.12) and (6.13) satisfies (6.6).

Proof. When $p = 0$ we have $v_j^n = 0 (0 \leqslant j < n)$,

$$v_{-1}^n = \frac{r_{-1} C_{n0}}{r_{-1} C_{n0} + 1} \qquad v_n^n = \frac{1}{r_{-1} C_{n0} + 1}$$

Equation (6.6) follows from direct identification.

Let $p > 0$. By direct identification, we know that the last expression in (6.6) is valid. By (6.12) and (6.13) we have

$$\begin{aligned} 1 - v_{n+1}^{n+1}(1 - C_{n+1,n}) &= \sum_{j=-1}^{n} v_j^{n+1} + v_{n+1}^{n+1} C_{n+1,n} \\ &= \frac{X_{n+1}}{A_{n+1}} \sum_{j=-1}^{n} r_j + Y_{n+1} C_{n+1,n} + \frac{X_n}{A_n} \sum_{j=n+1}^{\infty} r_j C_{jn} \\ &= \frac{X_{n+1}}{A_{n+1}} \left(r_{-1} + A_n + \frac{Y_{n+1} A_{n+1} C_{n+1,n}}{X_n} \right) \\ &= \frac{X_{n+1}}{A_{n+1}} \left(r_{-1} + A_n + \frac{d}{C_{n0}} \right) = \frac{X_{n+1}}{A_{n+1}} \frac{A_n}{X_n} \end{aligned} \tag{1}$$

$$\begin{aligned} v_n^{n+1} + v_{n+1}^{n+1} C_{n+1,n} &= \frac{X_{n+1}}{A_{n+1}} r_n + Y_{n+1} C_{n+1,n} + \frac{X_{n+1}}{A_{n+1}} \sum_{l=n+1}^{\infty} r_l C_{ln} \\ &= \frac{X_{n+1}}{A_{n+1}} \frac{A_n}{X_n} \left(Y_n + \frac{X_n}{A_n} \sum_{l=n}^{\infty} r_l C_{ln} \right) \end{aligned} \tag{2}$$

Therefore, (6) holds. QED

Lemma 2. Suppose that $V^n = \{v_j^n, 0 \leqslant j \leqslant n\} (n \geqslant 0)$ satisfies (6.6). Then there exists a probability space $(\Omega, \mathscr{F}, P)$ on which we may define a sequence of (Q, V^n) Doob processes $X^n = \{x^n(t), t < \sigma^n\} \in \mathscr{H}_s(Q) (n \geqslant 0)$ satisfying (6.4).

Proof. Fix a distribution (v_i) as an initial distribution. For each n, there exists a probability space $(\Omega_n, \mathscr{F}_n, P_n)$ on which a (Q, V^n) Doob process $\bar{X}_n = \{\bar{x}_n(t, \omega_n), t < \bar{\sigma}_n(\omega_n)\} (\omega_n \in \Omega_n)$ is defined. By Theorem 6.1, it follows that

for $m < n$, $Z_m = g_m(\bar{X}_n)$ is a (Q, V^n) Doob process on $(\Omega_n, \mathscr{F}_n, P_n)$. Let $k \geqslant 1$. For non-negative integers $n_i (1 \leqslant i \leqslant k)$ and non-negative real numbers $t_i \geqslant 0$ $(1 \leqslant i \leqslant k)$ and $j_i \in E (1 \leqslant i \leqslant k)$, we may choose $n > \max(n_1, n_2, \ldots, n_k)$ and define the k-dimensional distribution

$$F_{n_1 t_1, \ldots, n_k t_k}(j_1, \ldots, j_k) = P_n\{Z_{n_i}(t_i, \omega_n) = j_i, 1 \leqslant i \leqslant k\} \tag{3}$$

It is obvious that this distribution is independent of the choice of n and, furthermore, the family of finite-dimensional distributions $\{F_{n_1 t_1, \ldots, n_k t_k}\}$ is consistent. According to the Kolmogorov theorem (see Zi-kun Wang, 1965a, section 1.1, Theorem 1), there exists a probability space $(\Omega, \mathscr{F}, P)$ on which a sequence of processes $X^n = \{x^n(t), t < \sigma^n\}$ is defined and, moreover,

$$P\{x^{n_i}(t_i) = j_i, 1 \leqslant i \leqslant k\} = F_{n_1 t_1, \ldots, n_k t_k}(j_1, \ldots, j_k) \tag{4}$$

From the above and (3), $X^n \in \mathscr{H}_s(Q)$ is a (Q, V^n) Doob process.

Secondly, according to the theorem quoted above, we may choose $\Omega = (\omega)$, where $\omega = \omega(n, t)$ is a bivariate function taking values in E, $(n = 0, 1, 2, \ldots, t \in [0, \sigma_n), \sigma_n \leqslant \infty)$, and $x^n(t, \omega) = \omega(n, t), \sigma^n(\omega) = \sigma_n$. Therefore, (6.4) holds. The proof is terminated. QED

Let $X^n = \{x^n(t), t < \sigma^n\}$ be the sequence of processes in Lemma 2. From (6.4) it follows that $\sigma^n \leqslant \sigma^{n+1}$. Hence we may set $\sigma^n \uparrow \sigma$.

The quantities defined by the fashion in (6.1) and (6.2) for X^k are written as τ_m^{kn} and β_m^{kn}. From (6.4) it follows that $\beta_m^{n0} \leqslant \beta_m^{n+1,0} \leqslant \sigma^{n+1}$. Therefore, there exists the limit $\beta_m^0 = \lim_{n \to \infty} \beta_m^{n0} \leqslant \sigma$.

On account of (7.2.4) $\lim_{m \to \infty} \beta_m^0 \geqslant \lim_{m \to \infty} \beta_m^{n0} = \sigma^n$. Therefore

$$P\left\{ \lim_{m \to \infty} \beta_m^0 = \sigma \right\} = 1 \tag{5}$$

For $n > m$, put

$$L_{nm}^i = \begin{cases} \beta_i^{nm} - \tau_i^{nm} & \text{if } \beta_i^{nm} < \beta_1^{n0} \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

and

$$T_\varepsilon^{nm} = \sum_{\tau_1^n \leqslant \tau_{ij}^n < \beta_1^{nm}} f_\varepsilon(\tau_{i,j+1}^n - \tau_{ij}^n) \tag{7}$$

where τ_{ij}^n is the jth jump point after the ith leaping point of X^n. The process $f_\varepsilon(X)$ is defined as (5.1). By (6.4), we easily obtain

$$\begin{aligned} L_{nm} &= \sum_{i=1}^{\infty} L_{nm}^i \leqslant L_{n+1,m} \\ T_\varepsilon^{n0} &\leqslant T_\varepsilon^{n+1,0} \qquad T_{\varepsilon_1}^{n0} \leqslant T_{\varepsilon_2}^{n0} (\varepsilon_1 < \varepsilon_2) \end{aligned} \tag{8}$$

Consequently we may set

$$\begin{array}{ll} L_{nm}\uparrow L_m(n\uparrow\infty) & L_m\downarrow L(m\uparrow\infty) \\ T_\varepsilon^{n0}\uparrow T_\varepsilon(n\uparrow\infty) & T_\varepsilon\downarrow T(\varepsilon\downarrow 0)\end{array}\tag{9}$$

Lemma 3. $P\{L=0\}=P\{T=0\}=1$.

Proof. Put

$$\bar\beta_i^{nm}=\inf\{t\mid t\geqslant\tau_{i0}^n,\ x^n(t)\leqslant m\}$$

We consider

$$\bar L_{nm}^i=\begin{cases}\min(\tau_{i+1,0}^n,\bar\beta_i^{nm})-\tau_{i0}^n & \text{if } \tau_{i0}^n<\beta_1^{n0} \text{ and } m<x^n(\tau_{i0}^n)\\ 0 & \text{otherwise}\end{cases}$$

and

$$\bar T_{\varepsilon i}^{n0}=\begin{cases}\displaystyle\sum_{\tau_{i0}^n\leqslant\tau_{ij}^n<\min(\tau_{i+1,1}^n,\bar\beta_i^{n0})}f_\varepsilon(\tau_{i,j+1}^n-\tau_{ij}^n) & \text{if } \tau_{i0}^n<\beta_1^{n0}\\ 0 & \text{otherwise}\end{cases}$$

By the definition of L_{nm} and T_ε^{n0} we have

$$L_{nm}=\sum_{i=1}^\infty\bar L_{nm}^i\qquad T_\varepsilon^{n0}=\sum_{i=1}^\infty\bar T_{\varepsilon i}^{n0}$$

But by Lemma 5.1, (6.11) and (6.12), we have

$$\begin{aligned}EL_{nm}&=\sum_{i=1}^\infty E\bar L_{nm}^i\\ &=\sum_{i=1}^\infty\sum_{j=m+1}^n P\{\tau_{i0}^n<\beta_1^{n0},x^n\{\tau_{i0}^n)=j\}E\{\min(\tau_{i+1,0}^n,\bar\beta_i^{nm})-\tau_{i0}^n\mid x^n(\tau_{i0}^n)=j\}\\ &=\sum_{i=1}^\infty\sum_{j=m+1}^n\left\{\sum_{k=1}^n v_k^n(1-C_{k0})\right\}^{i-1}v_j^nH_{jm}^\infty\\ &=\frac{\sum_{j=m+1}^n v_j^nH_{jm}^\infty}{v_{-1}^n+\sum_{k=0}^n v_k^nC_{k0}}\\ &=\frac{\sum_{j=m+1}^{n-1}r_jH_{jm}^\infty+(d/C_{n0}+\sum_{l=n}^\infty r_lC_{ln})H_{nm}^\infty}{r_{-1}+A_0+d}\end{aligned}$$

and

$$\begin{aligned}ET_\varepsilon^{n0}&=\sum_{i=1}^\infty E\bar T_{\varepsilon i}^{n0}\\ &=\sum_{i=1}^\infty\sum_{k=1}^n P\{\tau_{i0}^n<\beta_1^{n0},x^n(\tau_{i0}^n)=k\}\\ &\quad\times E\left\{\sum_{\tau_{i0}^n\leqslant\tau_{ij}^n<\min(\tau_{i+1,0}^n,\bar\beta_i^{n0})}f_\varepsilon(\tau_{i,j+1}^n-\tau_{ij}^n)x_n(\tau_{i0}^n)=k\right\}\end{aligned}$$

$$\begin{aligned}&=\sum_{i=1}^\infty\sum_{k=1}^n\left\{\sum_{j=1}^n v_j^n(1-C_{j0})\right\}^{i-1}v_k^nH_{k0}^\varepsilon\\ &=\frac{\sum_{k=1}^n v_k^nH_{k0}^\varepsilon}{v_{-1}^n+\sum_{j=0}^n v_j^nC_{j0}}\\ &=\frac{\sum_{k=1}^{n-1}r_kH_{k0}^\varepsilon+(d/C_{n0}+\sum_{l=n}^\infty r_lC_{ln})H_{n0}^\varepsilon}{r_{-1}+A_0+d}\end{aligned}$$

By (6.2.6) we may verify $C_{ln}N_n\leqslant N_l(l\geqslant n)$. By Lemma 5.1, (6.10) and (6.12), it follows that $H_{jm}^\varepsilon\leqslant N_j$, $\sum_{j=0}^\infty r_jN_j<\infty$ and $\lim_{\varepsilon\downarrow 0}H_{k0}^\varepsilon=0$. If $s=\infty$, then $d=0$; if $s<\infty$, then

$$\lim_{m\to\infty}\lim_{n\to\infty}\frac{H_{nm}^\infty}{C_{n0}}=\lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\frac{H_{n0}^\varepsilon}{C_{n0}}=0$$

Therefore, by the above two expressions, we obtain

$$\begin{aligned}EL&=\lim_{m\to\infty}\lim_{n\to\infty}EL_{nm}\\ &=\frac{\lim_{m\to\infty}\sum_{j=m+1}^\infty r_jH_{jm}^\infty+\lim_{m\to\infty}\lim_{n\to\infty}(d/C_{n0}+\sum_{l=n}^\infty r_lC_{ln})H_{nm}^\infty}{r_{-1}+A_0+d}=0\end{aligned}$$

and

$$\begin{aligned}ET&=\lim_{\varepsilon\downarrow 0}\lim_{n\to 0}ET_\varepsilon^{n0}\\ &=\frac{\lim_{\varepsilon\downarrow 0}\sum_{k=1}^\infty r_kH_{k0}^\varepsilon+\lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}(d/C_{n0}+\sum_{l=n}^\infty r_lC_{ln})H_{n0}^\varepsilon}{r_{-1}+A_0+d}=0\end{aligned}$$

On the basis of this we have proved this lemma, and the proof is completed. QED

Theorem 4. For the sequence of (Q,V^n) Doob processes $X^n=\{x^n(t),t<\sigma^n\}$ in Lemma 2, its strong limit process $X=\{x(t),t<\sigma\}$ exists. $X\in\mathscr{H}_s(Q)$ is a non-minimal process. X is the unique Q process satisfying (6.11), that is, X is the unique Q process having the characteristic sequence p,q and $\gamma_n(n\geqslant -1)$.

Proof. (a) Obviously the limit $\sigma^n(\omega)\uparrow\sigma(\omega)$ exists. We now proceed to prove that for almost all $\omega\in\Omega$, and almost all $t\in[0,\sigma(\omega))$ in Lebesgue measure L, $x^n(t,\omega)$ is convergent to some state in E; for other $t\in[0,\sigma(\omega))$, $x^n(t,\omega)$ it is convergent to ∞.

Without loss of generality let us suppose that, for each $\omega\in\Omega$, $X^n(\omega)$ has only a finite number of i intervals ($i\in E, n\geqslant 0$) in any finite interval $[0,t)(t\leqslant\sigma(\omega))$. If every constant interval (i.e. general i interval) of each X^n is translated towards the left and, furthermore, the distance covered by translation of every interval

is not greater than ε, then the total length of the intervals composed of the points t which are in the interval $[0, \beta_1^{n0}(\omega))$ and make $x_n(t,\omega) \neq x_m(t,\omega)(n > m)$ is not more than $\varepsilon + T_\varepsilon^n(\omega) < \varepsilon + T_\varepsilon(\omega)$. Fixing k, taking $n > m > l(> k)$, on account of $\beta_1^{n0}(\omega) > \beta_1^{k0}(\omega)$ we obtain

$$L\{t \mid t \in [0, \beta_1^{k0}(\omega)), x_n(t,\omega) \neq x_m(t,\omega)\} \leqslant L_l(\omega) + T_{L_l(\omega)}(\omega) \tag{10}$$

Set $\Omega_0 = \{L_l + T_{L_l} \downarrow 0, l \uparrow \infty\}$. From Lemma 3 it follows that $P\{\Omega_0\} = 1$. Given $\omega \in \Omega_0$, from (10) we know that $x_n(t,\omega)$ converges in $[0, \beta_1^{k0}(\omega))$ in accordance with the Lebesgue measure L and hence there exists a subsequence $n_i \to \infty$ such that $x_{n_i}(t,\omega)$ converges for almost all t in $[0, \beta_1^{k0}(\omega))$. Fixing a convergence point t_0, since the Doob processes do not take the value '∞', consequently there exists $M \in E$ such that $x_{ni}(t_0,\omega) \to M(i \to \infty)$. As E is discrete, there is a positive number N such that

$$n_{n_i}(t_0,\omega) = M \qquad (i \geqslant N) \tag{11}$$

Now we start to prove that there exists a positive number N' such that, if $n > N'$, $x_n(t_0,\omega) = M$; hence $x_n(t_0,\omega)$ converges to M. Otherwise, there must exist $m_i \to \infty$ such that

$$x_{m_i}(t_0,\omega) \neq M \tag{12}$$

From this formula and (11) and by $g_m(X^n) = X^m (m < n)$ we know that in $[0, t_0]$, X^m has infinitely many M intervals. And this contradicts the hypothesis at the beginning of the proof.

Consequently, provided that $\omega \in \Omega_0$, then for almost all $t \in [0, \beta_1^{k0}(\omega)), x_n(t,\omega)$ is convergent to the states in E. Setting $k \to \infty$ we find that the same conclusion is valid for $[0, \beta_1^0(\omega))$. We can verify in the same way that the same conclusion is true for $[0, \beta_i^0(\omega))$. From (5) we know that $x^n(t,\omega)$ converges to the states in E for almost all $t \in [0, \sigma(\omega))$. As for the exceptional $t \in [0, \sigma(\omega))$, if $x^n(t,\omega)$ does not converge to ∞, then there surely exist two subsequences n_i and m_i and $M \in E$ so that (11) and (12) hold, which will likewise lead to contradictions.

(b) We are going to prove that $P\{x(t) = \infty\} = 0 (t \geqslant 0)$. On account of (a), $L\{t \mid x(t,\omega) = \infty\} = 0$. By the Fubini theorem there exists a set T, $L(T) = 0$, such that, if $t \notin T$, $P\{x(t) = \infty\} = 0$. Evidently $0 \notin T$.

Assume that $t_0 \in T$. Then $t_0 > 0$. We may take t_1 such that $t_1 \notin T, t_0 - t_1 \notin T$. Hence,

$$\begin{aligned} P\{x(t_0) \geqslant N\} &= \lim_{n\to\infty} P\{x^n(t_0) \geqslant N\} \\ &= \lim_{n\to\infty} E\{Px^n(t_1)[x^n(t_0 - t_1) \geqslant N]\} \\ &= E\left\{\lim_{n\to\infty} Px^n(t_1)[x^n(t_0 - t_1) \geqslant N]\right\} \\ &= E\left\{\lim_{n\to\infty} Px(t_1)[x^n(t_0 - t_1) \geqslant N]\right\} \\ &= E\{Px(t_1)[x(t_0 - t_1) \geqslant N]\} \end{aligned}$$

Letting $N \to \infty$ we have $P\{x(t_0) = \infty\} = E\{Px(t_1)[x(t_0 - t_1) = \infty]\} = 0$.

(c) From (a) and (b) it is easily shown that $X \in \mathscr{H}_s$ and that it is a non-minimal Q process.

(d) We proceed to prove that X satisfies (6.11). In actual fact, let β_1^{kn} be defined for X^k in fashion (6.1) and (6.2). Because of (6.4), $\beta_1^{kn} \uparrow (n \leqslant k \uparrow)$. It is easily seen that there exist limits $\lim_{k\to\infty} \beta_1^{kn} = \beta_1^n$, $\lim_{k\to\infty} x^n(\beta_1^{kn}) = x(\beta_1^n)$. Furthermore on account of (6.4), $x^k(\beta_1^{kn}) = x^n(\beta_1^{nn})$. Thus, for $0 \leqslant j \leqslant n$,

$$P\{x(\beta_1^n) = j\} = P\{x^n(\beta_1^{nn}) = j\} = v_j^n$$

Hence the above formula holds true for $j = -1$, too.

(e) Suppose that $\bar{X} = \{\bar{x}(t), t < \bar{\sigma}\} \in \mathscr{H}_s(Q)$ also satisfies (6.11). By Corollary 2 to Theorem 10.5.2, $\bar{X}$ is the strong limit of the (Q, V^n) Doob processes $\bar{X}^n = g_n(\bar{X})$.

It follows that both $\bar{X}$ and X take the limit of the transition probability $p_{ij}^n(t)$ of the (Q, V^n) Doob processes as their transition probability, that is $\bar{X}$ and X have the same transition probability, and so they belong to the same process. The proof is terminated. QED

12.8 SUMMARY

Theorem 1. Assume that $X \in \mathscr{H}_s(Q)$ is a non-minimal Q process. Then its characteristic sequence $p, q, \gamma_n (n \geqslant -1)$ satisfies (6.10).

Conversely, given a sequence of non-negative numbers $p, q, \gamma_n (n \geqslant -1)$ satisfying (6.10), then there exists the unique non-minimal process $X \in \mathscr{H}_s(Q)$ whose characteristic sequence is just the given sequence $p, q, \gamma_n (n \geqslant -1)$. Moreover, we may take X as the strong limit of a sequence of (Q, V^n) Doob processes X^n, where $V^n = (v_j^n, 0 \leqslant j \leqslant n)$ is determined by (6.12) and (6.13) under $p, q, \gamma_n (n \geqslant -1)$.

For X to be honest it is necessary and sufficient that $\gamma_{-1} = 0$. X satisfies the system of forward equations if and only if $p = 0$.

It is still necessary to prove the last sentence.

If X satisfies the system of forward equations, by Theorem 10.4.4, the U intervals in $[\tau, \sigma)$ are all ${}_\infty U$ intervals and hence $P\{x(\eta) = j\} = 0$ and from (6.14) it follows that $\gamma_j = 0$ $(j \geqslant 0)$. Consequently $p = 0$. Conservely, we assume $p = 0$. From (6.14) follows $P\{\xi_{k0} < \sigma\} = 0 (k \in E)$, where ξ_{kj} is determined by (10.4.1).

In addition, for $j \in E$,

$$P\{\xi_{kj} < \sigma\} \prod_{i=1}^{j} \frac{a_i}{a_i + b_i} \leqslant P\{\xi_{k0} < \sigma\}$$

Therefore, on account of Lemma 10.4.2,

$$P\{\xi_{EE} < \sigma\} \leqslant \sum_{k,j \in E} P\{\xi_{kj} < \sigma\} = 0$$

By Theorem 10.4.4, X satisfies the system of forward equations.

CHAPTER 13

Relation Between Two Kinds of Construction Theories of Birth–Death Processes

13.1 INTRODUCTION

In Chapter 6 and Chapter 12, we have constructed conservative ($a_0 = 0$) birth–death processes by using analytical methods and probability methods respectively. We naturally ask: What is the relationship between the results for the two methods? We have completely solved the problem (Xiang-qun Yang, 1965b).

In this chapter, we suppose that $a_0 = 0$ and the boundary point z is regular or exit. Moreover, we write $X^2(\lambda)$ in (6.5.1) as $X(\lambda)$. Let $X_i(\lambda) = E_i(e^{-\lambda\tau})$, τ being the first leaping point for the Q process.

13.2 CORRESPONDING THEOREMS

Theorem 1. Let the characteristic sequence of the non-minimal process $X \in \mathscr{H}_s(Q)$ be p, q and r_n ($n \geqslant -1$). Then the resolvent operators of X are

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + X_i(\lambda) \frac{\sum_k r_k \phi_{kj}(\lambda) + dzX_j(\lambda)\mu_j}{r_{-1} + \sum_k r_k[1 - X_k(\lambda)] + dz\lambda \sum_k X_k(\lambda)\mu_k} \tag{1}$$

where d is determined by (12.6.13).

Proof. Let $\psi^n_{ij}(\lambda)$ be the resolvent operators of (Q, v^n) Doob process X^n. Then by (11.4.5), we have

$$\psi^n_{ij}(\lambda) = \phi_{ij}(\lambda) + X_i(\lambda) \frac{\sum_{k=0}^n v^n_k \phi_{kj}(\lambda)}{1 - \sum_{k=0}^n v^n_k + \sum_{k=0}^n v^n_k[1 - X_k(\lambda)]} \tag{2}$$

We denote the fraction in the above expression as $H^n_j(\lambda)$. Substituting (12.6.12)

into $H_j^n(\lambda)$ and noticing $\sum_{k=-1}^{n} v_k^n = 1$, we have

$$H_j^n(\lambda) = \frac{(X_n/A_n)\sum_{k=0}^{n-1} r_k\phi_{kj}(\lambda) + [Y_n + (X_n/A_n)\sum_{l=n}^{\infty} r_l C_{ln}]\phi_{nj}(\lambda)}{(X_n/A_n)r_{-1} + (X_n/A_n)\sum_{k=0}^{n-1} r_k[1 - X_k(\lambda)] + [Y_n + (X_n/A_n)\sum_{l=n}^{\infty} r_l C_{ln}][1 - X_n(\lambda)]}$$

$$= \frac{\sum_{k=0}^{n-1} r_k\phi_{kj}(\lambda) + (d/C_{n0} + \sum_{l=n}^{\infty} r_l C_{ln})\phi_{nj}(\lambda)}{r_{-1} + \sum_{k=0}^{n-1} r_k[1 - X_k(\lambda)] + (d/C_{n0} + \sum_{l=n}^{\infty} r_l C_{ln})[1 - X_n(\lambda)]}$$

$$= \frac{\sum_{k=0}^{n-1} r_k\phi_{kj}(\lambda) + (d + \sum_{l=n}^{\infty} r_l C_{l0})\phi_{nj}(\lambda)/C_{n0}}{r_{-1} + \sum_{k=0}^{n-1} r_k[1 - X_k(\lambda)] + (d + \sum_{l=n}^{\infty} r_l C_{l0})[1 - X_n(\lambda)]/C_{n0}}$$

Note that $A_0 = \sum_{l=0}^{\infty} r_l C_{l0} < \infty$, and $d = 0$ if z is exit. By Lemmas 6.9.1 and 6.9.2, we obtain

$$\lim_{n\to\infty} H_j^n(\lambda) = \frac{\sum_k r_k\phi_{kj}(\lambda) + dzX_j(\lambda)\mu_j}{r_{-1} + \sum_k r_k[1 - X_k(\lambda)] + dz\lambda\sum_k X_k(\lambda)\mu_k}.$$

Since $X = \lim_{n\to\infty} X^n$, it follows that $\psi_{ij}(\lambda) = \lim_{n\to\infty}\psi_{ij}^n(\lambda)$, hence the theorem holds. And the proof is terminated. QED

By Theorems 6.6.1, 6.9.3 and 6.9.4, it follows that each non-minimal Q process $\psi(\lambda)$ has the following representation:

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + X_i(\lambda)\frac{\sum_k \alpha_k\phi_{kj}(\lambda) + DX_j(\lambda)\mu_j}{c + \sum_i \alpha_k[1 - X_k(\lambda)] + D\lambda\sum_k X_k(\lambda)\mu_k} \tag{3}$$

where the row vector $\alpha \geqslant 0$ satisfies (6.9.5), constant $D \geqslant 0$, and furthermore $D = 0$ if z is exist. Moreover, $\sum_k \alpha_k\phi_{kj}(\lambda) + DX_j(\lambda)\mu_j \not\equiv 0$, and the constant $c \geqslant 0$.

We point out that, except for a constant factor, the vector α, the constants c and D are uniquely determined by the process. In fact, suppose that α, c, D and $\bar{\alpha}, \bar{c}, \bar{D}$ correspond to the same process, then

$$\frac{\alpha\phi(\lambda) + DX(\lambda)\mu}{A_\lambda} = \frac{\bar{\alpha}\phi(\lambda) + \bar{D}X(\lambda)\mu}{\bar{A}_\lambda} \tag{4}$$

where $A_\lambda = c + [\alpha, 1 - X(\lambda)] + D\lambda[X(\lambda)\mu, 1]$, $\bar{A}_\lambda$ being the quantity corresponding to $\bar{\alpha}$, $\bar{c}$ and $\bar{D}$. Multiplying both sides of (4) by $\lambda I - Q$, we obtain

$$\alpha/A_\lambda = \bar{\alpha}/\bar{A}_\lambda$$

hence

$$D/A_\lambda = \bar{D}/\bar{A}_\lambda$$

Therefore $K = A_\lambda/\bar{A}_\lambda > 0$ and is independent of λ. Hence $\alpha = K\bar{\alpha}$ and $D = K\bar{D}$. Again substituting into (4) we get $c = K\bar{c}$.

Definition 1. The process in (3) is called a (Q, α, c, D) process.

Theorem 2. The characteristic sequence p, q and r_n $(n \geqslant -1)$ of a (Q, α, c, D) process is

$$r_{-1} = c \qquad r_n = \alpha_n\,(n \geqslant 0)$$

$$p = \begin{cases} 0 & \text{if } \alpha = 0 \\ A_0 z/(A_0 z + D), & \text{if } \alpha \neq 0 \end{cases} \qquad q = \begin{cases} 1 & \text{if } \alpha = 0 \\ D/(A_0 z + D) & \text{if } \alpha \neq 0 \end{cases} \tag{5}$$

where $A_0 = \sum_{l=0}^{\infty} r_l C_{l0}$.

Proof. Comparing (1) and (3), we know that r_n and α_n differ by a constant factor. We may as well consider $r_n = \alpha_n$ $(n \geqslant 0)$, so that $r_{-1} = c$ and $dz = D$. From this (5) follows. The proof is over. QED

13.3 PROPERTIES OF THE PROCESS AT THE FIRST LEAPING POINT

On account of the corresponding theorem, it becomes clear how the (Q, α, c, D) process constructed by means of the analytical method moves and, moreover, some probability quantities of the process can be computed.

Theorem 1. Let $X \in \mathscr{H}_s(Q)$ be a (Q, α, c, D) process, and β_1^n be determined by (12.6.1). Then the probability $v_j^n = P\{x(\beta_1^n) = j\}$ $(-1 \leqslant j \leqslant n)$ is calculated as follows:

$$v_{-1}^n = (X_n/A_n)c \qquad v_j^n = (X_n/A_n)\alpha_j\,(0 \leqslant j < n)$$

$$v_n^n = Y_n + (X_n/A_n)\sum_{l=n}^{\infty}\alpha_l C_{ln} \tag{1}$$

where

$$A_n = \sum_{l=0}^{\infty}\alpha_l C_{ln} \qquad d = \begin{cases} D/z & \text{if } \alpha \neq 0 \\ 1 & \text{if } \alpha = 0 \end{cases}$$

$$X_n = \frac{A_n C_{n0}}{(c + A_n)C_{n0} + d} \qquad Y_n = \frac{d}{(c + A_n)C_{n0} + d} \tag{2}$$

$$\frac{X_n}{A_n} = \frac{C_{n0}}{cC_{n0} + d} \qquad \text{if } \alpha = 0$$

Proof. The conclusion of this theorem follows from Theorem 2.2 and (12.6.3). QED

Theorem 2. Let $X \in \mathscr{H}_\delta(Q)$ be a (Q, α, c, D) process, τ be the first leaping point. That $D > 0$ or $[\alpha, 1] = \infty$ is called case A; that $D = 0$ and $[\alpha, 1] < \infty$ is called case B. Then for $0 \leqslant i < \infty$,

$$P\{x(\tau) = i\} = \begin{cases} 0 & \text{case A} \\ \alpha_i/(c + [\alpha, 1]) & \text{case B} \end{cases}$$

$$P\{\tau = \sigma < \infty\} = P\{x(\tau) = -1\} = \begin{cases} 0 & \text{case A} \\ c/(c + [\alpha, 1]) & \text{case B} \end{cases}$$

$$P\{x(\tau) = \infty\} = \begin{cases} 1 & \text{case A} \\ 0 & \text{case B} \end{cases}$$

Proof. By (12.5.11) and the right-continuity of X, we obtain

$$\{x(\tau) = i\} = \lim_{n \to \infty} \{x(\beta_1^n) = i\} \qquad -1 \leqslant i < \infty$$

By Theorem 1, we have

$$P\{x(\tau) = i\} = \lim_{n \to \infty} \frac{X_n}{A_n} a_i \qquad i \geqslant 0$$

$$P\{x(\tau) = -1\} = P\{\tau = \sigma < \infty\} = \lim_{n \to \infty} \frac{X_n}{A_n} c$$

But

$$\frac{X_n}{A_n} = \frac{zC_{n0}}{z(c + A_n)C_{n0} + D}$$

If $D > 0$, obviously, $X_n/A_n \to 0$. If $D = 0$, then

$$\frac{X_n}{A_n} = \frac{1}{c + A_n} \to \frac{1}{c + [\alpha, 1]} \tag{3}$$

In fact, since $A_n = \sum_{l=0}^{n} \alpha_l + \sum_{l=n}^{\infty} \alpha_l C_{ln}$, it follows that, if $[\alpha, 1] = \infty$, obviously (3) holds. If $[\alpha, 1] < \infty$, then (3) follows from $\sum_{l=n}^{\infty} \alpha_l C_{ln} \leqslant \sum_{l=n}^{\infty} \alpha_l \to 0$. Again noticing $P\{x(\tau) = \infty\} = 1 - \sum_{i=-1}^{\infty} P\{x(\tau) = i\}$ we obtain the conclusion of this theorem. The proof is concluded. QED

PART VI PROPERTIES OF MARKOV PROCESSES RELATED TO CONSTRUCTION THEORY

CHAPTER 14

Properties of Birth–Death Processes

14.1 INTRODUCTION

Birth–death processes constructed with the probability method have clear path structures. Every birth–death process is the strong limit of a sequence of Doob processes. Therefore, in order to study the properties of a Q process we only need to study the limits. In this chapter we shall carry out this procedure by studying distributions of first return times. We assume $a_0 = 0$ and adopt the notations given in Chapter 6, such as the increasing solutions $u(\lambda)$ and the decreasing solutions $v(\lambda)$; we also write

$$X(\lambda) = X^2(\lambda) = \frac{u(\lambda)}{u(z, \lambda)}$$

14.2 SOME FINE RESULTS OF THE MINIMAL PROCESS

Assume that $X = \{x(t), t < \sigma\} \in \mathscr{H}_s(Q)$, and that τ is the first leaping point and τ_1 is the first jumping point. Let

$$\eta_i = \begin{cases} \inf\{t \mid \tau_1 \leqslant t \leqslant \sigma, x(t) = i\} \\ \sigma \qquad \text{if the above set is empty} \end{cases} \tag{1}$$

be the time of the process returning to i for the first time. Zi-kun Wang and Xiang-qun Yang (1988, section 5.2) have pointed out that there exists $h = h(j) > 0$ such that for $\lambda > -h$, $E_k\{e^{-\lambda\eta_j}\}(k < j)$ are finite and are the unique solutions of the equations

$$\begin{aligned} D_\mu u_k^+ &= \lambda\mu_k \qquad k < j \\ u_j &= 1 \end{aligned} \tag{2}$$

For $k < j$ the moments ${}_1N_{kj}^l = E_k\{\eta_j^l\} = E_k\{\eta_j^l, \eta_j < \tau\}$ $(k < j)$ are finite and satisfy the relation

$$\begin{aligned} {}_1N_{kj}^l &= l\sum_{i=k}^{j-1}(z_{j+1} - z_j)\sum_{s=0}^{i} {}_1N_{sj}^{l-1}\mu_s \qquad (k < j) \\ {}_1N_{kj}^0 &= 1 \end{aligned} \tag{3}$$

Moreover $N_k^l = E_k\{\tau^l\}$ satisfy

$$N_k^l = l\sum_{i=k}^{\infty}(z_{i+1}-z_i)\sum_{s=0}^{i} N_s^{l-1}\mu_s \leqslant l!R^l \qquad k\geqslant 0$$
$$N_k^0 = 1 \tag{4}$$

Hence, one has

$$E_k\{e^{-\lambda\eta_j}\} = \frac{u_k(\lambda)}{u_j(\lambda)} \qquad (k<j)$$
$$E_k\{e^{-\lambda\tau}\} = X_k(\lambda) \qquad \lambda>0 \tag{5}$$

and

$$E_k\{e^{-\lambda\eta_j}\} = \sum_{l=0}^{\infty}(-\lambda)^l\frac{{}_1N_{kj}^l}{l!} \qquad (|\lambda|<h, k<j) \tag{6}$$

When $R<\infty$,

$$E_k\{e^{-1\tau}\} = \sum_{l=0}^{\infty}(-\lambda)^l\frac{N_k^l}{l!} \qquad |\lambda|<\frac{1}{R} \tag{7}$$

Now, for $i\leqslant k<j$ and $i+1<j$, let

$$\begin{aligned}{}_1\varphi_{kij}(\lambda) &= E_k\{e^{-\lambda\eta_i}, \eta_i<\eta_j\}\\ {}_2\varphi_{kij}(\lambda) &= E_k\{e^{-\lambda\eta_j}, \eta_j<\eta_i\}\end{aligned} \tag{8}$$

$${}_1N_{kij}^l = E_k\{\eta_i^l, \eta_i<\eta_j\} \qquad {}_2N_{kij}^l = E_i\{\eta_j^l, \eta_j<\eta_i\} \tag{9}$$

$${}_1\varphi_{kj}(\lambda) = E_k\{e^{-\lambda\eta_i}, \eta_i<\tau\} \qquad {}_2\varphi_{ki}(\lambda) = E_k\{e^{-\lambda\tau}, \tau\leqslant\eta_i\} \tag{10}$$

$${}_1N_{ki}^l = E_k\{\eta_i^l, \eta_i<\tau\} \qquad {}_2N_{ki}^l = E_k\{\tau^l, \tau\leqslant\eta_i\} \tag{11}$$

Clearly, $P_k\{\eta_j\uparrow\tau\}=1$ as $j\uparrow\infty$ and, for $i\leqslant k$ as $j\uparrow\infty$

$${}_1\varphi_{kij}(\lambda)\uparrow{}_1\varphi_{ki}(\lambda) \qquad {}_1\varphi_{kij}(\lambda)+{}_2\varphi_{kij}(\lambda)\uparrow{}_1\varphi_{ki}(\lambda)+{}_2\varphi_{ki}(\lambda) \tag{12}$$

$${}_1N_{kij}^l\uparrow{}_1N_{ki}^l \qquad {}_1N_{kij}^l+{}_2N_{kij}^l\uparrow{}_1N_{ki}^l+{}_2N_{ki}^i \tag{13}$$

Theorem 1. (i) There exists $h=h(j)>0$ such that for $\lambda>-h$ ${}_a\varphi_{kij}(\lambda)$ $(a=1,2,$ $i\leqslant k<j)$ are finite, all moments ${}_aN_{kij}^l$ $(a=1,\ 2,\ l\geqslant 0,\ i\leqslant k<j)$ are finite and they satisfy the relations:

$${}_aN_{kij}^l = l\left(\frac{z_j-z_k}{z_j-z_i}\sum_{s=i+1}^{k}(z_s-z_i)_aN_{sij}^{l-1}\mu_s + \frac{z_k-z_i}{z_j-z_i}\sum_{s=k+1}^{j-1}(z_j-z_s)_aN_{sij}^{l-1}\mu_s\right)$$
$$i<k<j$$

$${}_1N_{kij}^0 = \frac{z_j-z_k}{z_j-z_i} \qquad {}_2N_{kij}^0 = \frac{z_k-z_i}{z_j-z_i} \qquad i<k<j \tag{14}$$

$$\begin{aligned}{}_1N_{iij}^l &= \frac{1}{q_i}(l_1N_{iij}^{l-1}+a_{i1}N_{i-1,i}^l+b_{i1}N_{i+1,ij}^l)\\ {}_1N_{iij}^0 &= \frac{a_i}{q_i}+\frac{b_i}{q_i}\frac{z_j-z_{i+1}}{z_j-z_i}\end{aligned} \tag{15}$$

$$\begin{aligned}{}_2N_{iij}^l &= \frac{1}{q_i}(l_2N_{iij}^{l-1}+b_{i2}N_{i+1,ij}^l)\\ {}_2N_{iij}^0 &= \frac{b_i}{q_i}\frac{z_{i+1}-z_i}{z_j-z_i}\end{aligned} \tag{16}$$

$${}_a\varphi_{kij}(\lambda) = \sum_{l=0}^{\infty}(-\lambda)^l\frac{{}_aN_{kij}^l}{l!} \qquad (i\leqslant k<j, |\lambda|<h) \tag{17}$$

$$\begin{aligned}{}_1\varphi_{kij}(\lambda) &= \frac{u_j(\lambda)v_k(\lambda)-u_k(\lambda)v_j(\lambda)}{u_j(\lambda)v_k(\lambda)-u_i(\lambda)v_j(\lambda)}\\ {}_2\varphi_{kij}(\lambda) &= \frac{u_k(\lambda)v_i(\lambda)-u_i(\lambda)v_k(\lambda)}{u_j(\lambda)v_i(\lambda)-u_i(\lambda)v_j(\lambda)}\end{aligned} \qquad (\lambda>0, i<k<j) \tag{18}$$

$$\begin{aligned}{}_1\varphi_{iij}(\lambda) &= \frac{1}{\lambda+q_i}\left(a_i\frac{u_{i-1}(\lambda)}{u_i(\lambda)}+b_{i1}\varphi_{i+1,ij}(\lambda)\right)\\ {}_2\varphi_{iij}(\lambda) &= \frac{1}{\lambda+q_i}b_{i2}\varphi_{i+1,ij}(\lambda)\end{aligned} \qquad (\lambda>0) \tag{19}$$

(ii) The following also hold:

$${}_1\varphi_{ki}(\lambda) = \frac{v_k(\lambda)}{v_i(\lambda)} \qquad {}_2\varphi_{ki}(\lambda) = X_k(\lambda)-X_i(\lambda)\frac{v_k(\lambda)}{v_i(\lambda)} \qquad (i<k, \lambda>0) \tag{20}$$

$$\begin{aligned}{}_1\varphi_{ii}(\lambda) &= \frac{1}{\lambda+q_i}\left(a_i\frac{u_{i-1}(\lambda)}{u_i(\lambda)}+b_i\frac{v_{i+1}(\lambda)}{v_i(\lambda)}\right) \qquad \lambda>0\\ {}_2\varphi_{ii}(\lambda) &= \frac{b_i}{\lambda+q_i}\left(X_{i+1}(\lambda)-X_i(\lambda)\frac{v_{i+1}(\lambda)}{v_i(\lambda)}\right) \qquad \lambda>0\end{aligned} \tag{21}$$

$$\begin{aligned}{}_aN_{ki}^l &= l\left(\frac{z-z_k}{z-z_i}\sum_{s=i+1}^{k}(z_s-z_i)_aN_{si}^{l-1}\mu_s+\frac{z_k-z_i}{z-z_i}\sum_{s=k+1}^{\infty}(z-z_s)_aN_{si}^{l-1}\mu_s\right)\\ &\leqslant l!R^{l-1}N_k\leqslant lR^l \qquad (i<k,\ a=1 \text{ or } R<\infty \text{ as } a=2)\end{aligned} \tag{22}$$

$${}_1N_{ki}^0 = \frac{z-z_k}{z-z_i} \qquad {}_2N_{ki}^0 = \frac{z_k-z_i}{z-z_i} \qquad (i<k)$$

$$
{}_1N_{ii}^l = \frac{1}{q_i}(l_1N_{ii}^{l-1} + a_{i1}N_{i-1,i}^l + b_{i1}N_{i+1,i}^l) \leqslant l!R^l
$$

$$
{}_1N_{ii}^0 = \frac{a_i}{q_i} + \frac{b_i}{q_i}\frac{z - z_{i+1}}{z - z_i} \tag{23}
$$

$$
{}_2N_{ii}^l = \frac{1}{a_i}(l_2N_{ii}^{l-1} + b_{i2}N_{i+1,i}^l) \leqslant l!R^l
$$

$$
{}_2N_{ii}^0 = \frac{b_i}{q_i}\frac{z_{i+1} - z_i}{z - z_i} \tag{24}
$$

When $R < \infty$,

$$
{}_a\varphi_{ki}(\lambda) = \sum_{l=0}^{\infty} (-\lambda)^l \frac{{}_aN_{ki}^l}{l!} \qquad (i \geqslant k, |\lambda| < 1/R) \tag{25}
$$

Proof. (i) Since for $\lambda > -h$, $E_k\{e^{-\lambda\eta_j}\}$ $(k > j)$ and $E_k\{\eta_j^l\}$ $(k < j)$ are finite, it follows that ${}_a\varphi_{kij}(\lambda)$ and ${}_aN_{kij}^l$ are finite, and by Wilks (1962, p. 14),

$$
{}_aN_{kij}^l = (-1)^l \frac{d^l}{d\lambda^l}\, {}_a\varphi_{kij}(\lambda)\bigg|_{\lambda=0} \tag{26}
$$

and (17) holds.

The formula (19) follows from the strong Markov property. Multiplying both sides of (19) by $(\lambda + q_i)$, differentiating them l times and noting (26) and (5)–(6), we obtain the first expressions of (15) and (16). Obviously,

$$
{}_1N_{iij}^0 = P_i\{\eta_i < \eta_j\} = \frac{a_i}{q_i}P_{i-1}\{\eta_i < \eta_j\} + \frac{b_i}{q_i}P_{i+1}\{\eta_i < \eta_j\}
$$

$$
{}_2N_{iij}^0 = \frac{b_i}{q_i}P_{i+1}\{\eta_j < \eta_i\}
$$

Therefore the second expressions of (15)–(16) follow from (12.2.8).

Making use of the strong Markov property we find that ${}_a\varphi_{kij}(\lambda)$ $(i < k < j$, $\lambda > 0)$ satisfy the following equations:

$$
\begin{aligned}
u_i &= [1 - (-1)^a]/2 \\
(\lambda + a_k + b_k)u_k - a_ku_{k-1} - b_ku_{k+1} &= 0 \qquad (i < k < j) \\
u_j &= [1 + (-1)^a]/2
\end{aligned}
$$

Solve the above equations to obtain (18). Differentiating the above equations and noting (26), we see that $u_k = {}_aN_{kij}^l$ $(i < k < j$, $l \geqslant 1)$ satisfy equation (5.4.11) with $f_i = 0$, $f_k = l_aN_{kij}^{l-1}$ $(i < k < j)$ and $f_j = 0$. According to Theorem 6.2.3 and (5.4.12) the first expression of (14) is justified. The second one follows from (12.2.8).

(ii) On account of (12), let $j \uparrow \infty$ in (18)–(19) to obtain (20)–(21).

When $R < \infty$, by (7) for $k \geqslant i$ we have

$$
{}_1N_{ki}^l + {}_2N_{ki}^l = E_k\{\min(\eta_i, \tau)\}^l \leqslant E_k\tau^l \leqslant l!R^l < \infty
$$

Hence (25) holds.

The second expression in (22) follows from (12.2.16). From (13) we can obtain (23) and (24) merely by taking limits in the corresponding expressions in (i). Let $j \uparrow \infty$ in the first expression of (14); then according to the monotone convergence theorem we find that (22) holds for $a = 1$ and that if we substitute ${}_1N_{ki}^l + {}_2N_{ki}^l$, ${}_1N_{si}^{l-1} + {}_2N_{si}^{l-1}$ for ${}_1N_{ki}^l$, ${}_1N_{si}^{l-1}$ respectively in (22), the corresponding formulae still hold and ${}_1N_{ki}^l + {}_2N_{ki}^l \leqslant l!R^l < \infty$. Therefore for $a = 2$, (22) also holds. The proof is completed. QED

Theorem 2. The minimal solution is recurrent if and only if $z = \infty$; more precisely,

$$
\int_0^{\infty} f_{ij}(t)\,dt = \lim_{\lambda \downarrow 0} \phi_{ij}(\lambda) = \Gamma_{ij} = \begin{cases} (z - z_j)\mu_j & \text{if } j \geqslant i \\ (z - z_i)\mu_j & \text{if } j < i \end{cases} \tag{27}
$$

Proof. It follows from (6.3.2) that

$$
u_i(\lambda) \downarrow 1 \qquad (\lambda \downarrow 0) \tag{28}
$$

By (6.3.4) $v_i(\lambda) \to \sum_{j=i}^{\infty}(z_{j+1} - z_j) = z - z_i$ as $\lambda \downarrow 0$. So (27) follows and the proof is completed. QED

Theorem 3. Suppose $z = \infty$. Then the minimal solution is ergodic if and only if $\sum_{i=0}^{\infty} \mu_i < \infty$.

Proof. According to (12.2.17) we have ${}_1N_{i-1,i}^1 = m_{i-1}$. When $z = \infty$, by (22)–(23) it follows that ${}_1N_{ii}^0 = 1$ and

$$
E_i\eta_i = {}_1N_{1i}^i = \frac{1}{q_i}(1 + a_im_{i-1} + b_{i1}N_{i+1,i}^1)
$$

$$
{}_1N_{i+1,i}^1 = (z_{i+1} - z_i)\sum_{s=i+1}^{\infty} u_s
$$

Hence $E_i\{\eta_i\} < \infty$ is equivalent to $\sum_{s=0}^{\infty} u_s < \infty$. QED

Theorem 4. Suppose $R < \infty$ and $S < \infty$. Then

$$
\lim_{n \to \infty} \frac{{}_1\varphi_{nj}(\lambda)}{z - z_n} = \frac{X_0(\lambda)}{v_j(\lambda)} \qquad \lambda > 0 \tag{29}
$$

$$
\lim_{n \to \infty} \frac{1 - {}_2\varphi_{nj}(\lambda)}{z - z_n} = \lambda\sum_k X_k(\lambda)\mu_k + \frac{X_0(\lambda)X_j(\lambda)}{v_j(\lambda)} \qquad \lambda > 0 \tag{30}
$$

Proof. According to Lemmas 6.9.1 and 6.9.2, (29)–(30) are deduced from (20) immediately.

QED

14.3 INVARIANT MEASURES OF THE MINIMAL PROCESSES

Suppose $R < \infty$. If a process is recurrent, it must be honest. Conversely, according to Theorem 12.4.2 an honest process is recurrent and ergodic. We will find the invariant measure.

Theorem 1. Suppose X is a $(Q, a, 0, D)$ process. Then its invariant measure is

$$\pi_j = \lim_{t\to\infty} p_{ij}(t) = \frac{\sum_k a_k \Gamma_{kj} + D\mu_j}{\sum_k a_k N_k + D \sum_k \mu_k} \tag{1}$$

and

$$m_{ii} = E_i \eta_i = \frac{\sum_k a_k \Gamma_{ki} + D\mu_i}{q_i\left(\sum_k a_k N_k + D\sum_k \mu_k\right)} \tag{2}$$

Here Γ_{kj} are determined by (2.27) and N_k by (6.2.6).

Proof. By (7.7.10) we know that the limits in (1) exist. By Tauber's theorem (Hardy, 1949, Theorem 98), $\lim_{\lambda\downarrow 0} \lambda\psi_{ij}(\lambda) = \lim_{t\to\infty} p_{ij}(t)$. Because $P_i\{\tau < \infty\} = 1$, $X_i(\lambda) = E_i\{e^{-\lambda\tau}\} \uparrow 1$ as $\lambda \downarrow 0$ and

$$\frac{1 - X_i(\lambda)}{\lambda} \sum_j \phi_{ij}(\lambda) \uparrow \sum_j \Gamma_{ij} = N_i \qquad \text{as } \lambda\downarrow 0$$

Note that $D = 0$ if z is exist, and $\sum_k u_k < \infty$ if z is regular. Therefore (1) follows from (13.2.3), and (2) follows from (1) and (7.7.10). We conclude the proof.

QED

Theorem 2. Suppose that $X \in \mathscr{X}_s(Q)$ is a $(Q, a, 0, D)$ process and π is its invariant measure. Suppose that functions f and g satisfy $\sum_i |f(i)| \pi_i < \infty$, $\sum_i |g(i)| \pi_i < \infty$ and $\sum_i g(i)\pi_i \neq 0$. Then

$$P\left\{\lim_{t\to\infty} \frac{\int_0^t f[x(u)]\,du}{\int_0^t g[x(u)]\,du} = \frac{\sum_i f(i)\pi_i}{\sum_i g(i)\pi_i}\right\} = 1 \tag{3}$$

Proof. See Zhang-nan Li nd Rong Wu (1964), Theorem 3.1). QED

14.4 DISTRIBUTION OF THE FIRST RETURNING TIME

Theorem 1. Suppose that $X \in \mathscr{H}_s(Q)$ is a $(Q, \alpha, 0, D)$ process and η_j is defined by (2.1). Then for $\lambda > 0$,

$$E_i\{e^{-\lambda\eta_j}\} = \frac{u_i(\lambda)}{u_j(\lambda)} \qquad (i < j) \tag{1}$$

$$E_i\{e^{-\lambda\eta_j}\} = 1 - m(\lambda)(\lambda + q_i)^{-1}\left[m(\lambda)\phi_{ii}(\lambda) + X_i(\lambda)\left(\sum_k a_k\phi_{ki}(\lambda) + DX_i(\lambda)u_i\right)\right]^{-1} \tag{2}$$

$$E_i\{e^{-\lambda\eta_j}\} = \left[m(\lambda)\phi_{ij}(\lambda) + X_i(\lambda)\left(\sum_k a_k\phi_{kj}(\lambda) + DX_j(\lambda)\mu_j\right)\right]$$
$$\cdot\left[m(\lambda)\phi_{jj}(\lambda) + X_j(\lambda)\left(\sum_k a_k\phi_{kj}(\lambda) + DX_j(\lambda)u_j\right)\right]^{-1} \qquad (i > j) \tag{3}$$

where

$$m(\lambda) = \sum_k a_k[1 - X_k(\lambda)] + D\lambda\sum_k X_k(\lambda)\mu_k \tag{4}$$

or equivalently,

$$E_i\{e^{-\lambda\eta_j}\} = {}_1\varphi_{ij}(\lambda) + {}_2\varphi_{ij}(\lambda)\left(\sum_{k=0}^{j} a_k \frac{u_k(\lambda)}{u_j(\lambda)} + \sum_{k=j+1}^{\infty} a_k \frac{v_k(\lambda)}{v_i(\lambda)} + D\frac{X_0(\lambda)}{v_j(\lambda)}\right)$$
$$\cdot\left[\sum_{k=0}^{j} a_k + \sum_{k=j+1}^{\infty} a_k\left(1 - X_k(\lambda) + X_j(\lambda)\frac{X_0(\lambda)}{v_j(\lambda)}\right)\right]^{-1} \qquad (i \geqslant j) \tag{5}$$

where ${}_a\varphi_{ij}(\lambda)$ are determined by (2.20) and (2.21).

Proof. The formula (1) is just (2.5). By the well known formulae in Chung (1969a, pp. 192–3),

$$p_{ii}(t) = e^{-q_i t} + \int_0^t p_{ii}(t-u)\,dP_i \qquad (\eta_i \leqslant u)$$

$$p_{ij}(t) = \int_0^t p_{jj}(t-u)\,dP_i \qquad (\eta_i \leqslant u, i \neq j)$$

Taking Laplace transformations, noting (13.2.3) and making some simple rearrangements, we can obtain (2)–(3).

Now consider (Q, V^n) Doob processes. Let σ_h be the hth leaping point of X. By recurrence $P_i\{\eta_j < \infty\} = 1$ for $i \geqslant j$. It follows from the structure of Doob

processes that

$$E_i\{e^{-\lambda\eta_j}\} = E_i\{e^{-\lambda\eta_j}, \eta_j < \sigma_1\} + \sum_{h=1}^{\infty} E_i\{e^{-\lambda\eta_j}, \sigma_h \leqslant \eta_j < \sigma_{h+1}\}$$

$$= {}_1\varphi_{ij}(\lambda) + \sum_{h=1}^{\infty} E_i\{e^{-\lambda\sigma_1}, \sigma_1 \leqslant \eta_j\}\left(\sum_{k=i+1}^{\infty} v_k E_k(e^{-\lambda\sigma_1}, \sigma_1 \leqslant \eta_j)\right)^{h-1}$$

$$\cdot\left(\sum_{h=0}^{j-1} v_k E_k(e^{-\lambda\eta_j}) + v_i + \sum_{k=j+1}^{\infty} v_k E_k(e^{-\lambda\eta_j}, \eta_j \leqslant \sigma_1)\right)$$

$$= {}_1\varphi_{ij}(\lambda) + {}_2\varphi_{ij}(\lambda)\left(\sum_{k=0}^{j} v_k \frac{u_k(\lambda)}{u_j(\lambda)} + \sum_{k=j+1}^{\infty} v_k \frac{v_k(\lambda)}{v_j(\lambda)}\right)\left(1 - \sum_{k=j+1}^{\infty} v_{k\,2}\varphi_{kj}(\lambda)\right)^{-1}$$

$$= {}_1\varphi_{ij}(\lambda) + {}_2\varphi_{ij}(\lambda)\left(\sum_{k=0}^{j} v_k \frac{u_k(\lambda)}{u_j(\lambda)} + \sum_{k=j+1}^{\infty} v_k \frac{v_k(\lambda)}{v_j(\lambda)}\right)$$

$$\cdot\left(\sum_{k=0}^{j} v_k + \sum_{k=j+1}^{\infty} v_k(1 - {}_2\varphi_{kj}(\lambda))\right)^{-1} \tag{6}$$

For a general $(Q, a, 0, D)$ process X, it is the strong limit of a sequence of honest (Q, V^n) Doob processes X^n. Define η_i and η_i^n for X and X^n, respectively. Then $\eta_i^n \uparrow \eta_i$ as $n \uparrow \infty$.

Substituting V^n in (13.3.1) into (6), letting $n \to \infty$ and noting (6.9.2)–(6.9.3), (2.30) and (2.20) we can obtain (5) by a simple calculation. The proof is terminated. QED

CHAPTER 15

Recurrence and Ergodic Properties

15.1 INTRODUCTION

The recurrence and ergodic properties of processes are extremely useful for the study of approximation properties of their transition functions at infinity. They are also important for the study of excessive functions, zero–one law (see Zi-kun Wang, 1964, 1965c, 1966, 1980) and reversibility (see Min Qian and Zhen-ting Hou, 1979). General studies for the classification of Markov processes have been summed up in Chung (1967, II, section 10). Li-de Wu (1965) and Miller (1963) studied the classification of states of minimal Q processes and found the necessary and sufficient conditions related to the Q matrix, under which Q processes are recurrent or ergodic.

This chapter studies the classification of states of Q processes, which is closely related to construction theory. This kind of classification depends not only on Q but also on the construction of processes. Hence, different constructions need different treatments. The content of this chapter is derived from Xiang-qun Yang (1980d).

15.2 TWO LEMMAS

We call i, $\psi(\lambda)$-recurrent or ergodic if the state i is recurrent or ergodic with respect to $\psi(\lambda)$. We say a process is recurrent or ergodic if all its states are recurrent or ergodic. Obviously, i is $\psi(\lambda)$-recurrent if and only if $\lim_{\lambda\downarrow 0}\psi_{ii}(\lambda) = \infty$. By (7.7.9) and Tauber's theorem, i is $\psi(\lambda)$-ergodic if and only if $\lim_{\lambda\downarrow 0}\lambda\psi_{ii}(\lambda) = \pi_i > 0$. And in terms of m_{ii} defined in (7.7.8), we have

$$\pi_i = 1/q_i m_{ii} \tag{1}$$

In the following, the notation $i \overset{\psi(\lambda)}{\Rightarrow} j$ means that i may reach j with respect to $\psi(\lambda)$ (i.e. there exists $\lambda > 0$ such that $\psi_{ij}(\lambda) > 0$). We always have $i \overset{\psi(\lambda)}{\Rightarrow} i$.

Suppose $X \in \mathscr{H}_s(Q)$, τ_1 and τ are the first point and leaping point, respectively.

Lemma 1. As $\lambda \downarrow 0$,

$$\phi_{ij}(\lambda) \uparrow \Gamma_{ij} = \sum_{n=0}^{\infty} \pi_{ij}^n \frac{1}{q_j} \leqslant \Gamma_{jj} \tag{2}$$

$$\xi_i(\lambda) \equiv E_i\{e^{-\lambda\tau}\} \uparrow \xi_i = P_i\{\tau < \infty\} \tag{3}$$

$$\frac{1-\xi_i(\lambda)}{\lambda} \uparrow N_i = \sum_j \Gamma_{ij} = E_i \tau \tag{4}$$

where (π_{ij}) is defined as in (2.9.7) and

$$\xi_i(\lambda) = 1 - \lambda \sum_j \phi_{ij}(\lambda) \tag{5}$$

Proof. The formula (2) is just (2.10.28). The inequality $\Gamma_{ij} \leqslant \Gamma_{jj}$ follows from Theorem 7.2.1. Imitate the proof of Theorem 7.12.3, and we obtain

$$\lambda \sum_j \phi_{ij}(\lambda) = 1 - E_i\{e^{-\lambda\tau}\} \tag{6}$$

Comparing this with (5), we obtain $\xi_i(\lambda) = E_i\{e^{-\lambda\tau}\}$. Moreover (3) is proved. Finally,

$$\frac{1-\xi_j(\lambda)}{\lambda} = \sum_i \phi_{ij}(\lambda) = E_i\left\{\int_0^\tau e^{-\lambda t}\,dt\right\} \tag{7}$$

From this (4) follows. QED

Lemma 2. Suppose X is a $\psi(\lambda)$-process and i is $\psi(\lambda)$-recurrent. Then $P_i\{\sigma = \infty\} = 1$. If i is $\psi(\lambda)$-recurrent and $\phi(\lambda)$-non-recurrent, then $P_i\{\tau < \infty\} = 1$.

Proof. The first claim is clear because if i is $\psi(\lambda)$-recurrent, then on $\{x(0) = i\}$ x returns to i for infinitely many times. But according to Theorem 7.7.3, the nth sojourn times at i, ρ_i^n $(n \geqslant 0)$, are independent of each other. So $\sigma \geqslant \sum_{n=0}^{\infty} \rho_i^n = \infty$.

By Theorem 7.7.4, if i is $\psi(\lambda)$-recurrent then

$$P_i\{X(\omega) \text{ has infinitely many } i\text{-intervals in } [0, \infty)\} = 1$$

If i is $\phi(\lambda)$-non-recurrent, then

$$P_i\{X(\omega) \text{ has only finitely many } i\text{-intervals in } [0, \tau(\omega))\} = 1$$

Therefore $P_i\{\tau < \infty\} = 1$. We conclude the proof. QED

15.3 DOOB PROCESSES

Theorem 1. Suppose $\psi(\lambda)$ is a (Q, π) Doob process and i is $\phi(\lambda)$- non-recurrent. Then i is $\psi(\lambda)$-recurrent if and only if $\xi_i = 1$, $\sum_k \pi_k \xi_k = 1$ and there exists k such that $\pi_k > 0$ and

$$k \overset{\phi(\lambda)}{\Rightarrow} i \tag{1}$$

If i is $\psi(\lambda)$-recurrent, then i is ergodic if and only if

$$\sum_k \pi_k N_k < \infty \tag{2}$$

Remark

Because $\sum_k \pi_k \leqslant 1$ and $\xi_k \leqslant 1$, if (1) holds, necessarily $\sum_k \pi_k = 1$, i.e. $\psi(\lambda)$ is honest.

Proof. From the hypothesis, $\Gamma_{ii} < \infty$. By Lemma 2.1 let $\lambda \downarrow 0$ in (11.4.5) so that we find that

$$\lim_{\lambda\downarrow 0} \psi_{ij}(\lambda) = \Gamma_{ii} + \xi_i \frac{\sum_k \pi_k \Gamma_{kl}}{1 - \sum_k \pi_k \xi_k} \tag{3}$$

Suppose that i is $\psi(\lambda)$-recurrent. Then the left-hand side of (3) is infinite. According to Lemma 2.2, $\xi_i = 1$. Moreover, surely

$$\sum_k \pi_k \Gamma_{k_i} \leqslant (\textstyle\sum_k \pi_k)\Gamma_{ii} < \infty$$

and is positive, for otherwise (3) becomes $\lim_{\lambda\downarrow 0} \psi_{ii}(\lambda) = \Gamma_{ii} < \infty$, a contradiction. Since $\sum_k \pi_k \Gamma_{ki} > 0$, it follows that there exist $\pi_k > 0$ and $\Gamma_{ki} > 0$. But the latter is equivalent to $k \overset{\phi(\lambda)}{\Rightarrow} i$. Hence (1) holds. Conversely, by (1) it follows that the right-hand sde of (3) is infinite.

Suppose (1) hold. It is derived from Lemma 2.1, the remark of this theorem and (11.4.5) that

$$\lim_{\lambda\downarrow 0} \lambda \psi_{ii}(\lambda) = \lim_{\lambda\downarrow 0} \lambda \phi_{ii}(\lambda) + \lim_{\lambda\downarrow 0} \xi_i(\lambda) \frac{\sum_k \pi_k \phi_{ki}(\lambda)}{\sum_k \pi_k [1 - \xi_k(\lambda)]/\lambda}$$

It has been pointed out that the denominator of the right-hand side of the above formula is finite and positive. Hence the $\psi(\lambda)$-ergodic property of i is equivalent to (2). The proof is finished. QED

Corollary 1

Suppose for any $i, j, i \overset{\phi(\lambda)}{\Rightarrow} j$ and $\phi(\lambda)$ is non-recurrent. Then a (Q, π)Doob process is recurrent if and only if $\sum_k a_k \xi_k = 1$. If the process is recurrent,then it is ergodic if and only if (2) holds.

Proof. The necessity for recurrence is clear. We now prove the sufficiency. As $\sum_k \pi_k \xi_k = 1$, there surely exists some i such that $\pi_i > 0$ and $\xi_i = 1$. Hence for this i, (1) is satisfied, and so i is $\psi(\lambda)$-recurrent. On the other hand, obviously $i \overset{\phi(\lambda)}{\Leftrightarrow} j$ for any i, j for any i, j; therefore $\psi(\lambda)$ is recurrent.

Corollary 2

Suppose for any i, j, $i \overset{\phi(\lambda)}{\Leftrightarrow} j$ and Q is single exit. Then a (Q, π) Doobprocess is recurrent if and only if it is honest, i.e. Q is conservative and $\sum_k \pi_k = 1$.

Proof. For single exit Q, $P_i\{\tau < \infty \mid x(\tau - 0) \in B_e\} = 1$. Therefore by Theorem 7.12.8, $\xi_i = P_i\{\tau < \infty\} = 1$ and the condition $\sum_k \pi_k \xi_k = 1$ becomes $\sum_k \pi_k = 1$.

QED

15.4 SINGLE EXIT PROCESSES

Lemma 1. For $\eta(\lambda)$ and ξ given in Lemma 2.11.4

$$\lambda[\eta(\lambda), \xi] \downarrow 0 \qquad \text{as } \lambda \downarrow 0 \tag{1}$$

Proof. It follows from (2.11.40) and the property $\xi(\lambda) \uparrow \xi$ as $\lambda \downarrow 0$. QED

Theorem 2. Suppose $\psi(\lambda)$ is a process in Theorem 3.2.1 and i is $\phi(\lambda)$-non-recurrent. Then i is $\psi(\lambda)$-recurrent if and only if

$$\bar{X}_i > 0 \qquad \sum_k \alpha_k \Gamma_{kj} + \bar{\eta}_j > 0 \qquad c = [a, X^0] + \bar{\sigma}^0 = 0 \tag{2}$$

If i is $\psi(\lambda)$-recurrent, then i is $\psi(\lambda)$-ergodic if and only if

$$\sum_k \left(\alpha_k \sum_j \Gamma_{kj} \bar{X}_j + \bar{\eta}_k \bar{X}_k \right) < \infty \tag{3}$$

where $\bar{\eta}_j(\lambda) \uparrow \bar{\eta}_j$, as $\lambda \downarrow 0$.

Proof. Imitate the proof of Theorem 3.1. It suffices to note that from (2.11.7)

$$[\bar{X}_i - \bar{X}_i(\lambda)]/\lambda = \sum_j \phi_{ij}(\lambda) \bar{X}_j \uparrow \sum_j \Gamma_{ij} \bar{X}_j \qquad (\lambda \downarrow 0)$$

and to apply Lemma 1. QED

Corollary

Assume that Q is conservative and single exit. If for any $i, j \in E$, $i \overset{\phi(\lambda)}{\Leftrightarrow} j$ and $\phi(\lambda)$ is non-recurrent, then $\psi(\lambda)$ is recurrent if and only if $c = [a, X^0] + \bar{\sigma}^0 = 0$.

15.5 FIRST-ORDER PROCESSES

Suppose $X = \{x(t), t < \sigma\} \in \mathscr{H}_1(Q)$, i.e. X is a first-order process. It is a $\{\pi(a, \cdot), a \in B_e\}$-generalized D-type extension process of the minimal Q process $X^0 = \{x(t), t < \tau\}$, where $\pi(a, \cdot)$ satisfy (11.6.2). For this reason we call X a $\{Q, \pi(a, \cdot), a \in B_e\}$ first-order process.

For simplicity, in this section we assume that the non-atomic exit boundary B_{e_2} induced by Q is empty. That is, the exit boundary B_e entirely consists of atomic boundary points; simply put $\mathscr{A} = B_e$.

By (11.5.26) the resolvent operator $\psi(\lambda)$ of a $\{Q, \pi(a, \cdot), a \in \mathscr{A}\}$ first-order process has the following representation:

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \sum_{a \in \mathscr{A}} \sum_{b \in \mathscr{A}} X_i^a(\lambda) G_{ab}(\lambda) A_j^b(\lambda) \tag{1}$$

where

$$X_i^a(\lambda) = E_i\{e^{-\lambda\tau}, x(\tau - 0) = a\} \qquad A_j^b(\lambda) = \sum_k \pi(b, k) \phi_{kj}(\lambda).$$

$$\begin{aligned} &V_{ab}(\lambda) = \sum_k \pi(a, k) X_k^b(\lambda) \\ &\mathscr{V}(\lambda) = \{V_{ab}(\lambda)\} \qquad \{\mathscr{V}(\lambda)\}^l = \{V_{ab}^l(\lambda)\} \qquad \text{are } \mathscr{A} \times \mathscr{A} \text{ matrices} \\ &G_{ab}(\lambda) = \sum_{l=0}^{\infty} V_{ab}^l(\lambda) \end{aligned} \tag{2}$$

If we set

$$A = \{i \mid i \in E, \text{ there exists } a \in \mathscr{A} \text{ such that } \pi(a, i) > 0\} \tag{3}$$

then obviously

$$\pi(a, E - A) = 0 \qquad a \in \mathscr{A} \tag{4}$$

and the formula (1) can be rewritten as

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \sum_{r \in A} \sum_{q \in A} Z_{ir}(\lambda) D_{rq}(\lambda) \phi_j(\lambda) \tag{5}$$

where

$$\begin{aligned} &Z_{iq}(\lambda) = \sum_{a \in \mathscr{A}} X_i^a(\lambda) \pi(a, q) \qquad i \in E, q \in A \\ &\mathscr{Z}(\lambda) = \{Z_{rq}(\lambda)\} \qquad \{\mathscr{Z}(\lambda)\}^l = \{Z_{rq}^l(\lambda)\} \qquad \text{are } A \times A \text{ matrices} \\ &D_{rq}(\lambda) = \sum_{l=0}^{\infty} Z_{rq}^l(\lambda) \end{aligned} \tag{6}$$

Clearly, as $\lambda\downarrow 0$

$$X_i^a(\lambda)\uparrow X_i^a = P_i\{x(\tau-0)=a\} \qquad A_j^a(\lambda)\uparrow A_j^a=\sum_k \pi(a,k)\Gamma_{kj}$$

$$\mathscr{V}(\lambda)\uparrow\mathscr{V}=\left\{\sum_k \pi(a,k)X_k^b\right\} \qquad \{\mathscr{V}(\lambda)\}^l\uparrow\mathscr{V}^l$$

$$Z_{iq}(\lambda)\uparrow Z_{iq}=\sum_{a\in A} X_i^a\pi(a,q) \qquad \{\mathscr{Z}(\lambda)\}^l\uparrow\mathscr{Z}^l \tag{7}$$

$$\mathscr{G}(\lambda)\uparrow\mathscr{G}=\sum_{l=0}^{\infty}\mathscr{V}^l \qquad \mathscr{D}(\lambda)\uparrow\mathscr{D}=\sum_{l=0}^{\infty}\mathscr{Z}^l$$

Again obviously $\mathscr{Z}(\lambda)$, $\mathscr{Z}$ and $\mathscr{V}(\lambda)$, $\mathscr{V}$ may be considered as one-step transition matrices of Markov chains on the state spaces $\mathscr{A}$ and A, respectively. Hence we can say a is $\mathscr{V}(\lambda)$-recurrent, and so on. Thus it follows from (2.2.10) that

$$\begin{aligned} G_{ab}(\lambda)&\leqslant G_{bb}(\lambda) \qquad & G_{ab}&\leqslant G_{bb}\\ D_{rq}(\lambda)&\leqslant D_{qq}(\lambda) \qquad & D_{rq}&\leqslant D_{qq}\end{aligned} \tag{8}$$

Lemma 1. (i) Suppose for some $a,b\in\mathscr{A}$, $\lim_{\lambda\downarrow 0}\lambda G_{bb}(\lambda)=0$ and $a\overset{\mathscr{V}}{\Rightarrow}b$. Then

$$\lim_{\lambda\downarrow 0}\lambda G_{ab}(\lambda)=\lim_{\lambda\downarrow 0}\lambda G_{ba}(\lambda)=\lim_{\lambda\downarrow 0}\lambda G_{aa}(\lambda)=0$$

(ii) Suppose for some $r,q\in A$, $\lim_{\lambda\downarrow 0}\lambda D_{qq}(\lambda)=0$ and $r\overset{\mathscr{Z}}{\Rightarrow}q$. Then

$$\lim_{\lambda\downarrow 0}\lambda D_{rq}(\lambda)=\lim_{\lambda\downarrow 0}\lambda D_{qr}(\lambda)=\lim_{\lambda\downarrow 0}\lambda D_{rr}(\lambda)=0$$

Proof. It suffices to prove (i). By the hypothesis, there exist α,β such that $V_{ab}^{\alpha}V_{ba}^{\beta}>0$. However, by Chung (1966a, remark on p. 22)

$$V_{bb}^{\alpha+\beta+l}(\lambda)\geqslant V_{ba}^{\beta}(\lambda)V_{aa}^{l}(\lambda)V_{ab}^{\alpha}(\lambda)$$

It follows that

$$\lambda\left(G_{bb}(\lambda)-\sum_{l=0}^{\alpha+\beta}V_{bb}^{l}(\lambda)\right)\geqslant V_{ba}^{\beta}(\lambda)\lambda G_{aa}(\lambda)V_{ab}^{\alpha}(\lambda)$$

Noting (7) we have $V_{ba}^{\beta}(\lambda)V_{ab}^{\alpha}(\lambda)\uparrow V_{ba}^{\beta}V_{ab}^{\alpha}$ $(\lambda\downarrow 0)$. Therefore

$$\lim_{\lambda\downarrow 0}\lambda G_{bb}(\lambda)\geqslant V_{ba}^{\beta}\lim_{\lambda\downarrow 0}\lambda G_{aa}(\lambda)V_{ab}^{\alpha}$$

And so $\lim_{\lambda\downarrow 0}\lambda G_{aa}(\lambda)=0$. Then by (8) we have completed the proof. QED

Lemma 2. Suppose $\psi(\lambda)$ is a $\{Q,\pi(a,\cdot),a\in\mathscr{A}\}$ first-order process and i is $\phi(\lambda)$-non-recurrent and $\psi(\lambda)$-recurrent. Then:

(i) if $X_i^a>0$, a is $\mathscr{V}$-recurrent;

(ii) if $Z_{ir}>0$, r is $\mathscr{Z}$-recurrent.

Proof. (i) Suppose X is a $\psi(\lambda)$ process and τ^n is its nth leaping point for each n. It follows from the structure of the first-order processes that $x(\tau^n-0)$ $(n\geqslant 1)$ and $x(\tau^n)$ are $\mathscr{V}$-chains and $\mathscr{Z}$-chains, respectively. By the hypothesis and Lemma 2.2

$$P_i\{\tau<\infty\}=P_i\{a=\infty\}=1 \tag{9}$$

Let $\delta_0=0,\beta_0=\tau$, δ_1 be the time that X first returns to i after β_0, and β_1 be the first leaping point after δ_1. Then by (9) and the $\psi(\lambda)$-recurrence of i, $P_i\{\delta_1<\infty\}=P_i\{\beta_1<\infty\}=1$. Let δ_n be the time after β_{n-1} that X returns to i for the first time. Let β_n be the first leaping point after δ_n. Then it can be proved that $P_i\{\delta_n<\infty \text{ for all } n\}=1$.

It is easy to see that $\{x(\beta_n-0)=a\}\in\mathscr{F}'_{\delta_n}\cap\mathscr{F}_{\delta_{n+1}}$. By Theorem 7.7.3 they are independent. By the strong Markov property

$$P_i\{x(\beta_l-0)=a\}=P_i\{\delta_l<\infty,\theta_{\delta_l}[x(\tau-0)=a]\}=X_i^a>0 \tag{10}$$

Hence it follows from the Borel–Cantelli lemma that

$$P_i\{x(\beta_l-0)=a \text{ for infinitely many } l\}=1$$

and of course

$$\begin{aligned}&P_i\{x(\tau^l-0)=a \text{ for infinitely many } l\}\\ &\quad=\sum_{b\in\mathscr{A}}X_i^bP\{x(\tau^l-0)=a \text{ for infinitely many } l/x(\tau^1)=b\}=1\end{aligned}$$

Since $X_i^a>0$, it follows that $P\{x(\tau^l-0)=a$ for infinitely many $l/x(\tau^l-0)=a\}=1$, i.e. a is $\mathscr{V}$-recurrent.

(ii) The proof is similar to (i). The left-hand side of (10) is replaced by $P_i\{x(\beta_l)=r\}=Z_{ir}>0$. Hence $P_i\{x(\beta_l)=r$ for infinitely many $l\}=1$ and it goes without saying that

$$\begin{aligned}&P_i\{x(\tau^l)=r \text{ for infinitely many } l\}\\ &\quad=\sum_{q\in A}Z_{iq}P_i\{x(\tau^l)=r \text{ for infinitely many } l|x(\tau^1)=q\}=1\end{aligned}$$

Since $Z_{ir}>0$, it follows that $P_i\{x(\tau^l)=r$ for infinitely many $l|x(\tau^1)=r\}=1$, i.e. r is $\mathscr{Z}$-recurrent. And the proof is completed. QED

Theorem 3. Suppose $\psi(\lambda)$ is a $\{Q,\pi(a,\cdot),a\in\mathscr{A}\}$ first-order process and i is

$\phi(\lambda)$-non-recurrent. Then:

(i) i is $\psi(\lambda)$-recurrent if and only if there exist $a, b \in \mathscr{A}$ such that $X_i^a > 0$, $A_i^b > 0$, a is $\mathscr{V}$-recurrent and $a \overset{\mathscr{V}}{\Rightarrow} b$[1].

(ii) i is $\psi(\lambda)$-recurrent if and only if there exist $r, q \in A$ such that $Z_{ir} > 0$, $r \overset{\mathscr{Z}}{\Rightarrow} q \overset{\phi(\lambda)}{\Rightarrow} i$ and r is $\mathscr{Z}$-recurrent.

Proof. We need only prove (i). Let $\lambda \downarrow 0$ in (1) so that

$$\lim_{\lambda \downarrow 0} \psi_{ii}(\lambda) = \Gamma_{ii} + \sum_{a \in \mathscr{A}} \sum_{b \in \mathscr{A}} X_i^a G_{ab} A_i^b$$

$$= \Gamma_{ii} + \sum_{a,b \in \mathscr{A}}^{+} X_i^a g_{ab} G_{bb} A_i^b \tag{11}$$

where $\sum^+$ means that the summation is taken over positive summands and g_{ab} is the probability that a $\mathscr{V}$-chain starting from a reaches b after finite steps ($\geqslant 0$).

If i is $\psi(\lambda)$-recurrent, then the left-hand side of (11) is infinite, and so there must exist $a, b \in \mathscr{A}$ such that $X_i^a > 0, g_{ab} > 0$ and $A_i^b > 0$. By Lemma 2, a is $\mathscr{V}$-recurrent. But $g_{ab} > 0$ precisely demonstrates $a \overset{\mathscr{V}}{\Rightarrow} b$.

Suppose the sufficient conditions are satisfied. If $a = b$, then $g_{ab} = 1$, $G_{aa} = \infty$ and the right-hand side of (11) is infinite. If $a \neq b$, then since a is $\mathscr{V}$-recurrent and $a \overset{\mathscr{V}}{\Rightarrow} b$, i.e. $g_{ab} > 0$, it follows that b is also recurrent, i.e. $G_{bb} = \infty$. Therefore the right-hand side of (11) is also finite. We conclude the proof. QED

Theorem 4. Suppose for any i, j, $i \overset{\phi(\lambda)}{\Leftrightarrow} j$ and $\phi(\lambda)$ is non-recurrent. And suppose $\psi(\lambda)$ is a $\{Q, \pi(a, \cdot), a \in \mathscr{A}\}$ first-order process.

(i) If $\psi(\lambda)$ is recurrent, then naturally

$$Q \text{ is conservative} \qquad \xi_i = 1 \ (i \in E) \qquad \pi(a, E) = 1 \ (a \in \mathscr{A}) \tag{12}$$

where $\xi_i = P_i\{\tau < \infty\}$ is the same as (2.3).

(ii) If (12) holds and so does one of the following: (1°) $\mathscr{A}$ is finite; (2°) A is finite; (3°) $\mathscr{A}$ has at least one $\mathscr{V}$-recurrent state; (4°) A has at least one $\mathscr{Z}$-recurrent state; then $\psi(\lambda)$ is recurrent.

Proof. (i) By Lemma 2.2 $\xi_i = P_i\{\tau < \infty\} = 1$ $(i \in E)$ and $P_i\{\sigma = \infty\} = 1$. Hence Q must be conservative. Therefore, for $a \in \mathscr{A}$, by (11.6.2) we have

$$\sum_k \pi(a, k) = \sum_k P\{x(\tau) = k \mid x(\tau - 0) = a\}$$

[1] a and b may be the same, $a \overset{\mathscr{V}}{\Rightarrow} a$.

(ii) Suppose (12) holds. Under the hypotheses of the theorem, surely $X_i^a > 0$ and $A_i^a > 0$ for all $a \in \mathscr{A}, i \in E$, and so $Z_{ir} > 0$ for all $i \in E, r \in A$. Because

$$\sum_{b \in \mathscr{A}} V_{ab} = \sum_k \pi(a, k) \sum_{b \in \mathscr{A}} X_k^b = \sum_k \pi(a, k) \xi_k = 1$$

$$\sum_{q \in A} Z_{rq} = \sum_{a \in \mathscr{A}} X_r^a \sum_{q \in A} \pi(a, q) = \sum_{a \in \mathscr{A}} X_r^a = \xi_r = 1$$

if one of the conditions (1°)–(4°) holds, we can deduce that a ($\in A$) is $\mathscr{V}$-recurrent and γ ($\in A$) is $\mathscr{Z}$-recurrent. Thus the sufficient conditions in Theorem 3 hold. Hence $\psi(\lambda)$ is recurrent. The proof is completed. QED

Theorem 5. Suppose i is $\phi(\lambda)$-non-recurrent and $\psi(\lambda)$ is a $\{Q, \pi(a, \cdot), a \in \mathscr{A}\}$ first-order process. Suppose $\delta \equiv \sup_{a \in \mathscr{A}} \pi(a, E) < 1$, then i is $\psi(\lambda)$-non-recurrent.

Proof. By the hypothesis we know that $\Gamma_{ki} \leqslant \Gamma_{ii} < \infty$. Since

$$\sum_{a \in \mathscr{A}} X_i^a(\lambda) \leqslant \sum_{a \in \mathscr{A}} X_i^a \leqslant 1$$

it easily follows by (7) that

$$A_i^b(\lambda) = \sum_k \pi(b, k) \Gamma_{ki} \leqslant \delta \Gamma_{ii}$$

$$\sum_{b \in \mathscr{A}} T_{ab}^l(\lambda) \leqslant \delta^l \qquad \sum_{b \in \mathscr{A}} G_{ab}(\lambda) \leqslant \sum_{l=0}^{\infty} \delta^l = \frac{1}{1-\delta} < \infty$$

It follows from (1) that

$$\psi_{ii}(\lambda) \leqslant \Gamma_{ii} + \sum_{a \in \mathscr{A}} X_i^a(\lambda) \frac{1}{1-\delta} \delta \Gamma \leqslant \frac{\Gamma_{ii}}{1-\delta} < \infty$$

From this we know that i is $\psi(\lambda)$-non-recurrent. The proof is finished. QED

Lemma 6. Suppose $\psi(\lambda)$ is a $\{Q, \pi(a, \cdot), a \in \mathscr{A}\}$ first-order process and for any i, j, $i \overset{\psi(\lambda)}{\Leftrightarrow} j$. Then

(i) if $\pi(a, E) > 0$, then $a \overset{\mathscr{V}}{\Rightarrow} b$ $(b \in \mathscr{A})$;

(ii) for any $r, q \in A$, $r \overset{\mathscr{Z}}{\Rightarrow} q$.

Proof. Suppose X is a $\psi(\lambda)$ process and η_i^* is defined as in (7.6.7). Because i and j communicate with each other, with respect to $\psi(\lambda)$, we have

$$u_i = P_i\{\eta_j^* < \sigma\} > 0 \tag{13}$$

(i) By the hypothesis there exists i such that $\pi(a, i) > 0$ and for $b \in \mathscr{A}$ there

exists j such that $X_j^b > 0$. Thus

$$P\{\text{there exists } l \geqslant 1 \text{ such that } x(\tau^l - 0) = b \mid x(\tau^1 - 0) = a\} \geqslant \pi(a, i) u_{ij} X_i^b > 0$$

i.e. $a \overset{\gamma}{\Rightarrow} b$.

Claim (ii) can be proved similarly and the proof is completed. QED

Theorem 7. Suppose that $\psi(\lambda)$ is a $\{Q, \pi(a, \cdot), a \in \mathscr{A}\}$ first-order process, for any $i, j, i \overset{\phi(\lambda)}{\Leftrightarrow} j$, $\phi(\lambda)$ is non-recurrent and $\psi(\lambda)$ is recurrent. Then

(i) if A is finite, $\psi(\lambda)$ is ergodic if and only if

$$N_k = E_k \tau < \infty \qquad k \in A \tag{14}$$

(ii) if $\mathscr{A}$ is finite, $\psi(\lambda)$ is ergodic if and only if

$$\sum_k \pi(a, k) N_k < \infty \qquad a \in \mathscr{A} \tag{15}$$

Proof. We only prove (i). Claim (ii) can be proved similarly. Since $\psi(\lambda)$ is recurrent, by Lemma 2.2 $\psi(\lambda)$ is honest, i.e.

$$\lambda \sum_j \psi_i(\lambda) = 1 \qquad i \in E \tag{16}$$

Q is conservative since $\psi(\lambda)$ satisfies the system of backward equations. According to Theorem 3 there exists $r \in A$ such that r is $\mathscr{Z}$-recurrent. By Lemma 6 A is a $\mathscr{Z}$-recurrent class. Substituting (5) into (16), considering (1.5) and noting that $\xi_i(\lambda) = \sum_{a \in \mathscr{A}} X_i^a(\lambda)$ if Q is conservative, we arrive at

$$\sum_{a \in \mathscr{A}} X_i^a(\lambda) = \sum_{a \in \mathscr{A}} X_i^a(\lambda) \sum_{r, q \in \mathscr{A}} \pi(a, r) \lambda D_{rq}(\lambda) \sum_j \phi_q(\lambda).$$

It follows from the linear independence of $X^a(\lambda)$ that

$$1 = \sum_{r, q \in \mathscr{A}} \pi(a, r) \lambda D_{rq}(\lambda) \sum_j \phi_g(\lambda) \qquad a \in \mathscr{A} \tag{17}$$

(a) If there exists some $q \in A$ such that $N_q = \infty$, then for this q there exists $a \in \mathscr{A}$ such that $\pi(a, q) > 0$. By (17)

$$\pi(a, q) \lambda D_{qq}(\lambda) \sum_j \phi_{qj}(\lambda) \leqslant 1$$

$$\lambda D_{qq}(\lambda) \leqslant \frac{1}{\pi(a, q) \sum_j \phi_{qj}(\lambda)}$$

It follows that $\lim_{\lambda \downarrow 0} \lambda D_{qq}(\lambda) = 0$. By Lemma 1 for any $r, q \in A$, $\lim_{\lambda \downarrow 0} \lambda D_{rq}(\lambda) = 0$. Hence by (5) and the finiteness of A we have $\lim_{\lambda \downarrow 0} \lambda \psi_{ii}(\lambda) = 0$, i.e. $\psi(\lambda)$ is non-ergodic.

(b) Suppose $N_k < \infty$ for all k. We might as well assume that $\lim_{\lambda \downarrow 0} \lambda D_{rq}(\lambda)$ exist for all $r, q \in A$. Otherwise we can always choose a suitable subsequence of λ. Then by (17) and the finiteness of A,

$$\begin{aligned} 1 &= \sum_{r, q \in A} \pi(a, r) \lim_{\lambda \downarrow 0} \lambda D_{rq}(\lambda) \sum_j \Gamma_{qj} \\ &= \sum_j \sum_{r, q \in A} \pi(a, r) \lim_{\lambda \downarrow 0} \lambda D_{rq}(\lambda) \Gamma_{qj} \qquad a \in \mathscr{A} \end{aligned}$$

Consequently for fixed $a \in \mathscr{A}$ there exist $j \in E$ and $r, q \in A$ such that

$$\pi(a, r) \lim_{\lambda \downarrow 0} \lambda D_{rq}(\lambda) \Gamma_{qj} > 0$$

But for $a \in \mathscr{A}$ there must exist i such that $X_i^a > 0$. Therefore by (5)

$$\begin{aligned} \pi_i = \lim_{\lambda \downarrow 0} \lambda \psi_{ij}(\lambda) &\geqslant \lim_{\lambda \downarrow 0} z_{ir}(\lambda) \lambda D_{iq}(\lambda) \phi_{qj}(\lambda) \\ &\geqslant X_i^a \pi(a, r) \left(\lim_{\lambda \downarrow 0} \lambda D_{rq}(\lambda) \right) \Gamma_{qj} > 0 \end{aligned}$$

i.e. $\psi(\lambda)$ is ergodic. The proof is concluded. QED

Theorem 8. Suppose that the non-atomic exit boundary B_{e2} induced by Q is an empty set and $\mathscr{A} = B_e$ is a finite set. Then any D-type Q process $X \in \mathscr{H}_D(Q)$ is a $\{Q, \pi(a, \cdot), a \in \mathscr{A}\}$ first-order process, where

$$P\{x(\tau) = j \mid x(\tau - 0) = a\} = \pi(a, j) \tag{18}$$

Here τ is the first leaping point.

Proof. Since $X \in \mathscr{H}_D$, it follows that

$$\begin{aligned} P_{ij}(t) &= P_{ij}\{x(t) = j, t < \tau\} + P_i\{\tau \leqslant t, x(t) = j\} \\ &= f_{ij}(t) + \sum_{a \in \mathscr{A}} \sum_k P_i\{x(\tau - 0) = a, \tau \leqslant t, x(\tau) = k, x(t) = j\} \end{aligned}$$

Making use of the strong Markov property and applying Theorem 7.17.1, it is easy to prove that the summand of the above is equal to

$$\begin{aligned} \int_{\substack{x(\tau - 0) = a \\ x(\tau) = k, \tau \leqslant t}} p_{kj}(t - \tau) \mathrm{d}P_i &= \int_0^t p_{kj}(t - s) \mathrm{d}P_i\{x(\tau - 0) = a, x(\tau) = k, \tau \leqslant s\} \\ &\quad \times P_i\{x(\tau - 0) = a, x(\tau) = k, \tau \leqslant s\} \\ &= P_i\{x(\tau - 0) = a, \tau \leqslant s\} P\{x(\tau) = k \mid x(\tau - 0) = a\} \\ &= L_i^a(s) \pi(a, k) \end{aligned} \tag{19}$$

Thus (19) becomes

$$p_{ij}(t) = f_{ij}(t) + \sum_{a\in\mathscr{A}}\sum_k \pi(a,k)\int_0^t p_{kj}(t-s)\,\mathrm{d}L_i^a(s)$$

Taking Laplace transforms we have

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \sum_{a\in\mathscr{A}} X_i^a(\lambda)\sum_k \pi(a,k)\psi_{kj}(\lambda) \tag{20}$$

or

$$\psi(\lambda) = \phi(\lambda) + \{X^a(\lambda)\}'\{B^a(\lambda)\} \tag{21}$$

where $B_j^a(\lambda) = \sum_k \pi(a,k)\psi_{kj}(\lambda)$. Upon multiplying the above on the lefts by $\pi(a,\cdot)$ we obtain

$$\begin{aligned}\{B^a(\lambda)\} &= \{A^a(\lambda)\} + \mathscr{V}(\lambda)\{B^a(\lambda)\}\\ \{I-\mathscr{V}(\lambda)\}\{B^a(\lambda)\} &= \{A^a(\lambda)\}\end{aligned} \tag{22}$$

where $A^a(\lambda)$ is defined as (7) and $\mathscr{A}\times\mathscr{A}$ matrix $\mathscr{V}(\lambda)$ as (4.2). Because the matrices $\mathscr{V}(\lambda)$ have row sums $\sum_{b\in\mathscr{A}}\sum_k \pi(a,k)X_k^b(\lambda) < 1$ and $\mathscr{A}$ is finite, for each $\lambda > 0$, $\{I-\mathscr{V}(\lambda)\}^{-1}$ exists and is equal to $\mathscr{G}(\lambda) = \sum_{l=0}^{\infty}(\mathscr{V}(\lambda))^l$. Therefore by (22)

$$\{B^a(\lambda)\} = \{I-\mathscr{V}(\lambda)\}^{-1}\{A^a(\lambda)\} = \mathscr{G}(\lambda)\{A^a(\lambda)\}$$

Substituting this into (21) we find that $\psi(\lambda)$ have the form (1), i.e. $\psi(\lambda)$ is a $\{Q,\pi(a,\cdot),a\in\mathscr{A}\}$ first-order process. The proof is complete. QED

Remark

Many results in this section can also be extended to kth-order processes.

15.6 BILATERAL BIRTH–DEATH PROCESSES

In this section we shall utilize the notations given in Chapter 5.

Suppose that X is a bilateral birth–death process and τ is its first leaping point. Define the first hitting times ξ_i according to (12.2.6).

Lemma 1. Suppose $i \leqslant k \leqslant n$. Then

$$P_k\{\xi_i < \xi_n\} = \frac{z_n - z_k}{z_n - z_i} \qquad P_k\{\xi_n < \xi_i\} = \frac{z_k - z_i}{z_n - z_i} \tag{1}$$

Proof. Imitate the proof of Theorem 12.2.2. QED

Theorem 2. Let C_{kn} denote the probability that the process X starting from k reaches n after finite ($\geqslant 0$) jumps, i.e. $C_{kn} = P_k\{\xi_n < \tau\}$. Then

$$C_{kn} = \begin{cases}(r_2 - z_k)/(r_2 - z_n) & \text{if } n \leqslant k\\ (z_k - r_1)/(z_n - r_1) & \text{if } n > k\end{cases} \tag{2}$$

Proof. It follows by letting $n \to +\infty$ or letting $i \to -\infty$ respectively in (1). QED

Theorem 3. The minimal Q process is recurrent if and only if r_1 and r_2 are both infinite.

Proof. The probability that the process X starting from 0 returns to 0 is

$$f_0^* = \frac{a_0}{a_0+b_0}C_{-10} + \frac{b_0}{a_0+b_0}C_{10}$$

Hence by (2) we can see that $f_0^* = 1$ if and only if r_1 and r_2 are both infinite. The proof is completed. QED

Theorem 4. Suppose that the minimal Q process is recurrent. Then it is ergodic if and only if

$$\sum_k \mu_k < \infty \tag{3}$$

and in this case

$$m_{ii} = \left(\sum_k \mu_k\right)\Big/ q_i\mu_i \tag{4}$$

$$m_{ij} = (z_j - z_i)\sum_{s\leqslant t}\mu_s + \sum_{i<s<j}(z_j - z_s)\mu_s \qquad \text{if } j > i \tag{5}$$

$$m_{ij} = \sum_{j<s<i}(z_s - z_j)\mu_s + (z_i - z_j)\sum_{s\geqslant i}\mu_s \qquad \text{if } j < i \tag{6}$$

where $m_{ij} = E_i\eta_j^*$, where η_i^* is determined by (7.7.7).

Proof. Consider a minimal Q process $X = \{x(t), t < \infty\}$. Note that by recurrence we have $P_i\{\eta_i^* < \infty\} = P_i\{\xi_n < \infty\} = 1$. It is easy to see that for $i < k < n, u_k = \min(\xi_i, \xi_n)$ satisfy the equations (5.4.11) with $f_i = f_n = 0$ and $f_k = 1\,(i < k < n)$. Hence it follows from (5.4.12) that

$$E_k \min(\xi_i,\xi_n) = \sum_{i<s\leqslant k}\frac{(z_s - z_i)(z_n - z_k)}{z_n - z_i}\mu_s + \sum_{k<i<n}\frac{(z_k - z_i)(z_n - z_s)}{z_n - z_i}\mu_s \tag{7}$$

Letting $i \to -\infty, n \to +\infty$ in the above expression and noting that r_1 and r_2

are infinite, we obtain (5)–(6). By the strong Markov property

$$m_{ii} = \frac{1}{q_i} + \frac{a_i}{q_i} m_{i-1,i} + \frac{b_i}{q_i} m_{i+1,i}$$

Noting that $a_i(z_i - z_{i-1}) = b_i(z_{i+1} - z_i) = \mu_i^{-1}$ and substituting (5) and (6) into the above, we obtain (4). The proof is completed. QED

Lemma 5. If the minimal Q process is non-recurrent, then

$$N_k = E_k\tau = \sum_{s \leqslant k} \frac{(z_s - r_1)(r_2 - z_k)}{r_2 - r_1}\mu_s + \sum_{k<s} \frac{(z_k - r_1)(r_2 - z_s)}{r_2 - r_1}\mu_s$$

Proof. It follows by letting $i \to -\infty$ and $n \to +\infty$ in (7). QED

Theorem 6. Suppose that r_1 is entrance or natural and r_2 is exit or regular. Then a Q process $\psi(\lambda)$ is recurrent if and only if r_1 is inifinite and $\psi(\lambda)$ is honest.

Proof. This theorem is a special case of Theorem 4.2. Under the hypothesis, $\bar{X}_i = \bar{X}_i^2$ and $X_i^0 = X_i^1$. Thus the first two inequalities in (4.2) hold. Therefore (4.2) is equivalent to $c = 0$ and $X^0 = X^1 = 0$. By noting (5.7.2) our proof is finished. QED

Theorem 7. Suppose that r_1 is entrance or natural, r_2 is exit or regular, and $\psi(\lambda)$ is recurrent. Then

(i) if r_1 is exit, then $\psi(\lambda)$ is ergodic;

(ii) if r_1 is natural, (1*) when $\sum_{s\leqslant 0}\mu_s < \infty$, then $\psi(s)$ is non-ergodic; (2*) when $\sum_{s\leqslant 0}\mu_s < \infty$, and $\psi(\lambda)$ has the representation (5.8.3), then $\psi(\lambda)$ is ergodic if and only if

$$\sum_{s\leqslant 0} a_s N_s < \infty \tag{9}$$

In particular, if $\sum_{s\leqslant 0}\mu_s < \infty$ and $\sum_{s\leqslant 0} a_s(r_2 - z_s) < \infty$, then $\psi(\lambda)$ is ergodic.

Proof. Letting $\lambda \downarrow 0$ in (5.8.12) we have

$$\bar{\eta}_i = P_1(r_2 - z_j)\mu_j + P_2\mu_j \qquad \text{in which } P_1 = 0 \text{ if } r_1 \text{ is natural and } P_2 = 0 \text{ if } r_2 \text{ is exit} \tag{10}$$

But r_2 is regular or exit and, moreover,

$$\begin{aligned} &\sum_{j\leqslant 0}(r_2 - z_j)\mu_j < \infty \qquad \text{if } r_1 \text{ is entrance} \\ &\sum_{j\geqslant 0}\mu_j < \infty \qquad \text{if } r_2 \text{ is regular} \end{aligned} \tag{11}$$

Hence $\sum_i \bar{\eta}_i < \infty$ is equivalent to $P_2\sum_{s<0}\mu_s < \infty$. Note $\bar{X} = X^2 = 1$, so the ergodicity condition (4.3) becomes

$$\sum_s a_s N_s + P_2 \sum_{s<0}\mu_s < \infty \tag{12}$$

Because, by Theorem 6, r_1 is infinite, by (8)

$$N_k = (r_2 - z_k)\sum_{k\leqslant s}\mu_s + \sum_{s<k}(r_2 - z_s)\mu_s \tag{13}$$

We now proceed to prove that if $\sum_{s\leqslant 0}\mu_s < \infty$, then

$$\sum_{s\geqslant 0} a_s N_s < \infty \tag{14}$$

In fact, when $k \geqslant 0$

$$N_k = (r_2 - z_k)\sum_{s<0}\mu_s + M_k \tag{15}$$

where M_k is just N_k in (5.11.11). It follows from Theorems 5.11.3 and 5.11.4 that

$$\sum_{k\geqslant 0} a_k N_k < \infty$$

(i) If r_1 is entrance, then surely $\sum_{s\leqslant 0}\mu_s < \infty$. By (14) the ergodic property is equivalent to (9). But when $i \leqslant 0$, by (11) we have

$$N_i \leqslant \sum_{s\leqslant k}(r_2 - z_s)\mu_s + \sum_{k<s}(r_2 - z_s)\mu_s = \sum_s (r_2 - z_s)\mu_s < \infty$$

Hence by Theorem 5.11.5

$$\sum_{i\leqslant 0} a_i N_i \leqslant \left(\sum_{i\leqslant 0} a_i\right)\sum_s (r_2 - z_s)\mu_s < \infty$$

Therefore $\psi(\lambda)$ is ergodic.

(ii) Suppose r_1 is natural. (1*) Suppose $\sum_{s\leqslant 0}\mu_s < \infty$. By (13) $N_i = \infty$. If $a \neq 0$, then (12) is not satisfied. If $a = 0$, then necessarily $P_2 > 0$. Hence (12) is not satisfied, and so $\psi(\lambda)$ is not ergodic.

(2*) Suppose $\sum_{s\leqslant 0}\mu_s < \infty$. By (14), the ergodicity condition (12) becomes (9). Furthermore, suppose $\sum_{s\leqslant 0} a_s(r_2 - z_s) < \infty$. Then by (13) for $i \leqslant s$,

$$\begin{aligned} N_i &\leqslant (r_2 - z_i)\sum_{s<i}\mu_s + \sum_{i<s<0}(r_2 - z_s)\mu_s + \sum_{s\geqslant 0}(r_2 - z_s)\mu_s \\ &\leqslant (r_2 - z_i)\sum_{s<0}\mu_s + \sum_{s\geqslant 0}(r_2 - z_s)\mu_s \end{aligned}$$

By Theorem 5.11.5 we have $\sum_{k\leqslant 0} a_k<\infty$. Therefore $\sum_{s\leqslant 0} a_s N_s<\infty$, and so $\psi(\lambda)$ is ergodic. The proof is terminated. QED

Theorem 8. Suppose that r_1 and r_2 are regular or exit. Then:

(i) the Q process $\psi(\lambda)$ is recurrent if and only if it is honest;

(ii) any honest Q processes are ergodic.

Proof. By Lemma 2.2 $\psi(\lambda)$ is recurrent, so it must be honest.

First, we prove that if the vector $\alpha\geqslant 0$ satisfies $\alpha\phi(\lambda)\in l$, then

$$\sum_k a_k N_k<\infty \tag{16}$$

where N_k is determined acording to (8). When $k\geqslant 0$ from (8) it follows that

$$N_k\leqslant(r_2-z_k)\sum_{s<0}\frac{z_s-r_1}{r_2-r_1}\mu_s+M_k \tag{17}$$

where M_k is precisely N_k in (5.11.11). By Theorems 5.11.3 and 5.11.4 we have $\sum_{k\geqslant 0} a_k N_k<\infty$. Similarly $\sum_{k\leqslant 0} a_k N_k<\infty$ can be proved.

Next, for $\bar\eta_j(\lambda)$ defined in (5.7.12), let $\lambda\downarrow 0$. We obtain

$$\bar\eta=P_1X^1\mu+P_2X^2\mu \qquad \text{where } P_a=0 \text{ if } r_a \text{ is exit} \tag{18}$$

However, when r_2 is regular

$$\sum_j X_j^2\mu_j\leqslant\frac{1}{r_2-r_1}\sum_{j<0}(z_j-r_1)\mu_j+\sum_{j\geqslant 0}\mu_j<\infty$$

Similarly when r_1 is regular

$$\sum_j X_j^1\mu_j<\infty$$

Therefore we always have

$$\sum_j\bar\eta_j<\infty \tag{19}$$

Furthermore, it follows from Lemma 4.1 that

$$[\alpha^a, X^a-X^a(\lambda)]\downarrow 0 \qquad \lambda[\bar\eta(\lambda),X^a]\downarrow 0 \qquad (\lambda\downarrow 0) \tag{20}$$

If $\psi(\lambda)$ has the representation (5.10.16) ($c=0$ and $\alpha_1=\alpha_2$), then by (16) and (19), imitating Theorem 4.2, we can find that $\psi(\lambda)$ is recurrent and ergodic and, furthermore,

$$\pi_j=\lim_{\lambda\downarrow 0}\lambda\psi_{ij}(\lambda)=\frac{\sum_k a_k\Gamma_{kj}+P_1X_j^1\mu_j+P_2X_j^2\mu_j}{\sum_k a_kN_k+\sum_k(P_1X_k^1+P_2X_k^2)\mu_k} \tag{21}$$

Suppose that $\psi(\lambda)$ has the representation (5.10.1), (5.10.4) and (5.10.33) and that $\bar S^{12}=\bar S^{21}=1$. Write

$$\begin{aligned}\mathscr{G}(\lambda)=\mathscr{X}_\lambda^{-1}&=(I-\bar{\mathscr{S}}+\bar{\mathscr{H}}_\lambda+\bar{\mathscr{M}}\mathscr{U}_\lambda)^{-1}\\&=\sum_{l=0}^{\infty}\{\mathscr{F}(\lambda)\}^l\operatorname{diag}\left(\frac{1}{1-e_\lambda^{aa}}\right)\end{aligned} \tag{22}$$

where e_λ^{ab} are elements of the matrix $\bar{\mathscr{S}}-\bar{\mathscr{H}}_\lambda-\bar{\mathscr{M}}\mathscr{U}_\lambda$, and $\mathscr{F}(\lambda)=\{f^{ab}(\lambda)\}$ with

$$f^{aa}(\lambda)=0 \qquad f^{ab}(\lambda)=\frac{e_\lambda^{ab}}{1-e_\lambda^{aa}} \qquad (b\neq a) \tag{23}$$

It follows from (20) and $\bar S^{aa}=0$ that

$$1-e_\lambda^{aa}\uparrow 1 \qquad f^{ab}(\lambda)\uparrow\bar S^{ab} \qquad (\lambda\downarrow 0) \tag{24}$$

Thus

$$\mathscr{F}(\lambda)\uparrow\bar{\mathscr{S}} \qquad (\lambda\downarrow 0) \qquad \mathscr{G}(\lambda)\to\mathscr{G}=\sum_{l=0}^{\infty}\bar{\mathscr{S}}^l$$

and $\psi(\lambda)$ can be expressed as

$$\psi_{ij}(\lambda)=\phi_{ij}(\lambda)+\sum_{a=1}^{2}X_i^a(\lambda)G_{ab}(\lambda)\eta_i^b(\lambda) \tag{25}$$

where $\eta^b(\lambda)=\bar\alpha^b\phi(\lambda)+\bar M^{bb}X^b(\lambda)\mu$, in which $\bar M^{bb}=0$ when r_b is exit.

Because

$$\bar{\mathscr{S}}=\begin{pmatrix}0&1\\1&0\end{pmatrix} \tag{26}$$

let $\lambda\downarrow 0$ in (25) so that $\psi(\lambda)$ is recurrent. As $\psi(\lambda)$ is honest, it follows that

$$\lambda G_{ab}(\lambda)\sum_j\eta_j^b(\lambda)=1 \qquad a=1,2 \tag{27}$$

Paying attention to (16) and (18)–(19), and letting $\lambda\downarrow 0$ we obtain

$$\sum_j\eta_j^b(\lambda)\uparrow\sum_k(\bar\alpha_k^b N_k+\bar M^{bb}X_k^b\mu_k)<\infty$$

From (27) $\lim_{\lambda\downarrow 0}\lambda G_{ab}(\lambda)\sum_j\eta_j^b=1$. Hence there exists j such that $\lim_{\lambda\downarrow 0}\lambda G_{ab}(\lambda)\eta_j^b>0$.

From (25) we have

$$\lim_{\lambda\downarrow 0}\lambda\psi_{ij}(\lambda)\geqslant X_i^a\left(\lim_{\lambda\downarrow 0}\lambda G_{ab}(\lambda)\right)\eta_j^b>0$$

Therefore $\psi(\lambda)$ is ergodic. We conclude the proof. QED

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